BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36**(1) (2013), 173–177

Results of Formal Local Cohomology Modules

AMIR MAFI
Department of Mathematics, University of Kurdistan, P.O. Box 416, Sanandaj, Iran
Institute for Studies in Theoretical Physics and Mathematics, P. O. Box 19395-5746, Tehran, Iran
a_mafi@ipm.ir

Abstract. Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring, $\mathfrak{a}$ an ideal of R, and M a finitely generated R-module. We show that for a non-negative integer t the following cases are equivalent:

(a) The formal local cohomology modules $\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)$ are Artinian for all $i<t$;

(b) $\mathfrak{a}\subseteq \mathrm{Rad}(\mathrm{Ann}(\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)))$ for all $i<t$.

If one of the above cases holds, then $\varprojlim_n H^t_{\mathfrak{m}}(M/\mathfrak{a}^nM)/\mathfrak{a}\varprojlim_n H^t_{\mathfrak{m}}(M/\mathfrak{a}^nM)$ is Artinian. Also, there are some results concerning finiteness properties of formal local cohomology modules.

2010 Mathematics Subject Classification: 13D45, 13E99

Keywords and phrases: Artinian modules, formal local cohomology modules, local cohomology modules.

1. Introduction

Throughout this paper, we assume that $(R,\mathfrak{m})$ is a commutative Noetherian local ring with non-zero identity and $\mathfrak{a}$ an ideal of R. For an integer i and a finitely generated R-module M let $H^i_{\mathfrak{a}}(M)$ denote the local cohomology module of M with respect to $\mathfrak{a}$ (see [3] for the basic definitions). Huneke [11] asked the question: When the modules $H^i_{\mathfrak{a}}(M)$ are Artinian. In general, this question is not true see for example [14] and [10], also the question is still true in many situations (see [16], [6], [13] and [1]). Recently Schenzel [19] has examined the structure of the modules $\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)$ extensively. For each i, he called $\mathfrak{F}^i_{\mathfrak{a}}(M):=\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)$ the ith formal local cohomology module of M with respect to $\mathfrak{a}$. Not so much is known about these modules. In the case of a regular local ring they have been studied by Peskine and Szpiro (cf. [18], Chapter III) in relation to the vanishing of local cohomology modules. Another kind of investigations about formal cohomology has been done by Faltings (cf. [7]). For more details on the notion of formal cohomology, we refer the reader to [12] and [2]. Now it is natural to ask the following question for the formal cohomology: When are the formal local cohomology modules $\mathfrak{F}^i_{\mathfrak{a}}(M)$ Artinian?

Communicated by Sriwulan Adji.
Received: October 1, 2010; *Revised:* March 6, 2011.

The main aim of this paper is to prove the following theorems.

Theorem 1.1. *Let t be a non-negative integer and M be a finitely generated R-module. Then the following statements are equivalent:*

(a) $\mathfrak{F}^i_{\mathfrak{a}}(M)$ *is Artinian for all* $i < t$;

(b) $\mathfrak{a} \subseteq \mathrm{Rad}(\mathrm{Ann}(\mathfrak{F}^i_{\mathfrak{a}}(M)))$ *for all* $i < t$.

Moreover if one of the above cases holds, then $\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^t_{\mathfrak{a}}(M)$ *is Artinian.*

Note that if $R = M$ and R is Gorenstein, then the formal local cohomology is the Matlis dual of local cohomology this was observed in [9, see 7.1.1]. In this sense Theorem 1.1 seems to be the precise dual of a well known finiteness criterion for local cohomology [Proposition 9.1.2, see [3]].

The following result extends [19, Theorem 3.9].

Theorem 1.2. *Let M be an* $\mathfrak{a}$*-cofinite R-module. Then for all j, there are the following isomorphisms*

$$H^{\mathfrak{a}}_i(\mathfrak{F}^j_{\mathfrak{a}}(M)) \cong \begin{cases} \mathfrak{F}^j_{\mathfrak{a}}(M) & i = 0 \\ 0 & i \neq 0. \end{cases}$$

2. The results

For an R-module N, a prime ideal $\mathfrak{p}$ of R is said to be a *co-support* of N if the module $\mathrm{Hom}_R(R_{\mathfrak{p}}, N) \neq 0$. The set of all co-support prime ideals of N is denoted by $\mathrm{Cos}_R(N)$ (cf. [17]).

Proposition 2.1. *Let M be an R-module. Then for all i,* $\cap_{t>0}\mathfrak{a}^t\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$.

Proof. Note that for any inverse system $\{N_t\}$, $\mathfrak{a}\varprojlim_t N_t \subseteq \varprojlim_t \mathfrak{a}N_t$. Thus

$$\bigcap_{t>0}\mathfrak{a}^t\mathfrak{F}^i_{\mathfrak{a}}(M) \cong \varprojlim_t \mathfrak{a}^t \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \subseteq \varprojlim_t \varprojlim_n \mathfrak{a}^t H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \cong \varprojlim_n \varprojlim_t \mathfrak{a}^t H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0,$$

as $\mathfrak{a}^t H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0$ for all $t \geq n$. ∎

Lemma 2.1. *Let M be an R-module and S be a multiplicative set of R such that* $S \cap \mathfrak{a} \neq \varnothing$. *Then for all i,* $\mathrm{Hom}_R(S^{-1}R, \mathfrak{F}^i_{\mathfrak{a}}(M)) = 0$.

Proof. Since $S \cap \mathfrak{a} \neq \varnothing$, there is an element $s_1 \in S \cap \mathfrak{a}$. Assume that $f \in \mathrm{Hom}_R(S^{-1}R, \mathfrak{F}^i_{\mathfrak{a}}(M))$ and so $f(r/s) = {s_1}^t f(r/{s_1}^t s) \in \mathfrak{a}^t\mathfrak{F}^i_{\mathfrak{a}}(M)$ for all $r/s \in S^{-1}R$ and all $t > 0$. Therefore $f \in \cap_{t>0}\mathfrak{a}^t\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$. Hence $f = 0$, that means $\mathrm{Hom}_R(S^{-1}R, \mathfrak{F}^i_{\mathfrak{a}}(M)) = 0$. ∎

Corollary 2.1. *Let M be an R-module. Then for all i,* $\mathrm{Cos}(\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$.

Proof. Assume that $\mathfrak{p} \in \mathrm{Cos}(\mathfrak{F}^i_{\mathfrak{a}}(M))$. Then $\mathrm{Hom}_R(R_{\mathfrak{p}}, \mathfrak{F}^i_{\mathfrak{a}}(M)) \neq 0$ and hence, by Lemma 2.1, $\mathfrak{a} \cap (R \setminus \mathfrak{p}) = \varnothing$. Thus $\mathfrak{a} \subseteq \mathfrak{p}$. ∎

A module M is $\mathfrak{a}$-cofinite if $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all i.

Lemma 2.2. [15, Section 2]. *Let M be an* $\mathfrak{a}$*-cofinite R-module. Then the following cases hold:*

(1) $N \otimes_R M$ *is finitely generated for all finitely generated module N with* $\mathfrak{a} \subseteq \mathrm{Ann}(N)$.

(2) $M/\mathfrak{a}^n M$ *is finitely generated for all* $n \geq 1$.

Proof. See Section 2 of [15]. ∎

Let $L_i^{\mathfrak{a}}(-)$ denote the ith left derived functor of the $\mathfrak{a}$-adic completion functor $\varprojlim_n(R/\mathfrak{a}^n\otimes_R -)$ (cf.[8] and [20] for the basic results). Cuong and Nam [4], for an R-module M, define the ith local homology module $H_i^{\mathfrak{a}}(M)$ by $H_i^{\mathfrak{a}}(M)=\varprojlim_n \mathrm{Tor}_i^R(R/\mathfrak{a}^n,M)$. Furthermore they proved for an Artinian module M, $H_i^{\mathfrak{a}}(M)\cong L_i^{\mathfrak{a}}(M)$ [4, Proposition 4.1].

Theorem 2.1. (Compare with [19, Theorem 3.9]) *Let M be an $\mathfrak{a}$-cofinite R-module. Then for all j, there are the following isomorphisms*

$$H_i^{\mathfrak{a}}(\mathfrak{F}_{\mathfrak{a}}^j(M))\cong\begin{cases}\mathfrak{F}_{\mathfrak{a}}^j(M) & i=0\\ 0 & i\neq 0.\end{cases}$$

Proof of Theorem 1.2. Note that $H_{\mathfrak{m}}^i(M/\mathfrak{a}^nM)$ is Artinian for all i by Lemma 2.4 and [3, Exercise 7.1.4]. Hence, by [5, Proposition 3.4] we have

$$H_i^{\mathfrak{a}}(\mathfrak{F}_{\mathfrak{a}}^j(M))\cong\varprojlim_n H_i^{\mathfrak{a}}(H_{\mathfrak{m}}^j(M/\mathfrak{a}^nM)).$$

Let $\underline{x}=(x_1,\ldots,x_m)$ be a system of generators of $\mathfrak{a}$ and $\underline{x}(t)=(x_1^t,\ldots,x_m^t)$. Then by [4, Theorem 3.6] $H_i^{\mathfrak{a}}(\mathfrak{F}_{\mathfrak{a}}^j(M))\cong\varprojlim_n\varprojlim_t H_i(\underline{x}(t),H_{\mathfrak{m}}^j(M/\mathfrak{a}^nM))$. Since $\underline{x}(t)H_{\mathfrak{m}}^j(M/\mathfrak{a}^nM)=0$ for all $t\geq n$, we get

$$\varprojlim_t H_0(\underline{x}(t),H_{\mathfrak{m}}^j(M/\mathfrak{a}^nM))=H_{\mathfrak{m}}^j(M/\mathfrak{a}^nM)\quad\text{and}\quad\varprojlim_t H_i(\underline{x}(t),H_{\mathfrak{m}}^j(M/\mathfrak{a}^nM))=0$$

for all $i>0$. This finishes the proof. ∎

Remark 2.1. As the referee suggested, in the proof of Theorem 2.5, let $\mathfrak{U}_i^{\mathfrak{a}}(.)$ denote the left derived functor on the $\mathfrak{a}$-adic completion functor (see [20]). Then it seems that $\mathfrak{U}_j^{\mathfrak{a}}(\mathfrak{F}_{\mathfrak{a}}^i(M))=0$ for all $j>0$. Therefore $H_j^{\mathfrak{a}}(\mathfrak{F}_{\mathfrak{a}}^i(M))=0$ for all $j>0$ is a consequence of [21, Theorem 3.5.8].

Corollary 2.2. (Compare with [19, Corollary 3.10]) *Let M be an $\mathfrak{a}$-cofinite R-module. Let $j\in\mathbb{Z}$. Suppose that $\mathfrak{F}_{\mathfrak{a}}^j(M)=\mathfrak{a}\mathfrak{F}_{\mathfrak{a}}^j(M)$. Then $\mathfrak{F}_{\mathfrak{a}}^j(M)=0$.*

Proof. Set $X=\mathfrak{F}_{\mathfrak{a}}^j(M)$. Then the assumption provides $X=\mathfrak{a}^nX, n\in\mathbb{N}$. Therefore by Theorem 2.5 we have $\varprojlim_n X/\mathfrak{a}^nX=X$ and so $X=0$, as required. ∎

Theorem 2.2. *Let t be a non-negative integer and M be a finitely generated R-module. Then the following statements are equivalent:*

(a) $\mathfrak{F}_{\mathfrak{a}}^i(M)$ *is Artinian for all $i<t$;*
(b) $\mathfrak{a}\subseteq\mathrm{Rad}(\mathrm{Ann}(\mathfrak{F}_{\mathfrak{a}}^i(M)))$ *for all $i<t$.*

Proof of Theorem 1.1. $(a)\Longrightarrow(b)$. Let $i<t$. Since $\mathfrak{F}_{\mathfrak{a}}^i(M)$ is Artinian for all $i<t$, we have $\mathfrak{a}^s\mathfrak{F}_{\mathfrak{a}}^i(M)=0$ for some positive integer s by Proposition 2.1. Hence $\mathfrak{a}\subseteq\mathrm{Rad}(\mathrm{Ann}(\mathfrak{F}_{\mathfrak{a}}^i(M)))$ for all $i<t$.
$(b)\Longrightarrow(a)$. We use induction on t. Let $t=1$. Without loss of generality we may and do assume that R is complete with respect to $\mathfrak{m}$-adic completion (cf. [19, Proposition 3.3]). Then it follows that $\mathfrak{F}_{\mathfrak{a}}^0(M)$ is a finitely generated R-module. From [19, Lemma 4.1] we get that $\mathrm{Ass}(\mathfrak{F}_{\mathfrak{a}}^0(M))=\{\mathfrak{p}\in\mathrm{Ass}(M):\dim R/\mathfrak{a}+\mathfrak{p}=0\}$. Therefore, by the hypothesis we have

$\operatorname{Supp}(\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$ and so $\mathfrak{F}^0_{\mathfrak{a}}(M)$ has finite length. Hence in this case the claim holds. Now, let $t > 1$ and assume that the claim holds for all values less than $t-1$. Since $\Gamma_{\mathfrak{a}}(M)$ is annihilated by some power of $\mathfrak{a}$, by [19, Theorem 3.11] one has the following long exact sequence

$$\ldots \longrightarrow H^i_{\mathfrak{m}}(\Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \longrightarrow H^{i+1}_{\mathfrak{m}}(\Gamma_{\mathfrak{a}}(M)) \longrightarrow \ldots. \tag{2.1}$$

Hence, it is enough to prove that $\mathfrak{F}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ is Artinian for all $i < t$. Thus, we may and do assume that M is $\mathfrak{a}$-torsion free. Take $x \in \mathfrak{a} \setminus \cup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$ (cf. [3, Lemma 2.1.1]). Therefore, by the hypothesis there exists a positive integer s such that $x^s \mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. By [19, Theorem 3.11] the exact sequence $0 \longrightarrow M \xrightarrow{x^s} M \longrightarrow M/x^sM \longrightarrow 0$ implies the following exact sequence of formal local cohomology modules

$$0 \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M/x^sM) \longrightarrow \mathfrak{F}^{i+1}_{\mathfrak{a}}(M) \longrightarrow 0$$

for all $i < t-1$. It follows that $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}(\mathfrak{F}^i_{\mathfrak{a}}(M/x^sM)))$ and by the inductive hypothesis that $\mathfrak{F}^i_{\mathfrak{a}}(M/x^sM)$ is Artinian for all $i < t-1$. Hence $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian for all $i < t$. This finishes the inductive step. ∎

Theorem 2.3. *Let M be a finitely generated R-module and t be a non-negative integer such that $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian for all $i < t$. Then $\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^t_{\mathfrak{a}}(M)$ is Artinian.*

Proof. We proceed by induction on t. When $t = 0$, $\mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is Artinian by [2, Theorem 3.7] Now, let $t > 0$ and the claim has been proved for $t-1$. From the exact sequence (2.1) that used in the proof of Theorem 1.1, we deduce that $\mathfrak{F}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ is Artinian for all $i < t$. We split the exact sequence

$$H^t_{\mathfrak{m}}(\Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M) \xrightarrow{f} \mathfrak{F}^t_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \xrightarrow{g} H^{t+1}_{\mathfrak{m}}(\Gamma_{\mathfrak{a}}(M))$$

to the exact sequences

$$0 \longrightarrow \ker f \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M) \longrightarrow \operatorname{im} f \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} f \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \longrightarrow \operatorname{im} g \longrightarrow 0.$$

From these exact sequences, we deduce the following exact sequences

$$\ker f/\mathfrak{a}\ker f \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^t_{\mathfrak{a}}(M) \longrightarrow \operatorname{im} f/\mathfrak{a}\operatorname{im} f \longrightarrow 0 \tag{2.2}$$

and

$$\begin{aligned} \operatorname{Tor}^R_1(R/\mathfrak{a}, \operatorname{im} g) \longrightarrow \operatorname{im} f/\mathfrak{a}\operatorname{im} f \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))/\mathfrak{a}\mathfrak{F}^t_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \\ \longrightarrow \operatorname{im} g/\mathfrak{a}\operatorname{im} g \longrightarrow 0. \end{aligned} \tag{2.3}$$

Since $\ker f$ and $\operatorname{im} g$ are Artinian, in view of (2.2) and (2.3), it turn out that if $\mathfrak{F}^t_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))/\mathfrak{a}\mathfrak{F}^t_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ is Artinian, then $\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^t_{\mathfrak{a}}(M)$ is also Artinian. Hence we may and do assume that M is $\mathfrak{a}$-torsion free and so there exists $x \in \mathfrak{a} \setminus \cup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$. Thus by Theorem 1.1 there exists a positive integer s such that $x^s \mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. From the exact sequence $0 \longrightarrow M \xrightarrow{x^s} M \longrightarrow M/x^sM \longrightarrow 0$ we deduce the following exact sequence

$$0 \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M/x^sM) \longrightarrow \mathfrak{F}^{i+1}_{\mathfrak{a}}(M) \longrightarrow 0 \tag{2.4}$$

for all $i < t-1$. Hence $\mathfrak{F}^i_{\mathfrak{a}}(M/x^sM)$ is Artinian for all $i < t-1$ and so by the inductive hypothesis $\mathfrak{F}^{t-1}_{\mathfrak{a}}(M/x^sM)/\mathfrak{a}\mathfrak{F}^{t-1}_{\mathfrak{a}}(M/x^sM)$ is Artinian. By using the functor $R/\mathfrak{a} \otimes_R -$ on the exact sequence (2.4), we deduce that $\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^t(M)$ is Artinian. This complete the inductive step. ∎

The following consequence immediately follows by Theorem 2.1 and [19, Theorem 1.1].

Corollary 2.3. *Let M be a finitely generated R-module. Then*

$$\mathfrak{F}^{\mathrm{fgrade}(\mathfrak{a},M)}_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^{\mathrm{fgrade}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$$

is Artinian.

Acknowledgement. The author is deeply grateful to the referees for carefully reading of the manuscript and the helpful suggestions. This research was supported in part by a grant from IPM (No. 89130058)

References

[1] M. Aghapournahr and L. Melkersson, Local cohomology and Serre subcategories, *J. Algebra* **320** (2008), no. 3, 1275–1287.

[2] M. Asgharzadeh and K. Divaani-Aazar, Finiteness properties of formal local cohomology modules and Cohen-Macaulayness, *Comm. Algebra* **39** (2011), no. 3, 1082–1103.

[3] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge Studies in Advanced Mathematics, 60, Cambridge Univ. Press, Cambridge, 1998.

[4] Nguyen Tu Cuong and Tran Tuan Nam, The *I*-adic completion and local homology for Artinian modules, *Math. Proc. Cambridge Philos. Soc.* **131** (2001), no. 1, 61–72.

[5] N. T. Cuong and T. T. Nam, A local homology theory for linearly compact modules, *J. Algebra* **319** (2008), no. 11, 4712–4737.

[6] K. Divaani-Aazar and P. Schenzel, Ideal topologies, local cohomology and connectedness, *Math. Proc. Cambridge Philos. Soc.* **131** (2001), no. 2, 211–226.

[7] G. Faltings, Algebraisation of some formal vector bundles, *Ann. of Math. (2)* **110** (1979), no. 3, 501–514.

[8] J. P. C. Greenlees and J. P. May, Derived functors of *I*-adic completion and local homology, *J. Algebra* **149** (1992), no. 2, 438–453.

[9] M. Hellus, *Local Cohomology and Matlis Duality*, Habilitationsschrift, Leipzig, 2006.

[10] M. Hellus and J. Stückrad, Artinianness of local cohomology, *J. Commut. Algebra* **1** (2009), no. 2, 269–274.

[11] C. Huneke, Problems on local cohomology, in *Free Resolutions in Commutative Algebra and Algebraic Geometry (Sundance, UT, 1990)*, 93–108, Res. Notes Math., 2 Jones and Bartlett, Boston, MA.

[12] L. Illusie, Grothendieck's existence theorem in formal geometry, in *Fundamental Algebraic Geometry*, 179–233, Math. Surveys Monogr., 123 Amer. Math. Soc., Providence, RI, 2005.

[13] A. Mafi, Some results on local cohomology modules, *Arch. Math. (Basel)* **87** (2006), no. 3, 211–216.

[14] T. Marley and J. C. Vassilev, Local cohomology modules with infinite dimensional socles, *Proc. Amer. Math. Soc.* **132** (2004), no. 12, 3485–3490 (electronic).

[15] L. Melkersson, Modules cofinite with respect to an ideal, *J. Algebra* **285** (2005), no. 2, 649–668.

[16] L. Melkersson, Some applications of a criterion for Artinianness of a module, *J. Pure Appl. Algebra* **101** (1995), no. 3, 291–303.

[17] L. Melkersson and P. Schenzel, The co-localization of an Artinian module, *Proc. Edinburgh Math. Soc. (2)* **38** (1995), no. 1, 121–131.

[18] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck, *Inst. Hautes Études Sci. Publ. Math. No. 42* (1973), 47–119.

[19] P. Schenzel, On formal local cohomology and connectedness, *J. Algebra* **315** (2007), no. 2, 894–923.

[20] A.-M. Simon, Some homological properties of complete modules, *Math. Proc. Cambridge Philos. Soc.* **108** (1990), no. 2, 231–246.

[21] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, 38, Cambridge Univ. Press, Cambridge, 1994.