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A Strong Limit Theorem for Sequences of Blockwise and Pairwise Negative Quadrant M-Dependent Random Variables

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Abstract. In this paper, we establish a Marcinkiewicz-Zygmund type strong law for sequences of blockwise and pairwise negative quadrant m-dependent random variables. The sharpness of the results is illustrated by an example.

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1. Introduction

The concept of negative quadrant dependence was introduced by Lehmann [1]. The concept of blockwise m-dependence and blockwise quasiorthogonality for a sequence of random variables was introduced by Móricz [3]. The strong laws for blockwise independence case or blockwise orthogonal case then was studied by some authors. We refer to Rosalsky and Thanh [7], Quang and Thanh [5] for Banach spaces valued case and Quang and Thanh [6], Thanh [10] for multi-dimension case. Thanh and Anh [11] established a strong law of large numbers for blockwise and pairwise m-dependent random variables which extends the result of Thanh [8] to the arbitrary blocks case and also provided an example to illustrate the main result. In Thanh and Anh [11], authors considered a sequence of random variables which is blockwise and pairwise m-dependent with respect to the arbitrary blocks.

In this note, we consider a sequence of blockwise and pairwise negative quadrant m-dependent random variables $\{X_n, n \geq 1\}$ which is stochastically dominated by a random variable X. We establish a Marcinkiewicz-Zygmund type strong law of large numbers which extends the result of Thanh and Anh [11] to the blockwise and pairwise negative quadrant m-dependent case. We also provide an example to illustrate the main result.

Let X and Y be random variables. We say that X and Y are *negative quadrant dependent* if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)\,P(Y \leq y), \quad \forall x, y \in \mathbb{R}.$$

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A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *pairwise negative quadrant dependent* if for all $i \neq j$, X_i and X_j are negative quadrant dependent.

Let m be a fixed nonnegative integer. We say that a collection $\{X_j, 1 \leq j \leq n\}$ of n random variables is *pairwise negative quadrant m-dependent* if either $n \leq m+1$ or $n > m+1$ and X_i and X_j are negative quadrant dependent whenever $j - i > m$.

Let $\{\beta_k, k \geq 1\}$ be a strictly increasing sequence of positive integers with $\beta_1 = 1$ and set $B_k = [\beta_k, \beta_{k+1})$.

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *blockwise and pairwise negative quadrant m-dependent* with respect to the blocks $\{B_k, k \geq 1\}$ if for each $k \geq 1$, the random variables $\{X_i, i \in B_k\}$ are pairwise negative quadrant m-dependent.

For $\{\beta_k, k \geq 1\}$ and $\{B_k, k \geq 1\}$ as above, we introduce the following notation:

$$\begin{aligned}
B^{(l)} &= \{k : 2^l \leq k < 2^{l+1}\},\ l \geq 0,\\
B_k^{(l)} &= B_k \cap B^{(l)}, k \geq 1, l \geq 0,\\
I_l &= \{k \geq 1 : B_k^{(l)} \neq \emptyset\}, l \geq 0,\\
r_k^{(l)} &= \min\{r : r \in B_k^{(l)}\}, k \in I_l, l \geq 0,\\
c_l &= \operatorname{card} I_l, l \geq 0,\\
\varphi(n) &= \sum_{l=0}^{\infty} c_l I_{B^{(l)}}(n), n \geq 1,\\
\psi(n) &= \max_{k \leq n} \varphi(k), n \geq 1
\end{aligned}$$

where $I_{B^{(l)}}$ denotes the indicator function of the set $B^{(l)}, l \geq 0$.

Random variables $\{X_n, n \geq 1\}$ are said to be a *stochastically dominated* by random variable X if for some constant $C < \infty$

$$P(|X_n| > t) \leq CP(|X| > t), \forall t \geq 0, \forall n \geq 1.$$

2. Main result

Throughout this section, the logarithms are to the base 2, the symbol C denotes a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

Before establishing main result, we state two lemmas. The first lemma can be obtained by using a method similar to that used in the proof the Rademacher-Menshov inequality and Lemma 2.2 of Li, Rosalsky and Volodin [2].

Lemma 2.1. *If $\{X_n, n \geq 1\}$ is a sequence of pairwise negative quadrant dependent mean 0 random variables, then*

$$E\left(\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} X_j\right|\right)^2 \leq C(\log 4n)^2 \sum_{j=1}^{n} EX_j^2.$$

The second lemma can be obtained by using a method similar to that used in the proof the Lemma 3 of Thanh [8] and Lemma 2.2. It extends the Lemma 3 of Thanh [8] to the blockwise and pairwise negative quadrant m-dependent case.

Lemma 2.2. *If $\{X_j, 1 \le j \le n\}$ is a collection of pairwise negative quadrant m-dependent mean 0 random variables, then*

$$E\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right|\right)^2 \le C(m+1)(\log 4n)^2 \sum_{j=1}^{n} EX_j^2.$$

With the preliminaries accounted for, the main result may be established.

Theorem 2.1. *Let $1 \le r < 2$ and $\{X_n, n \ge 1\}$ be a sequence of random variables which is blockwise and pairwise negative quadrant m-dependent with respect to the blocks $\{B_k, k \ge 1\}$. Suppose that $\{X_n, n \ge 1\}$ is stochastically dominated by a random variable X. If*

(2.1) $$E(|X|^r(\log^+|X|)^2) < \infty,$$

then

(2.2) $$\lim_{n\to\infty} \frac{1}{n^{1/r}\psi^{1/2}(n)} \sum_{j=1}^{n}(X_j - EX_j) = 0 \ a.s.$$

Proof. Set

$$Y_n = X_n I(|X_n| \le n^{1/r}) + n^{1/r} I(X_n > n^{1/r}) - n^{1/r} I(X_n < -n^{1/r}),$$
$$Y_n^{(+)} = X_n^+ I(X_n \le n^{1/r}) + n^{1/r} I(X_n > n^{1/r}),$$
$$Y_n^{(-)} = X_n^- I(X_n \ge -n^{1/r}) + n^{1/r} I(X_n < -n^{1/r}),\ n \ge 1$$

and

$$T_{k(l)}^{(+)} = \max_{j\in B_k^{(l)}}\left|\sum_{i=r_k^{(l)}}^{j}(Y_i^{(+)} - EY_i^{(+)})\right|,\ k \in I_l,\ l \ge 0,$$
$$\tau_l^{(+)} = \frac{1}{\left(2^{\frac{l+1}{r}} - 2^{\frac{l}{r}}\right)\psi^{\frac{1}{2}}(2^l)} \sum_{k\in I_l} T_{k(l)}^{(+)},\ l \ge 0.$$

It follows from Lemma 2.1 of Li, Rosalsky and Volodin [2] that $\{Y_n^{(+)}, n \ge 1\}$ and $\{Y_n^{(-)}, n \ge 1\}$ are sequences of random variables which are blockwise and pairwise negative quadrant m-dependent with respect to the blocks $\{B_k, k \ge 1\}$.
Note at the outset that,

$$E(Y_n^{(+)})^2 \le 2\int_0^{n^{\frac{1}{r}}} xP(|X_n| > x)\,dx,$$
$$E|X_n - Y_n| \le C\left(n^{\frac{1}{r}}P(|X_n| > n^{\frac{1}{r}}) + \int_{n^{\frac{1}{r}}}^{\infty} P(|X_n| > x)\,dx\right),\ n \ge 1$$

and by using a method similar to that used in the proof of Theorem 1 of Thanh [8], we obtain

(2.3) $$\sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2/r}} E(Y_n^{(+)})^2 \le CE(|X|^r(\log^+|X|)^2) < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/r}} E|X_n - Y_n| \le CE(|X|^r \log^+ |X|) < \infty. \tag{2.4}$$

Note that for $l \ge 0$,

$$\begin{aligned}
E(\tau_l^{(+)})^2 &\le C \frac{1}{2^{\frac{2(l+1)}{r}} \psi(2^l)} c_l \sum_{k\in I_l} E(T_{k(l)}^{(+)})^2 \\
&\le C \frac{1}{2^{\frac{2(l+1)}{r}}} \sum_{k\in I_l} (\log(4\mathrm{card} B_k^{(l)}))^2 \sum_{i\in B_k^{(l)}} E|Y_i^{(+)} - EY_i^{(+)}|^2 \\
&\qquad \text{(by Lemma 2.3)} \\
&\le C \frac{1}{2^{\frac{2(l+1)}{r}}} (\log 2^{l+2})^2 \sum_{i=2^l}^{2^{l+1}-1} E|Y_i^{(+)} - EY_i^{(+)}|^2 \\
&\le C \sum_{i=2^l}^{2^{l+1}-1} \frac{(\log 4i)^2}{i^{\frac{2}{r}}} E(Y_i^{(+)})^2.
\end{aligned}$$

It follows from (2.3) that $\sum_{l=0}^{\infty} E(\tau_l^{(+)})^2 < \infty$ and so by the Markov inequality and the Borel-Cantelli lemma ensures that

$$\lim_{l\to\infty} \tau_l^{(+)} = 0 \ \text{ a.s.} \tag{2.5}$$

Note that for $n \ge 1$, letting $M \ge 0$ be such that $2^M \le n < 2^{M+1}$,

$$\begin{aligned}
\frac{|\sum_{i=1}^{n} (Y_i^{(+)} - EY_i^{(+)})|}{n^{\frac{1}{r}} \psi^{\frac{1}{2}}(n)} &\le \frac{\sum_{l=0}^{M} \sum_{k\in I_l} T_{k(l)}^{(+)}}{2^{\frac{M}{r}} \psi^{\frac{1}{2}}(2^M)} \\
&\le \sum_{l=0}^{M} \frac{2^{\frac{l+1}{r}} - 2^{\frac{l}{r}}}{2^{\frac{M}{r}}} \tau_l^{(+)}
\end{aligned}$$

and so (2.5) and Toeplitz lemma ensures that

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{r}} \psi^{\frac{1}{2}}(n)} \sum_{i=1}^{n} (Y_i^{(+)} - EY_i^{(+)}) = 0 \ \text{ a.s.}$$

Similarly,

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{r}} \psi^{\frac{1}{2}}(n)} \sum_{i=1}^{n} (Y_i^{(-)} - EY_i^{(-)}) = 0 \ \text{ a.s.}$$

and so $Y_n = Y_n^{(+)} - Y_n^{(-)}$, $n \ge 1$, we get

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{r}} \psi^{\frac{1}{2}}(n)} \sum_{i=1}^{n} (Y_i - EY_i) = 0 \ \text{ a.s.} \tag{2.6}$$

By (2.4), (2.6) and by using a method similar to that used in the proof of Theorem 2.1 of Thanh and Anh [11], we obtain (2.2). ∎

Note that if $\{X_n, n \geq 1\}$ is blockwise and pairwise m-dependent with respect to the blocks $\{B_k, k \geq 1\}$, then $\{X_n, n \geq 1\}$ is blockwise and pairwise negative quadrant m-dependent with respect to the blocks $\{B_k, k \geq 1\}$. So we get the following corollary which is the main result of Thanh and Anh [11].

Corollary 2.1. *Let $1 \leq r < 2$ and $\{X_n, n \geq 1\}$ be a sequence of random variables which is blockwise and pairwise m-dependent with respect to the blocks $\{B_k, k \geq 1\}$ and if (2.1) is satisfied, then (2.2) holds.*

Note that if $\beta_k = [q^{k-1}]$ for all large k and $q > 1$, then $c_l = \mathscr{O}(1)$, $\psi(n) = \mathscr{O}(1)$. So we get the following corollary.

Corollary 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of blockwise and pairwise negative quadrant m-dependent random variables with respect to the blocks $\{[2^{k-1}, 2^k), k \geq 1\}$ (or, more generally, with respect to the blocks $\{[\beta_k, \beta_{k+1}), k \geq 1\}$ where $\beta_k = [q^{k-1}]$ for all large k and $q > 1$) and if (2.1) is satisfied, then*

$$\lim_{n\to\infty} \frac{1}{n^{1/r}} \sum_{j=1}^{n} (X_j - EX_j) = 0 \ a.s. \tag{2.7}$$

The following example is a modify of Example 2.6 in Thanh and Anh [11]. However, we try to construct with large blocks.

Example 2.1. Let $\{Y_n, n \geq 1\}$ be a sequence of 0-dependent identically distributed of $N(0,1)$ random variables and let $3/2 \leq r < 2$. Let

$$X_n = Y_{n-k^3+1}, k^3 \leq n < (k+1)^3, k \geq 1.$$

Then $\{X_n, n \geq 1\}$ is blockwise and pairwise negative quadrant 0-dependent with respect to the blocks $\{[k^3, (k+1)^3), k \geq 1\}$ and (2.1) is satisfied, but $\{X_n, n \geq 1\}$ is not blockwise and pairwise negative quadrant m-dependent with respect to the blocks $\{[2^k, 2^{(k+1)}), k \geq 0\}$ for any non-negative integer m. Now, by noting that in $\beta_k = k^3, k \geq 1$ case $\psi(n) = \mathscr{O}(n^{1/3})$, so that by Corollary 2.4 we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n^{1/r+1/6}} = 0 \text{ a.s.}$$

Now, for $n = (M+1)^3 - 1$, we have

$$\begin{aligned} &\frac{\sum_{i=1}^{n} X_i}{n^{1/r}} \\ &= \frac{MY_1 + M(Y_2 + \cdots + Y_7) + \cdots + (Y_{M^3-(M-1)^3+1} + \cdots + Y_{(M+1)^3-M^3})}{((M+1)^3-1)^{1/r}} \\ &= S_{(M+1)^3-1}, \end{aligned}$$

where

$$S_{(M+1)^3-1} \sim N\Big(0, \frac{M^4 + 4M^3 + 7M^2 + 2M}{2((M+1)^3-1)^{2/r}}\Big),$$

so that (2.7) fails since $r \geq 3/2$.

Remark 2.1. Sequence $\{X_n, n \geq 1\}$ of random variables in Example 2.6 of Thanh and Anh [11] also is not blockwise and pairwise negative quadrant m-dependent with respect to the blocks $\{[2^{k-1},2^k),k\geq 1\}$ for any non-negative integer m and (2.1) is satisfied but (2.7) fails. So it also shows that Theorem 2.3 is sharp. More precisely, it shows that for all $\varepsilon>0$,

$$\limsup_{n\to\infty}\frac{|\sum_{i=1}^n X_i|}{n^{1/r-\varepsilon}\psi^{1/2}(n)}=\infty \text{ a.s.}$$

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References

[1] E. L. Lehmann, Some concepts of dependence, *Ann. Math. Statist.* **37** (1966), 1137–1153.
[2] D. Li, A. Rosalsky and A. I. Volodin, On the strong law of large numbers for sequences of pairwise negative quadrant dependent random variables, *Bull. Inst. Math. Acad. Sin. (N.S.)* **1** (2006), no. 2, 281–305.
[3] F. Móricz, Strong limit theorems for blockwise m-dependent and blockwise quasi-orthogonal sequences of random variables, *Proc. Amer. Math. Soc.* **101** (1987), no. 4, 709–715.
[4] V. H. Nguyen, V. Q. Nguyen and A. Volodin, Strong laws for blockwise martingale difference arrays in Banach spaces, *Lobachevskii J. Math.* **31** (2010), no. 4, 326–335.
[5] V. Q. Nguyen and L. V. Thanh, On the strong law of large numbers under rearrangements for sequences of blockwise orthogonal random elements in Banach spaces, *Aust. N. Z. J. Stat.* **49** (2007), no. 4, 349–357.
[6] V. Q. Nguyen and L. V. Thanh, On the strong laws of large numbers for two-dimensional arrays of blockwise independent and blockwise orthogonal random variables, *Probab. Math. Statist.* **25** (2005), no. 2, Acta Univ. Wratislav. No. 2784, 385–391.
[7] A. Rosalsky and L. Van Thanh, On the strong law of large numbers for sequences of blockwise independent and blockwise p-orthogonal random elements in Rademacher type p Banach spaces, *Probab. Math. Statist.* **27** (2007), no. 2, 205–222.
[8] L. V. Thanh, Strong laws of large numbers for sequences of blockwise and pairwise m-dependent random variables, *Bull. Inst. Math. Acad. Sinica* **33** (2005), no. 4, 397–405.
[9] L. V. Thanh, On the Brunk-Chung type strong law of large numbers for sequences of blockwise m-dependent random variables, *ESAIM Probab. Stat.* **10** (2006), 258–268 (electronic).
[10] L. V. Thanh, On the strong law of large numbers for d-dimensional arrays of random variables, *Electron. Comm. Probab.* **12** (2007), 434–441 (electronic).
[11] L. V. Thanh and N. V. Anh, A strong limit theorem for sequences of blockwise and pairwise m-dependent random variables, *Bull. Korean Math. Soc.* **48** (2011), no. 2, 343–351.