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# Multiplicity of Solutions for a Nonlinear Degenerate Problem in Anisotropic Variable Exponent Spaces

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**Abstract.** We study a nonlinear elliptic problem with Dirichlet boundary condition involving an anisotropic operator with variable exponents on a smooth bounded domain $\Omega \subset \mathbb{R}^N$. For that equation we prove the existence of at least two nonnegative and nontrivial weak solutions. Our main result is proved using as main tools the Mountain Pass Theorem and a direct method in Calculus of Variation.

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## 1. Introduction

Equations involving variable exponent growth conditions have been extensively studied in the last decade. The large number of papers studying problems involving variable exponent growth conditions is motivated by the fact that this type of equations can serve as models in the theory of electrorheological fluids (see, e.g. [15]), image processing (see, e.g. [3]), the theory of elasticity (see, e.g. [18]) or biology (see, e.g. [10]). In this context, we just refer to the survey paper [11] and the references therein.

Typical models of elliptic equations with variable exponent growth conditions appeal to the so called $p(x)$-Laplace operator, that is

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),$$

where $p(x)$ is a function satisfying $p(x) > 1$ for each $x$. Recently, Mihăilescu-Pucci-Rădulescu extended in [13] the study involving the $p(x)$-Laplace operator to the case of anisotropic equations with variable exponent growth conditions, where the differential operator considered has the form

$$\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u), \tag{1.1}$$

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with $p_i(x)$ functions satisfying $\inf_x p_i(x) > 1$ for each $i \in \{1,...,N\}$. Undoubtedly, in the particular case when $p_i(x) = p(x)$ for each $i \in \{1,...,N\}$ the above differential operator becomes $\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p(x)-2}\partial_{x_i} u)$ and has similar properties with the $p(x)$-Laplace operator. On the other hand, the anisotropic equations with variable exponent growth conditions enable the study of equations with more complicated nonlinearities since the differential operator (1.1) allows a distinct behavior for partial derivatives in various directions.

In this paper we study the existence of nontrivial solutions of an nonhomogeneous anisotropic problem of type

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) = \lambda f(x,u), & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \\ u \geq 0, & \text{for } x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $p_i$ are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x)$ for any $x \in \overline{\Omega}$ and $i \in \{1,...,N\}$.

## 2. Preliminary results

Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we define *the variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\,dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [12, Theorem 2.5], the Hölder inequality holds [12, Theorem 2.1], they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ [12, Corollary 2.7] and continuous functions are dense if $p^+ < \infty$ [12, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [12, Theorem 2.8]: if $0 < |\Omega| < \infty$ and $p_1$, $p_2$ are variable exponents, so that $p_1(x) \leq p_2(x)$ almost everywhere in $\Omega$, then there exists the continuous embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{q(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$ for any $x \in \overline{\Omega}$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(\cdot)}|v|_{q(\cdot)} \tag{2.1}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If $(u_n)$, $u \in L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+} \tag{2.2}$$

$$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-} \tag{2.3}$$

$$|u_n - u|_{p(\cdot)} \to 0 \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u_n - u) \to 0. \tag{2.4}$$

Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [4].

Next, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(\cdot)}.$$

The space $(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact and continuous, where $p^\star(x) = Np(x)/(N - p(x))$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \ge N$. We refer to [5, 6, 7, 8, 12] for further properties of variable exponent Lebesgue-Sobolev spaces.

Finally, we introduce a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ that will enable us to study with sufficient accuracy problem (1.2). For this purpose, let us denote by $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\overrightarrow{p} = (p_1, ..., p_N)$, where $p_1$,..., $p_N : \overline{\Omega} \to (1, \infty)$ are continuous functions. We define $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, the *anisotropic variable exponent Sobolev space,* as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}.$$

$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ endowed with the above norm is a reflexive Banach space (see [13]).

On the other hand, in order to facilitate the manipulation of the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we introduced $\overrightarrow{P}_+$, $\overrightarrow{P}_-$ in $\mathbb{R}^N$ as

$$\overrightarrow{P}_+ = (p_1^+, ..., p_N^+), \quad \overrightarrow{P}_- = (p_1^-, ..., p_N^-),$$

and $P_+^+, P_-^+, P_-^- \in \mathbb{R}^+$ as

$$P_+^+ = \max\{p_1^+, ..., p_N^+\}, \quad P_-^+ = \max\{p_1^-, ..., p_N^-\}, \quad P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^N \frac{1}{p_i^-} > 1 \tag{2.5}$$

and define $P_-^\star \in \mathbb{R}^+$ and $P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^\star = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1}, \quad P_{-,\infty} = \max\{P_-^+, P_-^\star\}.$$

Next, we recall Theorem 1 in [13], regarding a compactness embedding of $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ into *variable exponent Lebesgue space*:

**Theorem 2.1.** *Assume that $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary. Assume relation (2.5) is fulfilled. For any $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{-,\infty} \quad for\ all \ \ x \in \overline{\Omega}, \tag{2.6}$$

*the embedding*

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

*is continuous and compact.*

## 3. The main result

In this paper we study problem (1.2) in the particular case

$$f(x,t) = t^{\alpha(x)-1} - t^{\beta(x)-1}, \ \ t \geq 0, \ x \in \Omega,$$

where $\alpha : \overline{\Omega} \to \mathbb{R}$, $\beta : \overline{\Omega} \to \mathbb{R}$ are continuous functions such that

$$1 < \beta^- \leq \beta(x) \leq \alpha(x) \leq \alpha^+ < P_-^- \leq P_+^+ < P_{-,\infty} \ \text{ for } \ x \in \Omega$$

and there exists $x_0 \in \Omega$ such that

$$\beta(x_0) < \alpha(x_0).$$

More precisely, we consider the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) = \lambda(u^{\alpha(x)-1} - u^{\beta(x)-1}), & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \\ u \geq 0, & \text{for } x \in \Omega. \end{cases} \tag{3.1}$$

We seek solutions for problem (3.1) belonging to the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the sense below.

**Definition 3.1.** *We say that $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a weak solution for problem (3.1) if $u \geq 0$ almost everywhere in $\Omega$ and*

$$\int_\Omega \left\{ \sum_{i=1}^{N} \left( |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \, \partial_{x_i} v \right) - \lambda \left( u^{\alpha(x)-1} \, v - u^{\beta(x)-1} \, v \right) \right\} dx = 0,$$

*for all $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.*

The main result of this paper is given by the following theorem.

**Theorem 3.1.** *There exists $\overline{\lambda} > 0$ such that problem (3.1) has at least two distinct nonnegative, nontrivial weak solutions for all $\lambda \geq \overline{\lambda}$.*

Theorem 3.1 supplements [14] with the case of anisotropic spaces.

## 4. Proof of Theorem 3.1

We start by introducing the energy functional corresponding to problem (3.1) which is defined as $J_\lambda : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$,

$$J_\lambda(u) = \int_\Omega \sum_{i=1}^{N} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\ dx - \lambda \int_\Omega \frac{u_+^{\alpha(x)}}{\alpha(x)}\ dx + \lambda \int_\Omega \frac{u_+^{\beta(x)}}{\beta(x)}\ dx, \tag{4.1}$$

where $u_+(x) = \max\{u(x), 0\}$ for each $x \in \Omega$. Standard arguments assure that $J_\lambda \in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ and the Fréchet derivative is given by

$$\langle J_\lambda'(u), v\rangle = \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u\, \partial_{x_i} v\ dx - \lambda \int_\Omega u_+^{\alpha(x)-1}\ v\, dx + \lambda \int_\Omega u_+^{\beta(x)-1}\ v\, dx, \tag{4.2}$$

for all $u,\ v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

We prove some auxiliary results.

**Lemma 4.1.** *If $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ then $u_+, u_- \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and*

$$\frac{\partial\, u_+}{\partial x_i} = \begin{cases} 0, & \text{if } [u \le 0], \\ \dfrac{\partial\, u}{\partial x_i}, & \text{if } [u > 0], \end{cases} \qquad \frac{\partial\, u_-}{\partial x_i} = \begin{cases} 0, & \text{if } [u \ge 0], \\ \dfrac{\partial\, u}{\partial x_i}, & \text{if } [u < 0], \end{cases}$$

*where $u_\pm(x) = \max\{\pm u(x), 0\}$ for all $x \in \Omega$.*

*Proof.* The proof of this lemma runs similarly with the proof of Lemma 3.3 in [14] without any particular complication. Because of that fact we will omit it. ∎

**Lemma 4.2.** *If $u$ is a critical point of functional $J_\lambda$ then $u \ge 0$.*

*Proof.* If $u$ is a critical point of $J_\lambda$ then using Lemma 4.1 we obtain

$$\begin{aligned} 0 =& \langle J_\lambda'(u), u_-\rangle \\ =& \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u\, \partial_{x_i} u_-\ dx - \lambda \int_\Omega u_+^{\alpha(x)-1} u_-\ dx + \\ & \lambda \int_\Omega u_+^{\beta(x)-1} u_-\ dx \\ =& \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u\, \partial_{x_i} u_-\ dx \\ =& \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u_-|^{p_i(x)-2} \partial_{x_i} u_-\ \partial_{x_i} u_-\ dx \\ =& \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u_-|^{p_i(x)}\ dx, \end{aligned}$$

that is

$$\int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u_-|^{p_i(x)}\ dx = 0. \tag{4.3}$$

Applying Jensen's inequality to the convex function $g:[0,\infty)\to[0,\infty)$ defined by $g(s)=s^{P_-^-}$ it follows that for all $u\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we have

$$\frac{\|u_-\|^{P_-^-}_{\overrightarrow{p}(\cdot)}}{N^{P_-^- -1}}=N\left(\frac{\sum\limits_{i=1}^{N}|\partial_{x_i}u_-|_{p_i(\cdot)}}{N}\right)^{P_-^-}\leq\sum_{i=1}^{N}|\partial_{x_i}u_-|^{P_-^-}_{p_i(\cdot)}. \tag{4.4}$$

For each $i\in\{1,\ldots,N\}$ we define

$$\xi_i=\begin{cases} P_+^+, & \text{if } |\partial_{x_i}u_-|_{p_i(\cdot)}<1,\\ P_-^-, & \text{if } |\partial_{x_i}u_-|_{p_i(\cdot)}>1.\end{cases}$$

Thus, using relation (2.2), (2.3) and (4.4) we get

$$\int_\Omega\sum_{i=1}^{N}|\partial_{x_i}\,u_-|^{p_i(x)}\,dx\geq\sum_{i=1}^{N}|\partial_{x_i}\,u_-|^{\xi_i}_{p_i(\cdot)}.$$

By above pieces of information and relation (4.3) we deduce that $\|u_-\|_{\overrightarrow{p}(\cdot)}=\sum\limits_{i=1}^{N}|\partial_{x_i}u_-|_{p_i(\cdot)}=0$ which means that $u\geq 0$. ∎

We consider the functional $I:W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\to\mathbb{R}$ defined by

$$I(u)=\int_\Omega\sum_{i=1}^{N}\frac{|\partial_{x_i}\,u|^{p_i(x)}}{p_i(x)}\;dx$$

for every $u\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Standard arguments assure that $I$ is well-defined on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, $I\in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),\mathbb{R})$ and the Fréchet derivative is given by

$$\langle I'(u),v\rangle=\int_\Omega\sum_{i=1}^{N}|\partial_{x_i}u|^{p_i(x)-2}\,\partial_{x_i}u\,\partial_{x_i}v\;\;dx$$

for all $u,v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

**Lemma 4.3.** *The functional $I$ is weakly lower semicontinuous.*

*Proof.* The conclusion of this lemma is obvious since we deal with a functional $I$ which is continuous and convex on the Banach space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. ∎

**Lemma 4.4.** *There exists a positive constant $\mathscr{A}$ such that*

$$\sum_{i=1}^{N}\int_\Omega\frac{|\partial_{x_i}\,u|^{p_i(x)}}{p_i(x)}\,dx\geq\mathscr{A}\int_\Omega|u|^{P_-^-}\,dx$$

*for all $u\in S:=\{v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega):\ \|v\|_{\overrightarrow{p}(\cdot)}>N\}$.*

*Proof.* We fix arbitrary $u\in S$. Since $\|u\|_{\overrightarrow{p}(\cdot)}>N$ it follows that there exists $j\in\{1,...,N\}$ such that $|\partial_{x_j}u|_{p_j(\cdot)}>1$. Then we deduce that

$$\int_\Omega\frac{|\partial_{x_j}\,u|^{p_j(x)}}{p_j(x)}\,dx\geq\frac{1}{P_+^+}|\partial_{x_j}u|^{P_-^-}_{p_j(\cdot)}. \tag{4.5}$$

On the other hand, since $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we infer that $\partial_{x_j} u \in L^{p_j(\cdot)}(\Omega)$. Since $P_-^- \le p_j(x)$ for every $x \in \overline{\Omega}$ we get that $L^{p_j(\cdot)}(\Omega)$ is continuously embedded in $L^{P_-^-}(\Omega)$ and consequently, there exists a positive constant, say $C_j > 0$, such that

$$|\partial_{x_j} u|_{P_-^-} \le C_j |\partial_{x_j} u|_{p_j(\cdot)}\,,$$

or

$$|\partial_{x_j} u|_{p_j(\cdot)}^{P_-^-} \ge \frac{1}{C_j^{P_-^-}} \int_\Omega |\partial_{x_j} u|^{P_-^-}\, dx\,. \tag{4.6}$$

Now, using similar arguments as those used in the proof of relation (11) in [9] we obtain the existence of a positive constant, say $D_j > 0$, such that

$$\int_\Omega |\partial_{x_j} u|^{P_-^-}\, dx \ge D_j \int_\Omega |u|^{P_-^-}\, dx\,. \tag{4.7}$$

Relations (4.5), (4.6) and (4.7) imply the existence of a positive constant

$$\mathscr{A} := \min_{j\in\{1,\ldots,N\}} \frac{D_j}{P_+^+ C_j^{P_-^-}}$$

for which the conclusion of Lemma 4.4 holds true. ∎

**Remark 4.1.** By Lemma 4.4 we deduce that there exists $\lambda^\star > 0$ such that

$$\lambda^\star = \inf_{u\in S} \frac{\displaystyle\sum_{i=1}^N \int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\, dx}{\displaystyle\int_\Omega |u|^{P_-^-}\, dx} \tag{4.8}$$

where $S := \{v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) :\ \|v\|_{\overrightarrow{p}(\cdot)} > N\}$.

**Lemma 4.5.**

1° *The functional $J_\lambda$ is coercive and bounded from below.*
2° *The functional $J_\lambda$ is weakly lower semicontinuous.*

*Proof.* 1°. By relation $1 < \beta^- \le \beta(x) \le \alpha(x) \le \alpha^+ < P_-^-$ for every $x \in \Omega$ and $\beta(x_0) < \alpha(x_0)$, we get that

$$\lim_{t\to\infty} \frac{\dfrac{1}{\alpha(x)}\, t^{\alpha(x)} - \dfrac{1}{\beta(x)}\, t^{\beta(x)}}{t^{P_-^-}} = 0$$

for every $x \in \Omega$. Then for any $\lambda > 0$ there exists a positive constant $C_\lambda$ such that

$$\lambda\left(\frac{1}{\alpha(x)}\, t^{\alpha(x)} - \frac{1}{\beta(x)}\, t^{\beta(x)}\right) \le \frac{\lambda^\star}{2}\, t^{P_-^-} + C_\lambda$$

for all $t \ge 0$ and $x \in \Omega$, where $\lambda^\star$ is given by relation (4.8).
The above inequality shows that for any $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(\cdot)} > N$ we obtain

$$J_\lambda(u) = \sum_{i=1}^N \int_\Omega \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\, dx - \lambda \int_\Omega \left(\frac{1}{\alpha(x)}\, u_+^{\alpha(x)} - \frac{1}{\beta(x)}\, u_+^{\beta(x)}\right) dx$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{N} \int_{\Omega} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\ dx - \lambda \int_{\Omega} \left( \frac{1}{\alpha(x)}\ |u|^{\alpha(x)}\ - \frac{1}{\beta(x)}\ u_+^{\beta(x)} \right)\ dx \\
&\geq \sum_{i=1}^{N} \int_{\Omega} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\ dx - \frac{\lambda^\star}{2} \int_{\Omega} |u|^{P_-^-}\ dx - C_\lambda \cdot |\Omega| \\
&\geq \sum_{i=1}^{N} \frac{1}{2} \int_{\Omega} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\ dx - C_\lambda \cdot |\Omega| \\
&\geq \frac{1}{2P_+^+} \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u|^{p_i(x)}\ dx - C_\lambda \cdot |\Omega|.
\end{aligned}
$$

In order to go further, we define for each $i \in \{1, ..., N\}$ and each $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(\cdot)} > N$

$$
\kappa_{i,u} = \begin{cases} P_+^+, & \text{if } |\partial_{x_i} u|_{p_i(\cdot)} < 1, \\ P_-^-, & \text{if } |\partial_{x_i} u|_{p_i(\cdot)} > 1. \end{cases}
$$

Using relations (2.2) and (2.3) we infer that for each $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(\cdot)} > N$ we have

$$
\begin{aligned}
\sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u|^{p_i(x)}\ dx &\geq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}^{\kappa_{i,u}} \\
&\geq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}^{P_-^-} - \sum_{\{i:\ \kappa_{i,u} = P_+^+\}} \left( |\partial_{x_i} u|_{p_i(\cdot)}^{P_-^-} - |\partial_{x_i} u|_{p_i(\cdot)}^{P_+^+} \right) \\
&\geq \frac{1}{N^{P_-^-}} \|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-} - N.
\end{aligned}
$$

From the above pieces of information we find that for each $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|u\|_{\overrightarrow{p}(\cdot)} > N$ the following estimate holds true

$$
J_\lambda(u) \geq \frac{1}{2P_+^+ N^{P_-^-}}\ \|u\|_{\overrightarrow{p}(\cdot)}^{P_-^-} - \frac{N}{2P_+^+} - C_\lambda |\Omega|.
$$

This inequalities show that $J_\lambda$ is coercive and bounded from below.

$2^\circ$. By Lemma 4.3 we have that the functional $I : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$
I(u) = \int_{\Omega} \sum_{i=1}^{N} \frac{|\partial_{x_i}\ u|^{p_i(x)}}{p_i(x)}\ dx
$$

for every $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is weakly lower semicontinuous. Next, we prove that $J_\lambda$ is weakly lower semicontinuous. Let $\{u_n\} \subset W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ be a sequence that converges weakly to $u$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Since $I$ is weakly lower semicontinuous we deduce that

$$
I(u) \leq \liminf_{n\to\infty}\ I(u_n). \tag{4.9}
$$

On the other hand, as $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is continuously and compactly embedded in $L^{\alpha(\cdot)}(\Omega)$ and $L^{\beta(\cdot)}(\Omega)$ (by Theorem 2.1) it follows that $\{(u_n)_+\}$ converges strongly to $u_+$ in $L^{\alpha(\cdot)}(\Omega)$ and

$L^{\beta(\cdot)}(\Omega)$. This fact and relation (4.9) imply that

$$J_\lambda(u) \leq \liminf_{n\to\infty} J_\lambda(u_n),$$

that is the functional $J_\lambda$ is weakly lower semicontinuous. The proof of Lemma 4.5 is completed. ∎

*By Lemma 4.5 and Theorem 1.2 in* [16], *we deduce that there exists* $v_1 \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ **a global minimizer of** $J_\lambda$**,** *thus* $v_1 \geq 0$ *in* $\Omega$ *by Lemma 4.2.*

**Lemma 4.6.** *There exists* $\overline{\lambda} > 0$ *such that* $\inf_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)} J_\lambda < 0$*, for all* $\lambda \geq \overline{\lambda}$*.*

*Proof.* Since $\beta(x_0) < \alpha(x_0)$ we can choose a small neighborhood $\Omega_1 \subset \Omega$ of $x_0$ and we deduce that there exists an element $v_0 \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega_1) \subset W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$\int_\Omega \left( \frac{v_0^{\alpha(x)}}{\alpha(x)} - \frac{v_0^{\beta(x)}}{\beta(x)} \right) dx > 0.$$

Consequently, there exists $\overline{\lambda} > 0$ such that $J_\lambda(v_0) < 0$ for any $\lambda \geq \overline{\lambda}$. ∎

**Remark 4.2.** By Lemma 4.6 and the fact that $v_1$ is a global minimizer of $J_\lambda$ it follows that $J_\lambda(v_1) < 0$ for any $\lambda \geq \overline{\lambda}$ and thus, we find that $v_1$ is a nontrivial weak solution of problem (3.1) for $\lambda > 0$ large enough.

We fix $\lambda \geq \overline{\lambda}$ and consider function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$h(x,t) = \begin{cases} 0, & \text{if } t < 0, \\ t^{\alpha(x)-1} - t^{\beta(x)-1}, & \text{if } 0 \leq t \leq v_1(x), \\ v_1^{\alpha(x)-1}(x) - v_1^{\beta(x)-1}(x), & \text{if } t > v_1(x) \end{cases}$$

and function $H : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$H(x,t) = \int_0^t h(x,s)\ ds,$$

that is the primitive of function $h$ with respect to the second variable.

Define the functional $K_\lambda : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \mathbb{R}$ by

$$K_\lambda(v) = \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} v|^{p_i(x)}}{p_i(x)}\ dx - \lambda \int_\Omega H(x,v)\ dx.$$

Standard arguments assures that $K_\lambda \in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \mathbb{R})$ and its Frechét derivative is given by

$$\langle K_\lambda'(v), w\rangle = \int_\Omega \sum_{i=1}^N |\partial_{x_i} v|^{p_i(x)-2}\, \partial_{x_i} v\, \partial_{x_i} w\ dx - \lambda \int_\Omega h(x,v)\, w\ dx,$$

for all $v, w \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

**Remark 4.3.** We point out that if $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a critical point of functional $K_\lambda$ then $v \geq 0$. The proof is similar with the one considered in the case of $J_\lambda$.

**Lemma 4.7.** *If* $v$ *is a critical point of functional* $K_\lambda$ *then* $v \leq v_1$*.*

*Proof.* We have

$$
\begin{aligned}
0 =& \langle K'_\lambda(v),(v-v_1)_+\rangle - \langle J'_\lambda(v_1),(v-v_1)_+\rangle \\
=& \int_\Omega \sum_{i=1}^N \left[|\partial_{x_i} v|^{p_i(x)-2}\partial_{x_i} v - |\partial_{x_i} v_1|^{p_i(x)-2}\partial_{x_i} v_1\right]\partial_{x_i}(v-v_1)_+ dx - \\
& \lambda \int_\Omega \left[h(x,v) - \left(v_1^{\alpha(x)-1} - v_1^{\beta(x)-1}\right)\right](v-v_1)_+ dx \\
=& \int_{[v>v_1]} \sum_{i=1}^N \left[|\partial_{x_i} v|^{p_i(x)-2}\partial_{x_i} v - |\partial_{x_i} v_1|^{p_i(x)-2}\partial_{x_i} v_1\right]\partial_{x_i}(v-v_1) dx,
\end{aligned}
$$

that is

$$
\int_{[v>v_1]} \sum_{i=1}^N \left[|\partial_{x_i} v|^{p_i(x)-2}\partial_{x_i} v - |\partial_{x_i} v_1|^{p_i(x)-2}\partial_{x_i} v_1\right](\partial_{x_i} v - \partial_{x_i} v_1)\, dx = 0. \tag{4.10}
$$

Next, we recall that the following elementary inequality

$$
(|\eta|^{t-2}\eta - |\zeta|^{t-2}\zeta)(\eta-\zeta) \geq 2^{-t}\ |\eta-\zeta|^t \text{ for all } \eta,\zeta \in \mathbb{R}^N
$$

which is valid for all $t \geq 2$. By equality (4.10), applying the above inequality we get

$$
\sum_{i=1}^N \int_{[v>v_1]} |\partial_{x_i} v - \partial_{x_i} v_1|^{p_i(x)}\ \ dx = 0,
$$

so $\partial_{x_i}\ v(x) = \partial_{x_i}\ v_1(x)$ for all $i \in \{1,\ldots,N\}$ and $x \in \Omega_2 := \{y \in \Omega :\ v(y) > v_1(y)\}$.

Hence

$$
\sum_{i=1}^N \int_{\Omega_2} |\partial_{x_i} v - \partial_{x_i} v_1|^{p_i(x)}\ \ dx = 0,
$$

and thus,

$$
\sum_{i=1}^N \int_{\Omega} |\partial_{x_i} (v - v_1)_+|^{p_i(x)}\ \ dx = 0.
$$

By relations (2.2) and (2.3) we obtain $\|(v-v_1)_+\|_{\overrightarrow{p}(\cdot)} = 0$. Since $v - v_1 \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ by Lemma 4.1 we have that $(v-v_1)_+ \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Thus, we obtain that $(v-v_1)_+ = 0$ in $\Omega$ which means that $v \leq v_1$ in $\Omega$. ∎

Next, we will determinate a nontrivial critical point for functional $K_\lambda$ using as a main tool the Mountain Pass Theorem. In order to do this, we prove the following lemma.

**Lemma 4.8.** *There exist two constants $\theta \in (0, \|v_1\|_{\overrightarrow{p}(\cdot)})$ and $a > 0$ such that*

$$
K_\lambda(v) \geq a \text{ for all } v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \text{ with } \|v\|_{\overrightarrow{p}(\cdot)} = \theta.
$$

*Proof.* We fix $v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ arbitrary with $\|v\|_{\overrightarrow{p}(\cdot)} < 1$. Obviously we have that

$$
\frac{1}{\alpha(x)}\ s^{\alpha(x)} - \frac{1}{\beta(x)}\ s^{\beta(x)} \leq 0 \text{ for any } s \in [0,1] \text{ and any } x \in \Omega.
$$

We define $\Omega_3 := \big\{x \in \Omega :\ v(x) > \min\{1, v_1(x)\}\big\}$.
If $x \in \Omega \setminus \Omega_3$ we have that $v(x) \leq v_1(x)$ and $v(x) \leq 1$ and we deduce that

$$
H(x,v) = \frac{1}{\alpha(x)}\ v_+^{\alpha(x)} - \frac{1}{\beta(x)}\ v_+^{\beta(x)} \leq 0.
$$

If $x \in \Omega_3 \cap \{x \in \Omega;\ v_1(x) < v(x) < 1\}$ we have that

$$H(x,v) = \frac{1}{\alpha(x)}\ v_1^{\alpha(x)} - \frac{1}{\beta(x)}\ v_1^{\beta(x)} + (v_1^{\alpha(x)-1} - v_1^{\beta(x)-1}) \cdot (v - v_1) \leq 0.$$

Therefore, we conclude that $H(x,v) \leq 0$ on $(\Omega \setminus \Omega_3) \cup (\Omega_3 \cap \{x \in \Omega;\ v_1(x) < v(x) < 1\})$.

We define the set

$$\Omega_3' := \Omega_3 \setminus \{x \in \Omega;\ v_1(x) < v(x) < 1\}.$$

By relation (2.3), for all $w \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $\|w\|_{\overrightarrow{p}(\cdot)} < 1$, we obtain using Jensen's inequality

(4.11)

$$\frac{\|w\|_{\overrightarrow{p}(\cdot)}^{P_+^+}}{N^{P_+^+-1}} = N\left(\frac{\sum_{i=1}^{N} |\partial_{x_i} w|_{p_i(\cdot)}}{N}\right)^{P_+^+} \leq \sum_{i=1}^{N} |\partial_{x_i} w|_{p_i(\cdot)}^{P_+^+} \leq \sum_{i=1}^{N} |\partial_{x_i} w|_{p_i(\cdot)}^{p_i^+} \leq \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} w|^{p_i(x)}\, dx.$$

Provided that $\|v\|_{\overrightarrow{p}(\cdot)} < 1$ by relation (4.11) we get

$$K_\lambda(v) \geq \frac{1}{P_+^+}\ \frac{\|v\|_{\overrightarrow{p}(\cdot)}^{P_+^+}}{N^{P_+^+-1}} - \lambda \int_{\Omega_3'} H(x,v)\ dx. \tag{4.12}$$

We choose a constant $r$ such that $1 < P_+^+ < r < P_{-,\infty}$. By Theorem 2.1 it follows that $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is continuously embedded in $L^r(\Omega)$ that means there exists a positive constant $\mathfrak{C}_1$ such that

$$|v|_{L^r(\Omega)} \leq \mathfrak{C}_1\ \|v\|_{\overrightarrow{p}(\cdot)}\ \text{ for all }\ v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{4.13}$$

Using the definition of functional $H$ and (4.13) we have

$$\begin{aligned}
\lambda \int_{\Omega_3'} H(x,v)dx =& \lambda \int_{\Omega_3' \cap\, [v>v_1]} \left(\frac{v_1^{\alpha(x)}}{\alpha(x)} - \frac{v_1^{\beta(x)}}{\beta(x)}\right) dx + \\
& \lambda \int_{\Omega_3' \cap\, [v>v_1]} (v_1^{\alpha(x)-1} - v_1^{\beta(x)-1})(v - v_1)dx + \\
& \lambda \int_{\Omega_3' \cap\, [v<v_1]} \left(\frac{v_+^{\alpha(x)}}{\alpha(x)} - \frac{v_+^{\beta(x)}}{\beta(x)}\right)\ dx \\
\leq& \frac{\lambda}{\alpha^-} \int_{\Omega_3' \cap\, [v>v_1]}\ v_1^{\alpha(x)} dx + \lambda \int_{\Omega_3' \cap [v>v_1]} v_1^{\alpha(x)-1} v\, dx + \\
& \frac{\lambda}{\alpha^-} \int_{\Omega_3' \cap\, [v<v_1]} v_+^{\alpha(x)}\ dx \\
\leq& \lambda\ \mathfrak{C}_2 \int_{\Omega_3'}\ v_+^{\alpha(x)}\ dx \\
\leq& \lambda\ \mathfrak{C}_2 \int_{\Omega_3'}\ v_+^{r}\ dx \\
\leq& \lambda\ \mathfrak{C}_3\ \|v\|_{\overrightarrow{p}(\cdot)}^{r},
\end{aligned}$$

where $\mathfrak{C}_2$ and $\mathfrak{C}_3$ are positive constants.

By the above inequalities we deduce that for a $\theta \in (0, \min\{1, \|v_1\|_{\overrightarrow{p}(\cdot)}\})$ small enough we get

$$K_\lambda(v) \geq \left( \frac{1}{P_+^+ \, N^{P_+^+ - 1}} - \lambda \, \mathfrak{C}_3 \, \|v\|_{\overrightarrow{p}(\cdot)}^{r - P_+^+} \right) \|v\|_{\overrightarrow{p}(\cdot)}^{P_+^+}.$$

Since $r > P_+^+$ the proof of Lemma 4.8 is completed. ∎

**Lemma 4.9.** *Functional $K_\lambda$ is coercive.*

*Proof.* This proof can be carried out in a similar manner as the proof of Lemma 4.5 and for that reason we shall omit it. ∎

**Proof of Theorem 3.1.** By Lemma 4.8 and the Mountain Pass Theorem (see [1] with the variant given by [17, Theorem 1.15]), we deduce that there exists a sequence $\{v_n\} \subset W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$K_\lambda(v_n) \to c > 0 \ \text{ and } \ K'_\lambda(v_n) \to 0, \tag{4.14}$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} K_\lambda(\gamma(t)) \geq a > 0,$$

with $a$ given by Lemma 4.8 and

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)) : \ \gamma(0) = 0, \ \gamma(1) = v_1\}.$$

By relation (4.14) and Lemma 4.9, we obtain that $\{v_n\}$ is a bounded sequence and thus, passing eventually to a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$ we may assume that there exists $v_2 \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that $\{v_n\}$ converges weakly to $v_2$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

We will show that $\{v_n\}$ converges strongly to $v_2$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.
By relation (4.14) we have that

$$\lim_{n\to\infty} \langle K'_\lambda(v_n), v_n - v_2 \rangle = 0. \tag{4.15}$$

We get

$$\langle I'(v_n) - I'(v_2), v_n - v_2 \rangle = \langle K'_\lambda(v_n) - K'_\lambda(v_2) \rangle + \lambda \int_\Omega [h(x, v_n) - h(x, v_2)] \, (v_n - v_2) \, dx. \tag{4.16}$$

Since by Theorem 2.1 the anisotropic variable exponent space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is continuously and compactly embedded in the Lebesgue spaces $L^{\alpha(\cdot)}(\Omega)$ and $L^{\beta(\cdot)}(\Omega)$, we conclude that $\{v_n\}$ converges strongly to $v_2$ in $L^{\alpha(\cdot)}(\Omega)$ and $L^{\beta(\cdot)}(\Omega)$.

Then by (4.15) and (4.16) we deduce that

$$\langle I'(v_n) - I'(v_2), v_n - v_2 \rangle = o(1)$$

which is equivalent with

$$\lim_{n\to\infty} \sum_{i=1}^N \int_\Omega \left[ |\partial_{x_i} v_n|^{p_i(x)-2} \, \partial_{x_i} v_n - |\partial_{x_i} v_2|^{p_i(x)-2} \, \partial_{x_i} v_2 \right] \, (\partial_{x_i} v_n - \partial_{x_i} v_2) \ dx = 0. \tag{4.17}$$

Next, we recall again the inequality

$$(|\eta|^{t-2}\eta - |\zeta|^{t-2}\zeta)(\eta - \zeta) \geq 2^{-t} \, |\eta - \zeta|^t \text{ for all } \eta, \zeta \in \mathbb{R}^N$$

which is valid for all $t \geq 2$. Applying the above inequality in equality (4.17), we get

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} v_n - \partial_{x_i} v_2|^{p_i(x)} \ dx = 0,$$

and, consequently, the sequence $\{v_n\}$ converges strongly to $v_2$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Since $K_\lambda \in C^1(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),\mathbb{R})$ and relation (4.14) holds true, we find $K_\lambda(v_2) = c > 0$ and $K'_\lambda(v_2) = 0$ in $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)^\star$, the dual space of $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

By Lemma 4.7 and Remark 4.3 we deduce that $0 \leq v_2 \leq v_1$ in $\Omega$. Therefore, $h(x,v_2) = v_2^{\alpha(x)-1} - v_2^{\beta(x)-1}$ and

$$H(x,v_2) = \frac{v_2^{\alpha(x)}}{\alpha(x)} - \frac{v_2^{\beta(x)}}{\beta(x)}$$

and thus

$$K_\lambda(v_2) = J_\lambda(v_2) \ \text{ and } \ K'_\lambda(v_2) = J'_\lambda(v_2).$$

We conclude that $v_2$ is a critical point of functional $J_\lambda$ and thus a weak solution of Problem (3.1).

Moreover, since $J_\lambda(v_2) = c > 0 = J_\lambda(0)$ it follows that $v_2$ is nontrivial. On the other hand, by relation $J_\lambda(v_2) = c > 0 > J_\lambda(v_1)$, where the latter inequality is given by Remark 4.2, we have that $v_2 \neq v_1$.

In conclusion, we proved that problem (3.1) has two distinct nonnegative and nontrivial weak solutions for $\lambda$ large enough. The proof of our main result is complete. ∎

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