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Completely Continuous Linear Maps on Semigroup Algebras

ALI GHAFFARI
Department of Mathematics, Semnan University, Semnan, Iran
aghaffari@semnan.ac.ir

Abstract. For a locally compact group G, $L^1(G)$ is its group algebra and $L^\infty(G)$ is the dual of $L^1(G)$. Crombez and Govaerts introduced the notion of a uniformly measurable function in $L^\infty(G)$ and proved that such a function induces a completely continuous operator. The aim of this paper is to go further and generalize the above results to foundation semigroup algebras. We study completely continuous linear maps on semigroup algebras which commute with translations.

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1. Introduction and notations

Let S be a locally compact topological semigroup with convolution measure algebra $M(S)$ and let $M_p(S)$, as usual, be the convolution semigroup of probability measures on S. Recall that $M_a(S)$ denotes the space of all measures $\mu \in M_a(S)$ for which the mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M(S)$ are weakly continuous, where δ_x denotes the Dirac measure at x [1]. A *foundation* semigroup is a locally compact semigroup such that $\cup\{\text{supp}(\mu);\ \mu \in M_a(S)\}$ is dense in S. A trivial example is a topological group and in this case $M_a(S) = L^1(S)$ [6]. If S is a foundation topological semigroup with identity, then $M_a(S)$ is a closed two-sided L-ideal of $M(S)$. We also note that if S is a foundation semigroup with identity, then $M_a(S)$ has a bounded approximate identity and, for every μ in $M_a(S)$, both mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M_a(S)$ are norm continuous [5]. We denote by $\mathscr{K}$ the family of compact subsets of S.

For any foundation topological semigroup S, we denote by $M_a(S)^*$ and $M_a(S)^{**}$ its first and second dual. By definition, the first Arens product on $M_a(S)^{**}$ is induced by the left $M_a(S)$-module structure on $M_a(S)$. That is, for $E, F \in M_a(S)^{**}$, $f \in M_a(S)^*$, and $\mu, \nu \in M_a(S)$, we have

$$\langle EF, f\rangle = \langle E, Ff\rangle,$$

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where $\langle Ff, \mu\rangle = \langle F, f\mu\rangle$ and $\langle f\mu, \nu\rangle = \langle f, \mu * \nu\rangle$. Equipped with this multiplication, $M_a(S)^{**}$ is a Banach algebra and this multiplication agrees on $M_a(S)$ with the given product [2].

Let $\mathrm{LUC}(S)$ be the space of all $f \in C_b(S)$ such that the mapping $x \mapsto L_x f$ from S into $C_b(S)$, where $L_x f(y) = f(xy)$, $x, y \in S$, is continuous. If S is a foundation topological semigroup with identity, then $\mathrm{LUC}(S) = M_a(S)^* M_a(S)$, where $M_a(S)^* M_a(S) = \{f\mu;\ f \in M_a(S)^*,\ \mu \in M_a(S)\}$. Among the elements of $\mathrm{LUC}(S)^*$ are the unit point masses δ_x for $x \in S$. These do not appear in $M_a(S)^{**}$.

Let $\mathscr{A}$ be a Banach space. A bounded linear map $T : \mathscr{A} \to \mathscr{A}^*$ is called *completely continuous* or a *Dunford-Pettis* operator if every weakly convergent sequence $\{b_n\}$ is mapped into a norm convergent sequence $\{T(b_n)\}$ (see 1.6.1 in [13] or [15]). It is known that every completely continuous linear map $T : \mathscr{A} \to \mathscr{A}^*$ maps weak Cauchy sequences into norm convergent sequences. Dunford and Schwartz [4] made a systematic study of representation theorems for compact and weakly compact linear maps of L^1 into a Banach space $\mathscr{A}$ [4]. Let (X, σ, μ) be a positive measure space and let T be a weakly compact linear map of $L^1(X, \sigma, \mu)$ into a B-space. Then T maps weak Cauchy sequences into strongly convergent sequences, see p. 508 in [4].

Let S be a foundation topological semigroup with identity. For $f \in M_a(S)^*$, let $\lambda_f : M_a(S) \to M_a(S)^*$ be the bounded linear map of multiplication by f on left, i.e., $\lambda_f(\mu) = f\mu$. f is called completely continuous if λ_f maps any weakly convergent sequence onto a norm convergent sequence.

In this paper, among the other things, we introduce the notion of a uniformly measurable functional and prove that if $f \in M_a(S)^*$ is uniformly measurable then λ_f is completely continuous. Crombez and Govarets [3] proved that if G is a nondiscrete locally compact metrizable group, then $\lambda_f : L^1(G) \to L^\infty(G)$ is completely continuous if and only if f is uniformly measurable.

2. Main results

The following theorem shows that the space of all bounded linear maps $T : M_a(S) \to M_a(S)^*$ commuting with translations can be identified with $M_a(S)^*$. More information on linear operators which commute with translations can be found in [10] and [7].

Theorem 2.1. *Let S be a foundation topological semigroup with identity. Then the following statements hold:*

(i) *Let $T : M_a(S) \to M_a(S)^*$ be a bounded linear map such that $T(\mu * \delta_x) = T(\mu)\delta_x$ for every $\mu \in M_a(S)$ and $x \in S$. Then there exists a unique functional $f \in M_a(S)^*$ such that $T = \lambda_f$ and $\|T\| = \|f\|$;*

(ii) *Further suppose that S is commutative. Let $T : M_a(S) \to \mathrm{LUC}(S)^*$ be a bounded linear map such that $T(\mu * \delta_x) = T(\mu)\delta_x$ for every $\mu \in M_a(S)$ and $x \in S$. Then there exists a unique functional $F \in \mathrm{LUC}(S)^*$ such that $T = \lambda_F$ and $\|T\| = \|F\|$.*

Proof. (i) By [8, Theorem 3] and its proof, there exists a functional $f \in M_a(S)^*$ such that $T(\mu) = f\mu$ for all $\mu \in M_a(S)$. For any $\mu \in M_a(S)$, $\|T(\mu)\| = \|f\mu\| \le \|f\|\|\mu\|$. It follows that $\|T\| \le \|f\|$. Let $\mu \in M_a(S)$, and let $\{e_\alpha\}$ be a bounded approximate identity of $M_a(S)$ bounded by one [11]. Then

$$|\langle f, \mu\rangle| = \lim_\alpha |\langle f, e_\alpha * \mu\rangle| = \lim_\alpha |\langle fe_\alpha, \mu\rangle| = \lim_\alpha |\langle T(e_\alpha), \mu\rangle| \le \|T\|\|\mu\|.$$

This shows that $\|f\| \leq \|T\|$, and so $\|T\| = \|f\|$. It is easy to see that f is unique.

(ii) Let $\mu, \nu \in M_a(S)$ and $f \in \mathrm{LUC}(S)$. It is easy to see that any $f \in \mathrm{LUC}(S)$ can be written in the form $f = f_1\eta$ with $f_1 \in M_a(S)^*$ and $\eta \in M_a(S)$. Thus by [5]

$$\begin{aligned}\langle T(\mu * \nu), f\rangle &= \langle T^*(f), \mu * \nu\rangle = \int \langle T^*(f), \mu * \delta_x\rangle d\nu(x)\\ &= \int \langle T(\mu * \delta_x), f\rangle d\nu(x) = \int \langle T(\mu), f_1\eta * \delta_x\rangle d\nu(x)\\ &= \langle T(\mu), f_1\eta * \nu\rangle = \langle T(\mu)\nu, f\rangle.\end{aligned}$$

Since this holds for all $f \in \mathrm{LUC}(S)$, we conclude that $T(\mu * \nu) = T(\mu)\nu$ for every $\mu, \nu \in M_a(S)$. Let $\{e_\alpha\}$ be an approximate identity of norm one in $M_a(S)$. By passing to a subnet if necessary, we may assume that $\{T(e_\alpha)\}$ converges to some $F \in \mathrm{LUC}(S)^*$ in the weak* topology of $\mathrm{LUC}(S)^*$ [14]. Then for every $f \in \mathrm{LUC}(S)$ and $\mu \in M_a(S)$,

$$\langle T(\mu), f\rangle = \lim_\alpha \langle T(e_\alpha * \mu), f\rangle = \lim_\alpha \langle T(e_\alpha), f\mu\rangle = \langle F, f\mu\rangle = \langle F\mu, f\rangle.$$

This shows that $T = \lambda_F$. We will show that F is unique. Let $E \in \mathrm{LUC}(S)^*$. What we must show is that if $F\mu = E\mu$ for all $\mu \in M_a(S)$, then $F = E$. Let $f \in \mathrm{LUC}(S)$. Then $f = f_0\eta$ for some $f_0 \in M_a(S)^*$ and $\eta \in M_a(S)$. We have

$$\begin{aligned}\langle F, f\rangle &= \langle F, f_0\eta\rangle = \lim_\alpha \langle F, f_0\eta * e_\alpha\rangle = \lim_\alpha \langle Fe_\alpha, f_0\eta\rangle = \lim_\alpha \langle Ee_\alpha, f_0\eta\rangle\\ &= \lim_\alpha \langle E, f_0\eta * e_\alpha\rangle = \langle E, f_0\eta\rangle = \langle E, f\rangle.\end{aligned}$$

It is not hard to see that $\|T\| = \|F\|$. ∎

Definition 2.1. *If S is a foundation topological semigroup, a bounded linear functional f on $M_a(S)$ is said to be uniformly measurable if for all $\varepsilon > 0$ and $K \in \mathscr{K}$, there exists a Borel measurable partition $\mathfrak{K} = \{K_i\}_{i=1}^m$ of K such that, to each $\nu \in M_a(S)$ and $1 \leq i \leq m$, there corresponds a complex number $c_{\nu,i}$ such that $|\langle f, \delta_x * \nu\rangle - c_{\nu,i}| < \|\nu\|\varepsilon$ for all $x \in K_i$.*

Note that the following theorem generalizes [3, Theorem 3.2].

Theorem 2.2. *Let S be a foundation topological semigroup with identity. If $f \in M_a(S)^*$ is uniformly measurable, then λ_f is completely continuous.*

Proof. Let $\{\mu_n\}$ be a sequence in $M_a(S)$ such that $\{\mu_n\}$ converges to 0 in the weak topology of $M_a(S)$. Thus $\{\mu_n\}$ is weakly bounded and so bounded [14]. Let $c = \sup\{\|\mu_n\|;\, n \in \mathbb{N}\}$. If $\|f\| = 0$, we have trivially $f\mu_n \to 0$ in the norm topology. We now consider the case $\|f\| > 0$. Given $\varepsilon \in (0,1)$, there exists a compact subset K in S such that $|\mu_n|(S \setminus K) < \varepsilon/(3\|f\|)$ for every $n \in \mathbb{N}$. By assumption, there exists a Borel measurable partition $\mathfrak{K} = \{K_i\}_{i=1}^m$ of K such that, to each $\nu \in M_a(S)$ and $1 \leq i \leq m$, there corresponds a complex number $c_{\nu,i}$ such that $|\langle f, \delta_x * \nu\rangle - c_{\nu,i}| < (\|\nu\|\varepsilon)/(3c+1)$ for all $x \in K_i$. Clearly $|c_{\nu,i}| \leq (1 + \|f\|)\|\nu\|$ for all $\nu \in M_a(S)$ and $i \in \{1,\ldots,m\}$. For $\nu \in M_a(S)$ with $\|\nu\| \leq 1$ and $n \in \mathbb{N}$, by [5]

$$\begin{aligned}|\langle f\mu_n, \nu\rangle| &= \Big|\int_{S\setminus K} \langle f, \delta_x * \nu\rangle d\mu_n(x)\Big| + \Big|\int_K \langle f, \delta_x * \nu\rangle d\mu_n(x)\Big|\\ &\leq \frac{\varepsilon}{3} + \Big|\sum_{i=1}^m \int_{K_i} \langle f, \delta_x * \nu\rangle - c_{\nu,i} d\mu_n(x) + \sum_{i=1}^m \int_{K_i} c_{\nu,i} d\mu_n(x)\Big|\\ &\leq \frac{\varepsilon}{3} + \frac{\|\mu_n\|\varepsilon}{3c} + \Big|\sum_{i=1}^m \int_{K_i} c_{\nu,i} d\mu_n(x)\Big|\end{aligned}$$

$$\leq \frac{2\varepsilon}{3} + \Big| \sum_{i=1}^{m} \int_{K_i} c_{\nu,i} d\mu_n(x) \Big|.$$

Since $\{\mu_n\}$ is weakly convergent to zero in $M_a(S)$, the second member on the right hand side of the above inequality is bounded by $\varepsilon/3$ for sufficiently large values of n. Since this holds for all $\nu \in M_a(S)$ with $\|\nu\| \leq 1$, we have $\|f\mu_n\| < \varepsilon$ for sufficiently large values of n. We conclude that $\{f\mu_n\}$ is norm convergent to zero in $M_a(S)^*$. This shows that λ_f is completely continuous. ∎

Take the additive group $\mathbb{R}$ of real numbers. Since $\mathbb{R}$ is nondiscrete, it follows from [3, Theorem 3.5] that if $f \in M_a(\mathbb{R})^*$ is not uniformly measurable, then λ_f is not completely continuous.

Example 2.1. Take the additive group $\mathbb{R}$ of real numbers. For all $n \in \mathbb{N}$, we consider the measurable functions $\varphi_n(x) = e^{-inx}\chi_{[0,2\pi]}(-x)$ where $\chi_{[0,2\pi]}$ is the characteristic function of $[0,2\pi]$ in $\mathbb{R}$. The sequence $\{\varphi_n\}$ converges weakly to zero in $M_a(\mathbb{R})$. If $x = t + 2m\pi$ for $t \in (0,2\pi]$ and $m \in \mathbb{Z}$, we define $f(x) = f(t+2m\pi) = e^{imt}$. Then f is bounded and continuous. It is not hard to see that $\|f\varphi_n\| > \pi$ for every $n \in \mathbb{N}$. This shows that λ_f is not completely continuous [3]. For every $n \in \mathbb{N}$, choose a function h_n in $C_c(\mathbb{R})$ (the space of all complex-valued continuous function on $\mathbb{R}$ having compact support) such that $0 \leq h_n \leq 1$, and $h_n = 1$ on $[-n,n]$. Clearly $h_n f$ is completely continuous and $\{h_n f\}$ converges to f in the weak*-topology of $M_a(\mathbb{R})^*$.

By Example 2.1, it is not true that for given f in $M_a(S)^*$ the function $\mu \mapsto f\mu$ from $M_a(S)$ to $M_a(S)^*$ is always completely continuous. From the following theorem, we deduce that all λ_f are completely continuous if $f \in M_a(S)^* M_a(S)$.

Recall that a functional $f \in M_a(S)^*$ is called weakly almost periodic if the set $\{f\mu;\ \mu \in M_a(S), \|\mu\| \leq 1\}$ is relatively compact with respect to the weak topology on $M_a(S)^*$. The set of weakly almost periodic functionals of $M_a(S)^*$ is denoted by $\mathrm{wap}(M_a(S))$ [8].

Theorem 2.3. *Let S be a foundation topological semigroup with identity. Then the following statements hold:*

(i) *If $f \in M_a(S)^* M_a(S)$, then f is uniformly measurable;*
(ii) *If $f \in \mathrm{wap}(M_a(S))$, then λ_f is completely continuous.*

Proof. (i) Let $\varepsilon > 0$ and K be a compact subset of S. For every $x \in K$, there exists a neighbourhood U_x of x such that $\|f\delta_x - f\delta_y\| < \varepsilon/2$ whenever $y \in U_x$. Indeed, $f = h\mu$ for some $h \in M_a(S)^*$ and $\mu \in M_a(S)$. By [5], the mapping $x \mapsto \mu * \delta_x$ from S into $M_a(S)$ is continuous. Thus we can find an open neighbourhood U_x of x such that $\|f\delta_x - f\delta_y\| < \varepsilon/2$. Now cover K by $\{U_x;\ x \in K\}$. By compactness we may extract a finite subcover $\{U_{x_1}, \ldots, U_{x_k}\}$ of K. We can find a Borel measurable partition $\mathfrak{K} = \{K_i\}_{i=1}^m$ of K such that $\|f\delta_x - f\delta_y\| < \varepsilon$ whenever $x, y \in K_i$ and $1 \leq i \leq m$. For every $i \in \{1, \ldots, m\}$, we may choose $x_i \in K_i$. For every $\nu \in M_a(S)$ with $\|\nu\| \leq 1$, we have

$$|\langle f, \delta_x * \nu\rangle - \langle f, \delta_{x_i} * \nu\rangle| \leq \|f\delta_x - f\delta_{x_i}\| < \varepsilon,$$

whenever $x \in K_i$ and $i \in \{1, \ldots, m\}$. Let $\nu \in M_a(S)$ and $1 \leq i \leq m$, we put $c_{\nu,i} = \langle f, \delta_{x_i} * \nu\rangle$. This shows that f is uniformly measurable.

(ii) Let $f \in \mathrm{wap}(M_a(S))$, and let $\{e_\alpha\}$ be an approximate identity of norm one in $M_a(S)$ [5]. For every $\alpha \in I$, $fe_\alpha \in \{f\mu;\ \mu \in M_a(S), \|\mu\| \leq 1\}$. By passing to a subnet if necessary, we may assume that $\{fe_\alpha\}$ converges to some $h \in M_a(S)^*$ in the weak topology. For $\mu \in M_a(S)$,

$$\langle f,\mu\rangle = \lim_\alpha \langle f, e_\alpha * \mu\rangle = \lim_\alpha \langle fe_\alpha,\mu\rangle = \langle h,\mu\rangle.$$

We conclude that $f = h$, and so $\{fe_\alpha\}$ converges to f in the weak topology. Finally, since $M_a(S)^*$ is a locally convex space and $\{f\mu;\ \mu \in M_a(S), \|\mu\| \le 1\}$ is convex, the weak closure of $\{f\mu;\ \mu \in M_a(S), \|\mu\| \le 1\}$ equals the closure of $\{f\mu;\ \mu \in M_a(S), \|\mu\| \le 1\}$ in the norm topology on $M_a(S)^*$ [14]. Since $M_a(S)$ has a bounded approximate identity, by Cohen's factorization theorem, $M_a(S)^*M_a(S)$ is a Banach subspace of $M_a(S)^*$ (see [9, Theorem 32.22]). It follows that

$$f \in \overline{\{f\mu;\ \mu \in M_a(S),\ \|\mu\| \le 1\}} \subseteq M_a(S)^*M_a(S).$$

By Theorem 2.2, f is completely continuous. ∎

Corollary 2.1. *Let S be a foundation topological semigroup with identity. Then the following statements hold:*

(i) *If $f \in M_a(S)^*M_a(S)$, then λ_f is completely continuous;*
(ii) *If S is discrete, then for every $f \in M_a(S)^*$, λ_f is completely continuous.*

Proof. (i) From Theorem 2.2 and Theorem 2.3, we conclude that all λ_f $(f \in M_a(S)^*M_a(S))$ are completely continuous.

(ii) Let S be a discrete semigroup. Then $M_a(S)^*M_a(S) = M_a(S)^*$. By (i), λ_f is completely continuous for every $f \in M_a(S)^*$. ∎

Whenever S is a discrete semigroup, every λ_f is completely continuous. For non-discrete S this property is not true: for instance, the sequence of functions $t \mapsto e^{int}$ on the circle group $\mathbb{T}$ is not norm convergent in $M_a(\mathbb{T})$, but converges weakly to zero by the Riemann-Lebesgue lemma.

Theorem 2.4. *The set of all uniformly measurable functionals in $M_a(S)^*$ is a norm closed linear subspace of $M_a(S)^*$.*

Proof. It is trivial that the set of uniformly measurable functionals is closed under scalar multiplication. It suffices to prove that this set is closed for addition. Let f_1, f_2 be uniformly measurable functionals in $M_a(S)^*$, and let $\varepsilon > 0$, $K \in \mathscr{K}$. There exist $\mathfrak{K}^1 = \{K_i^1\}_{i=1}^{m_1}$ and $\mathfrak{K}^2 = \{K_j^2\}_{j=1}^{m_2}$ of Borel measurable partitions of K such that for every $\nu \in M_a(S)$ and $1 \le i \le m_1$ $(1 \le j \le m_2)$, there corresponds a complex number $c_{\nu,i}^1$ $(c_{\nu,j}^2)$ such that

$$|\langle f_1,\delta_x * \nu\rangle - c_{\nu,i}^1| < \frac{\|\nu\|\varepsilon}{2}, \quad \left(|\langle f_2,\delta_x * \nu\rangle - c_{\nu,j}^2| < \frac{\|\nu\|\varepsilon}{2}\right)$$

whenever $x \in K_i^1$ $(x \in K_j^2)$ and $1 \le i \le m_1, 1 \le j \le m_2$. Now define the partition $\mathfrak{K}$ wanted for $f_1 + f_2$ as $\mathfrak{K} = \{K_i^1 \cap K_j^2; 1 \le i \le m_1, 1 \le j \le m_2\}$. For $\nu \in M_a(S)$, we have

$$|\langle f_1 + f_2,\delta_x * \nu\rangle - c_{\nu,i}^1 - c_{\nu,j}^2| \le |\langle f_1,\delta_x * \nu\rangle - c_{\nu,i}^1| + |\langle f_2,\delta_x * \nu\rangle - c_{\nu,j}^2| < \|\nu\|\varepsilon$$

whenever $x \in K_i^1 \cap K_j^2$ and $1 \le i \le m_1$, $1 \le j \le m_2$. This shows that the set of uniformly measurable functionals is closed under addition.

Now let $\{f_n\}$ be a sequence of uniformly measurable functionals such that $\{f_n\}$ converges to some $f \in M_a(S)^*$ in the norm topology of $M_a(S)^*$. Let $\varepsilon > 0$ and K be a compact subset of S. There exists $n_0 \in \mathbb{N}$ such that $\|f_{n_0} - f\| < \varepsilon/2$. The triangle inequality shows that the partition of K for n_0 and the numbers $c_{\nu,i}$ for f_{n_0} corresponding to a given measure ν in $M_a(S)$ may also be used for f. This completes the proof. ∎

Definition 2.2. *For a compact subset K in S and a Borel measurable partition $\mathfrak{K} = \{K_i\}_{i=1}^m$ of K, we call a functional h in $M(S)^*$ an $\mathfrak{K}$-step functional if h is constant on each probability measure $\mu \in M_p(S)$ with $\mu(K_i^c) = \{0\}$. We write Step $\mathfrak{K}$ for the space of all $\mathfrak{K}$-step functionals in $M(S)^*$.*

Theorem 2.5. *For a functional f in $M_a(S)^*$, the following statements are equivalent:*

(i) *f is uniformly measurable;*

(ii) *For every $\varepsilon > 0$ and $K \in \mathscr{K}$, there exists a Borel measurable partition $\mathfrak{K} = \{K_i\}_{i=1}^m$ of K and a dense subset D_i of K_i $(1 \le i \le m)$ such that, to each $\nu \in M_a(S)$ and $1 \le i \le m$, there corresponds a complex number $c_{\nu,i}$ such that $|\langle f, \delta_x * \nu\rangle - c_{\nu,i}| < \|\nu\|\varepsilon$ for all $x \in D_i$;*

(iii) *For every $\varepsilon > 0$ and $K \in \mathscr{K}$, there exists a Borel measurable partition $\mathfrak{K} = \{K_i\}_{i=1}^m$ of K such that, to each $\nu \in M_a(S)$, there corresponds a functional h_ν in Step $\mathfrak{K}$ such that*
$$|\langle f, \mu * \nu\rangle - \langle h_\nu, \mu\rangle| < \|\nu\|\varepsilon$$
for all $1 \le i \le m$ and $\mu \in M_p(S)$ with $\mu(K_i^c) = \{0\}$.

Proof. (i) implies (ii): This is trivial.

(ii) implies (i): Let $\varepsilon > 0$ and K be a compact subset in S. There exists a Borel measurable partition $\mathfrak{K} = \{K_i\}_{i=1}^m$ of K such that corresponding to any $\nu \in M_a(S)$ and $1 \le i \le m$, a complex number $c_{\nu,i}$ may be found such that $|\langle f, \delta_d * \nu\rangle - c_{\nu,i}| < (\|\nu\|\varepsilon)/2$ for all $d \in D_i$ (D_i is dense in K_i). On the other hand, the mapping $x \mapsto \delta_x * \nu$ of S into $M_a(S)$ is continuous [5]. Given $x \in K_i$, there exists a neighbourhood U_x of x in S such that $|\langle f, \delta_x * \nu\rangle - \langle f, \delta_y * \nu\rangle| < (\|\nu\|\varepsilon)/2$ whenever $y \in U_x$. Choose $d \in U_x \cap D_i$, we have
$$|\langle f, \delta_x * \nu\rangle - c_{\nu,i}| \le |\langle f, \delta_x * \nu\rangle - \langle f, \delta_d * \nu\rangle| + |\langle f, \delta_d * \nu\rangle - c_{\nu,i}| < \|\nu\|\varepsilon.$$

(i) implies (iii): Let $\varepsilon > 0$, and let K be a compact subset of S. Choose a partition $\mathfrak{K} = \{K_i\}_{i=1}^m$ of K such that (i) is true. For $\nu \in M_a(S)$, we define h_ν on $M(S)$ by $h_\nu = \sum_{i=1}^m c_{\nu,i}\chi_{K_i}$. We have
$$|\langle f, \mu * \nu\rangle - \langle h_\nu, \mu\rangle| = \left|\int_{K_i} \langle f, \delta_x * \nu\rangle - c_{\nu,i} d\mu(x)\right| < \|\nu\|\varepsilon,$$
whenever $\mu \in M_p(S)$ and $\mu(K_i^c) = \{0\}$.

(iii) implies (i): If (i) did not hold, then there exist $\varepsilon_0 > 0$ and a compact subset K of S such that for any Borel measurable partition $\mathfrak{K} = \{K_i\}_{i=1}^m$ of K a measure $\nu \in M_a(S)$ and $1 \le i \le m$ may be found such that for every complex number $c_{\nu,i}$, there exist a point x in K_i such that $|\langle f, \delta_x * \nu\rangle - c_{\nu,i}| \ge \|\nu\|\varepsilon_0$. Defining h_ν on $M_a(S)^*$ by $h_\nu = c_{\nu,i}\chi_{K_i}$. Without loss of generality, we may assume that
$$|Re\langle f, \delta_x * \nu\rangle - c_{\nu,i}| > \frac{\sqrt{\varepsilon_0}}{2}.$$
Since the mapping $x \mapsto \delta_x * \nu$ is continuous [5], we can find a probability measure μ in $M_p(S)$ with $\mu(K_i^c) = \{0\}$ such that $|\langle f, \mu * \nu\rangle - \langle h_\nu, \mu\rangle| \ge (\sqrt{\varepsilon_0})/2$, which is a contradiction. ∎

Remark 2.1. If X is a linear subspace of $M_a(S)^*$ containing the constant functional 1, where $\langle 1,\mu\rangle = \mu(S)$, $\mu \in M_a(S)$. We say that X is topologically invariant if $f\mu \in X$ for any $\mu \in M(S)$ and $f \in X$. A functional M in X^* is called a *mean* if and only if $\|M\| = \langle M,1\rangle = 1$. A mean M is topologically left invariant if $\langle M, f\mu\rangle = \langle M, f\rangle$ for any $f \in X$ and $\mu \in M_p(S)\cap M_a(S)$ [16]. Further information on topologically invariant means can be found in [12]. Now, let M be a topologically left invariant mean on the set of all completely continuous functionals. Let μ_0 be in $M_p(S)\cap M_a(S)$. Define $M' \in M_a(S)^{**}$ by $\langle M', f\rangle = \langle M, f\mu_0\rangle$, $f \in M_a(S)^*$. Note that M' is well defined by Corollary 2.1. Clearly M' is a mean on $M_a(S)^*$. Let $\{e_\alpha\}$ be an approximate identity in $M_p(S)\cap M_a(S)$ for $M_a(S)$ [11]. For any $f \in M_a(S)^*$ and $\mu \in M_p(S)\cap M_a(S)$, we have

$$\begin{aligned}\langle M', f\mu\rangle &= \langle M, f\mu * \mu_0\rangle = \lim_\alpha \langle M, fe_\alpha * \mu * \mu_0\rangle = \lim_\alpha \langle M, fe_\alpha * \mu_0\rangle \\ &= \langle M, f\mu_0\rangle = \langle M', f\rangle.\end{aligned}$$

This shows that M' is a topologically left invariant mean on $M_a(S)^*$.

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