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Combinatorial Results for Certain Semigroups of Transformations Preserving Orientation and a Uniform Partition

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Abstract. Let $\mathscr{T}_X$ be the full transformation semigroup on a set X and E be a non-trivial equivalence on X. The set

$$T_E(X) = \{f \in \mathscr{T}_X : \forall (x,y) \in E, (f(x), f(y)) \in E\}$$

is a subsemigroup of $\mathscr{T}_X$. For a finite totally ordered set X and a convex equivalence E on X, the set of all the orientation-preserving transformations in $T_E(X)$ forms a subsemigroup of $T_E(X)$ denoted by $OP_E(X)$. In this paper, under the hypothesis that the totally ordered set X is of cardinality mn $(m,n \geq 2)$ and the equivalence E has m classes such that each E-class contains n consecutive points, we calculate the cardinality of the semigroup $OP_E(X)$, and that of its idempotents.

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1. Introduction

Let $X = \{1, 2, \cdots, n\}$ with the usual order and let $\mathscr{P}_X$ and $\mathscr{T}_X$ denote the partial transformation semigroup and the full transformation semigroup on X, respectively. A map $f \in \mathscr{T}_X$ is said to be *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$ for $x, y \in X$. The collection of all the order-preserving maps on X is denoted by O_X in [6] (the symbol O_X is replaced by O_n in [2]). A sequence $A = (a_1, a_2, \cdots, a_n)$ is said to be *cyclic* if there exists no more than one subscript i such that $a_i > a_{i+1}$. A map $f \in \mathscr{T}_X$ is said to be *orientation-preserving*, if $(f(1), f(2), \cdots, f(n))$ is cyclic, which implies that there exists some $j \in \{0, 1, \cdots, n-1\}$ such that

$$f(j+1) \leq f(j+2) \leq \cdots \leq f(n) \leq f(1) \leq \cdots \leq f(j)$$

(where we adopt the convention that $f(1) \leq f(2) \leq \cdots \leq f(n)$ if $j = 0$). Clearly, if f is order-preserving, then it is also orientation-preserving. The collection of all the orientation-preserving maps on X is denoted by OP_n and has been investigated by Catarino and Higgins in [2]. Combinatorial results of various classes of transformation subsemigroups of $\mathscr{P}_X$ and

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$\mathscr{T}_X$ have been studied over a long period and many interesting results have emerged. For example, Howie [6] calculated the cardinality of O_X and the number of idempotents. Catarino and Higgins [2] gave formulas for the cardinality of OP_n and the number of idempotents. Umar [15] considered the cardinality and the number of nilpotents and idempotents of the semigroup S_n^- of all the order-decreasing maps on X. Higgins [5] studied the combinatorial properties of $\mathscr{C}_n$, the semigroup of all the decreasing and order-preserving full transformations on X. Laradji and Umar [8] investigated the cardinality and the number of idempotents of $\mathscr{P}O_n$, the semigroup of all the order-preserving partial transformations on a finite chain X.

We may regard the elements of $X = \{1, 2, \cdots, n\}$ as being placed clockwise on a circle so that the integer i lies between $i-1$ and $i+1$ for $1 < i < n$, n between $n-1$ and 1, and 1 between n and 2. A *closed interval* $[i, j]$ for $i, j \in X$ may be expressed clockwise by $[i, j] = \{i, i+1, \cdots, j-1, j\}$. A subset Y of X is said to be *convex* if Y is a closed interval. An equivalence E on X is said to be *convex* if each E-class is convex. For two convex disjoint subsets P, Q of X, the *generalized interval* $[P, Q]$ can be expressed clockwise by $[P, Q] = \{p, p+1, \cdots, q-1, q\}$ where $p = \min P$ and $q = \max Q$. For example, let $n = 10$, then $[3, 8] = \{3, 4, \cdots, 8\}$ and $[9, 2] = \{9, 10, 1, 2\}$. If $P = [5, 6], Q = [10, 2]$, then the generalized intervals $[P, Q] = \{5, 6, 7, 8, 9, 10, 1, 2\}$, $[8, Q] = \{8, 9, 10, 1, 2\}$.

Let X be a set with $|X| \geq 3$ and E be an equivalence on X. Set

$$T_E(X) = \{f \in \mathscr{T}_X : \forall (x, y) \in E, (f(x), f(y)) \in E\}.$$

Clearly, $T_E(X)$ is a subsemigroup of $\mathscr{T}_X$ and if $E = \{(x, x) : x \in X\}$ or $E = X \times X$, then $T_E(X) = \mathscr{T}_X$. In [14], for a finite totally ordered set X and the convex equivalence E on X, the authors considered the subsemigroup of $T_E(X)$

$$OP_E(X) = \{f \in T_E(X) : f \text{ is orientation-preserving}\},$$

and under the supposition that all E-classes were of the same size, the regularity and Green's relations for the semigroup $OP_E(X)$ were described.

In this paper, as in [14], we always assume the totally ordered set $X = \{1 < 2 < \cdots < mn\}$ $(m, n \geq 2)$ and the equivalence E to be

$$E = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \cdots \cup (A_m \times A_m),$$

where $A_i = [(i-1)n + 1, in]$ for $1 \leq i \leq m$. We investigate combinatorial properties of the semigroup $OP_E(X)$. The paper is organized as follows. In Section 2, we determine the cardinality of $OP_E(X)$. In Section 3, we characterize the idempotents in the semigroup $OP_E(X)$ and calculate their number.

Denote by X/E the quotient set of X. The following result whose proof is routine describes an essential property of the transformations in the semigroup $T_E(X)$ where X is an arbitrary set and E is an arbitrary equivalence on X.

Lemma 1.1. *Let $f \in T_E(X)$, then for each $B \in X/E$, there exists $B' \in X/E$ such that $f(B) \subseteq B'$. Consequently, for each $A \in X/E$, the set $f^{-1}(A)$ is either $\emptyset$ or a union of some E-classes.*

For each $f \in T_E(X)$, let

$$E(f) = \{f^{-1}(A) : A \in X/E \text{ and } f^{-1}(A) \neq \emptyset\}.$$

Then $E(f)$ is a partition of X. The following result shows that each orientation-preserving transformation induces a partition of convex subsets.

Lemma 1.2. *Let $f \in OP_E(X)$. Then each $U \in E(f)$ is a convex subset of X.*

2. The cardinality of $OP_E(X)$

In this section, we focus our attention on the cardinality of $OP_E(X)$. We notice that for each $f \in OP_E(X)$, there exists some j such that $f(j+1) = \min f(X)$ and $f(j) = \max f(X)$ and j is unique if f is not constant. Therefore, there are two cases: $(j, j+1) \notin E$ or $(j, j+1) \in E$. We first consider subsets of $OP_E(X)$ consisting of those elements for which $(j, j+1) \notin E$. It is not hard to see that in this case, j is the greatest number in some E-class A_i while $j+1$ is the smallest number in the next E-class A_{i+1}. Define certain subsets $\mathscr{A}_1, \mathscr{A}_2, \cdots, \mathscr{A}_m$ of $OP_E(X)$ by:

$$\mathscr{A}_1 = \{f \in OP_E(X) : f(1) = \min f(X) \,\text{and}\, f(mn) = \max f(X)\},$$

$$\mathscr{A}_2 = \{f \in OP_E(X) : f(n+1) = \min f(X) \,\text{and}\, f(n) = \max f(X)\},$$

$$\cdots,$$

$$\mathscr{A}_m = \{f \in OP_E(X) : f((m-1)n+1) = \min f(X) \,\text{and}\, f((m-1)n) = \max f(X)\}.$$

Obviously, if $f \in \mathscr{A}_i\,(1 \le i \le m)$, then $|f(X)| \le mn$ and

$$f((i-1)n+1) \le f((i-1)n+2) \le \cdots \le f(mn) \le f(1) \le \cdots \le f((i-1)n).$$

Next, we consider anther subsets consisting of those elements for which $(j, j+1) \in E$ and $f(j+1) = \min f(X)$, $f(j) = \max f(X)$. For each $1 \le s \le m$, define certain subsets $\mathscr{B}_{s,1}, \mathscr{B}_{s,2}, \cdots, \mathscr{B}_{s,n-1}$ of $OP_E(X)$ by

$$\mathscr{B}_{s,1} = \{f \in OP_E(X) : f((s-1)n+2) = \min f(X) \text{ and } f((s-1)n+1) = \max f(X)\},$$

$$\mathscr{B}_{s,2} = \{f \in OP_E(X) : f((s-1)n+3) = \min f(X) \text{ and } f((s-1)n+2) = \max f(X)\},$$

$$\cdots,$$

$$\mathscr{B}_{s,n-1} = \{f \in OP_E(X) : f(sn) = \min f(X) \text{ and } f(sn-1) = \max f(X)\}.$$

If $f \in \mathscr{B}_{s,t}\,(1 \le t \le n-1)$, then f maps all the elements of X into some E-class and

$$f((s-1)n+t+1) \le f((s-1)n+t+2) \le \cdots \le f(mn) \le f(1) \le \cdots \le f((s-1)n+t).$$

Therefore,

$$OP_E(X) = \left(\bigcup_{s=1}^{m} \mathscr{A}_s\right) \bigcup \left(\bigcup_{s=1}^{m}\bigcup_{t=1}^{n-1} \mathscr{B}_{s,t}\right)$$

and for $s \ne s'$, $t \ne t'$,

$$\mathscr{A}_s \cap \mathscr{A}_{s'} = \mathscr{B}_{s,t} \cap \mathscr{B}_{s',t'} = \mathscr{B}_{s,t} \cap \mathscr{A}_{s'} = \{\langle 1\rangle, \langle 2\rangle, \cdots, \langle mn\rangle\},$$

where $\langle x\rangle$ denotes the constant map which maps all the elements of X into x.

We give two properties for the subsets $\mathscr{A}_1, \mathscr{A}_2, \cdots, \mathscr{A}_m$ and $\mathscr{B}_{s,1}, \mathscr{B}_{s,2}, \cdots, \mathscr{B}_{s,n-1} (1 \le s \le m)$.

Lemma 2.1. *Let $\mathscr{A}_1, \mathscr{A}_2, \cdots, \mathscr{A}_m$ be as defined above. Then*

$$|\mathscr{A}_1| = |\mathscr{A}_2| = \cdots = |\mathscr{A}_m|.$$

Proof. For $f \in \mathscr{A}_1$, define $\psi_1 : \mathscr{A}_1 \to \mathscr{A}_2$ by $\psi_1(f) = g$ where

$$g(x) = \begin{cases} f(mn+x-n) & 1 \le x \le n \\ f(x-n) & \text{otherwise.} \end{cases}.$$

Then ψ_1 is well defined. To see $g \in \mathscr{A}_2$, let $(x,y) \in E$, if $x,y \in A_1$, then $(mn+x-n, mn+y-n) \in E$ and $(g(x),g(y)) = (f(mn+x-n), f(mn+y-n)) \in E$. If $x,y \notin A_1$, then $(x-n, y-n) \in E$ and $(g(x),g(y)) = (f(x-n), f(y-n)) \in E$ which implies that $g \in T_E(X)$. Moreover,

$$\begin{aligned} g(n+1) = f(1) \le g(n+2) = f(2) \le \cdots \le g(mn) = f(mn-n) \\ \le g(1) = f(mn+1-n) \le g(2) = f(mn+2-n) \cdots \le g(n) = f(mn). \end{aligned}$$

So $g \in \mathscr{A}_2$. It is clear that ψ_1 is a bijection from $\mathscr{A}_1$ onto $\mathscr{A}_2$. Therefore, $|\mathscr{A}_1| = |\mathscr{A}_2|$. Similarly, we can define $\psi_2, \psi_3, \cdots, \psi_{m-1}$ and show that $|\mathscr{A}_2| = |\mathscr{A}_3|, |\mathscr{A}_3| = |\mathscr{A}_4|, \cdots, |\mathscr{A}_{m-1}| = |\mathscr{A}_m|$. Consequently, $|\mathscr{A}_1| = |\mathscr{A}_2| = \cdots = |\mathscr{A}_m|$. ∎

Lemma 2.2. *For $1 \le s \le m$, let $\mathscr{B}_{s,1}, \mathscr{B}_{s,2}, \cdots, \mathscr{B}_{s,n-1}$ be as defined above. Then*

(1) $|\mathscr{B}_{s,1}| = |\mathscr{B}_{s,2}| = \cdots = |\mathscr{B}_{s,n-1}|$.

(2) $|\mathscr{B}_{s,l}| = |\mathscr{B}_{s',l}|$ *for* $1 \le s, s' \le m$ *and* $1 \le l \le n-1$.

Proof. (1) For $f \in \mathscr{B}_{s,t}(1 \le t \le n-1)$, define $\rho : \mathscr{B}_{s,t} \to \mathscr{B}_{s,t+1}$ by $\rho(f) = g$ where

$$g(x) = \begin{cases} f(mn) & x = 1 \\ f(x-1) & \text{otherwise.} \end{cases}.$$

Since f maps X into some E-class and $g(X) = f(X)$, we have $g \in T_E(X)$. Moreover,

$$\begin{aligned} g((s-1)n+t+2) = f((s-1)n+t+1) \le g((s-1)n+t+3) = f((s-1)n+t+2) \le \cdots \\ \le g(mn) = f(mn-1) \le g(1) = f(mn) \le \cdots \le g((s-1)n+t+1) = f((s-1)n+t). \end{aligned}$$

Thus $g \in \mathscr{B}_{s,t+1}$. One easily verifies that ρ is a bijection from $\mathscr{B}_{s,t}$ onto $\mathscr{B}_{s,t+1}$. Hence $|\mathscr{B}_{s,t}| = |\mathscr{B}_{s,t+1}|$ and $|\mathscr{B}_{s,1}| = |\mathscr{B}_{s,2}| = \cdots = |\mathscr{B}_{s,n-1}|$.

(2) Similar to that of Lemma 2.1. ∎

As we know, the number of r-combinations of k distinct objects each available in unlimited supply is $\binom{r+k-1}{r}$ (see [1, Theorem 3.5.1, p. 72]).

We now can state and prove the main result of this section.

Theorem 2.1.

$$|OP_E(X)| = m \sum_{k_1+k_2+\cdots+k_m=m} \prod_{s=1}^{m} \binom{(k_s+1)n-1}{k_s n} + m^2(n-1)\binom{n(m+1)-1}{mm} - mn(mn-1),$$

where $(k_1, k_2, \cdots, k_m)$ is any non-negative integer solution to the equation $\sum_{s=1}^{m} k_s = m$.

Proof. By Lemmas 2.1 and 2.2, in order to calculate $|OP_E(X)|$, we need only consider $|\mathscr{A}_1|$ and $|\mathscr{B}_{1,1}|$. We first calculate $|\mathscr{A}_1|$. Suppose that

(2.1) $\quad f([A_1, A_{k_1}]) \subseteq A_1, f([A_{k_1+1}, A_{k_1+k_2}]) \subseteq A_2, \cdots, f([A_{k_1+k_2+\cdots+k_{m-1}+1}, A_m]) \subseteq A_m,$

where $(k_1, k_2, \cdots, k_m)$ is one non-negative integer solution to the equation $\sum_{s=1}^{m} k_s = m$. Then the number of maps f satisfying (2.1) is $\prod_{s=1}^{m} \binom{(k_s+1)n-1}{k_s n}$. Thus,

$$|\mathscr{A}_1| = \sum_{k_1+k_2+\cdots+k_m=m} \prod_{s=1}^{m} \binom{(k_s+1)n-1}{k_s n},$$

where $(k_1, k_2, \cdots, k_m)$ is any non-negative integer solution to the equation $\sum_{s=1}^{m} k_s = m$. Hence it follows from Lemma 2.1 that

$$|\mathscr{A}_1| = |\mathscr{A}_2| = \cdots = |\mathscr{A}_m| = \sum_{k_1+k_2+\cdots+k_m=m} \prod_{s=1}^{m} \binom{(k_s+1)n-1}{k_s n}.$$

Notice that, for any distinct s and s',

$$\mathscr{A}_s \cap \mathscr{A}_{s'} = \{\langle 1\rangle, \langle 2\rangle, \cdots, \langle mn\rangle\},$$

so the number of distinct maps $f \in \bigcup_{s=1}^{m} \mathscr{A}_s$ is

$$m \sum_{k_1+k_2+\cdots+k_m=m} \prod_{s=1}^{m} \binom{(k_s+1)n-1}{k_s n} - mn(m-1).$$

We now calculate $|\mathscr{B}_{1,1}|$. If $f \in \mathscr{B}_{1,1}$, then $f(X) \subseteq A$ for some $A \in X/E$. Set

$$\mathscr{F}_i = \{f \in \mathscr{B}_{1,1} : f(X) \subseteq A_i\},$$

where $1 \le i \le m$. It follows that $|\mathscr{F}_i| = \binom{n(m+1)-1}{mn}$ and so $|\mathscr{B}_{1,1}| = |\bigcup_{i=1}^{m} \mathscr{F}_i| = m\binom{n(m+1)-1}{mn}$. By virtue of Lemma 2.2, for $1 \le s \le m$ and $1 \le t \le n-1$, we have $|\mathscr{B}_{s,t}| = m\binom{n(m+1)-1}{mn}$. Since

$$\mathscr{B}_{s,t} \cap \mathscr{B}_{s',t'} = \mathscr{B}_{s,t} \cap \mathscr{A}_{s'} = \{\langle 1\rangle, \langle 2\rangle, \cdots, \langle mn\rangle\},$$

the number of distinct non-constant maps $f \in \bigcup_{s=1}^{m} \bigcup_{t=1}^{n-1} \mathscr{B}_{s,t}$ is

$$m^2(n-1)\binom{n(m+1)-1}{mn} - m^2 n(n-1).$$

Therefore,

$$|OP_E(X)| = m \sum_{k_1+k_2+\cdots+k_m=m} \prod_{s=1}^{m} \binom{(k_s+1)n-1}{k_s n} + m^2(n-1)\binom{n(m+1)-1}{mn} - mn(mn-1),$$

as required. ∎

Earlier the authors [12] considered the class of transformation semigroups

$$O_E(X) = \{f \in T_E(X) : \forall x, y \in X, x \le y \Rightarrow f(x) \le f(y)\},$$

where the set X and the equivalence E are as defined in this paper. It is clear that $O_E(X) \subset OP_E(X)$, and in fact, the semigroup $O_E(X)$ whose cardinality is not known hitherto, is exactly $|\mathscr{A}_1|$. Thus, an immediate consequence of Theorem 2.1 is the following corollary.

Corollary 2.1.

$$|O_E(X)| = \sum_{k_1+k_2+\cdots+k_m=m} \prod_{s=1}^{m} \binom{(k_s+1)n-1}{k_s n},$$

where $(k_1, k_2, \cdots, k_m)$ is any non-negative integer solution to the equation $\sum_{s=1}^{m} k_s = m$.

Remark 2.1. Recently I have been told that Fernandes and Quinteiro [4] had calculated the size of the semigroups $OP_E(X)$ and $O_E(X)$. However, the approach used differs greatly from that in this paper.

The following Tables 1 and 2 give the size of the semigroups $OP_E(X)$ and $O_E(X)$ for smaller m and n, respectively.

Table 1. The cardinality of $OP_E(X)$

$m\backslash n$	2	3	4	5	6
2	46	506	5034	51682	575268
3	447	9453	248823	8445606	349109532
4	4324	223852	17184076	1819339324	247307947608
5	42075	5555990	1207660095	387720453255	170017607919290
6	405828	136530144	83547682248	81341248206546	114804703283314542

Table 2. The cardinality of $O_E(X)$

$m\backslash n$	2	3	4	5	6
2	19	156	1555	17878	225820
3	138	2845	78890	2768760	115865211
4	1059	55268	4284451	454664910	61824611940
5	8378	1109880	241505530	77543615751	34003513468232
6	67582	22752795	13924561150	13556873588212	19134117191404027

3. The number of idempotents in $OP_E(X)$

For a given subset M of the semigroup $OP_E(X)$, we denote by $E(M)$ its set of idempotents. In this section, we aim to calculate the cardinality of $E(OP_E(X))$. Since the semigroup $OP_E(X)$ has been divided into some subsets $\mathscr{A}_1(=O_E(X)), \mathscr{A}_2, \cdots, \mathscr{A}_m$, $\mathscr{B}_{s,1}, \mathscr{B}_{s,2}, \cdots$, $\mathscr{B}_{s,n-1} (1 \leq s \leq m)$, that is,

$$OP_E(X) = \left(\bigcup_{s=1}^{m} \mathscr{A}_s \right) \bigcup \left(\bigcup_{s=1}^{m} \bigcup_{t=1}^{n-1} \mathscr{B}_{s,t} \right),$$

we need only calculate the cardinality of the sets $E(\mathscr{A}_1), E(\mathscr{A}_2), \cdots, E(\mathscr{A}_m), \bigcup_{t=1}^{n-1} E(\mathscr{B}_{s,t})$ $(1 \leq s \leq m)$, respectively.

We begin with considering the number of idempotents in the semigroup $O_E(X)$. Recall that, the Fibonacci numbers are recursively defined by

$$F_0 = 0, F_1 = 1, F_{k+1} = F_k + F_{k-1}, \quad k \geq 1.$$

The following lemma which comes from [6, Theorem 2.3] was reproved in [2, Lemma 2.9].

Lemma 3.1. $|E(O_n)| = F_{2n}$.

Lemma 3.2. *Let $f \in O_E(X)$ and $f^{-1}(A_j) = [A_{i+1}, A_{i+t}]$ for $1 \leq i, t \leq m-1$, $i+1 \leq j \leq i+t$. Then the restriction of f to $[A_{i+1}, A_{i+t}]$*

$$f|_{[A_{i+1}, A_{i+t}]} : [A_{i+1}, A_{i+t}] \to A_j$$

is an idempotent in $\mathscr{T}_{[A_{i+1}, A_{i+t}]}$ if and only if the restriction of f to the E-class A_j

$$f|_{A_j} : A_j \to A_j$$

is an idempotent in $\mathscr{T}_{A_j}$ and $f([A_{i+1}, A_{j-1}]) = f(a)$, $f([A_{j+1}, A_{i+t}]) = f(b)$ where $a = \min A_j = (j-1)n+1$ and $b = \max A_j = jn$.

Proof. It is immediate for an order-preserving transformation in $T_E(X)$. ∎

Remark 3.1. From Lemma 3.2, in order to construct an idempotent

$$f|_{[A_{i+1},A_{i+t}]} : [A_{i+1},A_{i+t}] \to A_j$$

in $\mathscr{T}_{[A_{i+1},A_{i+t}]}$, we go along the following line:

Step 1. Construct an idempotent $f|_{A_j} : A_j \to A_j$ in $\mathscr{T}_{A_j}$;

Step 2. Let $f([A_{i+1},A_{j-1}]) = f(a)$ and $f([A_{j+1},A_{i+t}]) = f(b)$ where $a = \min A_j = (j-1)n+1$ and $b = \max A_j = jn$.

From Lemma 3.2 and Remark 3.1, we can deduce

Lemma 3.3. *Let $f \in O_E(X)$ and $f^{-1}(A_j) = [A_{i+1},A_{i+t}]$ for $1 \le i,t \le m-1$, $i+1 \le j \le i+t$. Then the number of idempotents*

$$f|_{[A_{i+1},A_{i+t}]} : [A_{i+1},A_{i+t}] \to A_j$$

in $\mathscr{T}_{[A_{i+1},A_{i+t}]}$ equals that of idempotents in $\mathscr{T}_{A_j}$.

Theorem 3.1.

$$|E(O_E(X))| = \sum_{t=1}^{m}\left(\sum_{k_1+k_2+\cdots+k_t=m}\prod_{i=1}^{t} k_iF_{2n}\right),$$

where $(k_1, k_2, \cdots, k_t)$ is any positive integer solution to the equation $\sum_{i=1}^{t} k_i = m$.

Proof. Let $f \in E(O_E(X))$. Denote

$$t = |\{A \in X/E : A \cap f(X) \neq \emptyset\}|,$$

where $1 \le t \le m$. Suppose that

$$(3.1) \quad f([A_1,A_{k_1}]) \subseteq A_{s_1}, f([A_{k_1+1},A_{k_1+k_2}]) \subseteq A_{s_2}, \cdots, f([A_{k_1+k_2+\cdots+k_{t-1}+1},A_m]) \subseteq A_{s_t},$$

where $A_{s_i} \in X/E$ for $1 \le i \le t$, the subscript set $\{s_1,s_2,\cdots,s_t\} \subseteq \{1,2,\cdots,m\}$ and $(k_1,k_2,\cdots,k_t)$ is one positive integer solution to the equation $\sum_{i=1}^{t} k_i = m$. Then, for each i, there are k_i choices for A_{s_i}. By Lemma 3.3, for the fixed positive integer solution $(k_1, k_2, \cdots, k_t)$ to the equation $\sum_{i=1}^{t} k_i = m$, the number of idempotents f satisfying (3.1) is $\prod_{i=1}^{t} k_iF_{2n}$. So the number of idempotents f satisfying (3.1) is $\sum_{k_1+k_2+\cdots+k_t=m}\prod_{i=1}^{t} k_iF_{2n}$, where $(k_1, k_2, \cdots, k_t)$ is any positive integer solution to the equation $\sum_{i=1}^{t} k_i = m$. Noting that $1 \le t \le m$, we have

$$|E(O_E(X))| = \sum_{t=1}^{m}\left(\sum_{k_1+k_2+\cdots+k_t=m}\prod_{i=1}^{t} k_iF_{2n}\right). \quad \blacksquare$$

Remark 3.2. From Lemma 2.1, $|\mathscr{A}_1| = |\mathscr{A}_2| = \cdots = |\mathscr{A}_m|$. However, in general, the number of idempotents in $\mathscr{A}_1$ doesn't equal that of $\mathscr{A}_j$ for $j \neq 1$. For example, let $m = 2$, $n = 2$, that is, $A_1 = \{1,2\}$, $A_2 = \{3,4\}$. By Theorem 3.1, we have

$$|E(\mathscr{A}_1)| = 2F_4 + F_4F_4 = 15.$$

Denote by $(abcd)$ the map $f \in OP_E(X)$ which maps $1,2,3,4$ into a,b,c,d, respectively, and

$$E(\mathscr{A}_1) = \{\langle 1\rangle, (1222), \langle 2\rangle, (1133), (1134), (1144), (1233), (1234), (1244), (2233), (2234), (2244), \langle 3\rangle, (3334), \langle 4\rangle\}.$$

However, there are only 6 idempotents in $\mathscr{A}_2$, and

$$E(\mathscr{A}_2) = \{\langle 1\rangle, (1211), \langle 2\rangle, \langle 3\rangle, (4434), \langle 4\rangle\}.$$

Now we calculate the number of idempotents in $\mathscr{A}_l$ for $2 \le l \le m$.

Lemma 3.4. *Let $f \in \mathscr{A}_l$ $(2 \le l \le m)$ and $f^{-1}(A_p) = [A_l, A_{l'}]$ for some E-class A_p with $p \le l' < l$. Then the restriction of f to $[A_l, A_{l'}]$*

$$f|_{[A_l,A_{l'}]} : [A_l, A_{l'}] \to A_p$$

is an idempotent in $\mathscr{T}_{[A_l,A_{l'}]}$ if and only if the restriction of f to A_p

$$f|_{A_p} : A_p \to A_p$$

is an idempotent in $\mathscr{T}_{A_p}$ and $f([A_l, A_{p-1}]) = f(a)$, $f([A_{p+1}, A_{l'}]) = f(b)$ where $a = minA_p = (p-1)n+1$ and $b = maxA_p = pn$.

Remark 3.3. In Lemma 3.4, there are two special cases.

(1) if $p = l' = 1$, then the restriction of f to $[A_l, A_1]$

$$f|_{[A_l,A_1]} : [A_l, A_1] \to A_1$$

is an idempotent in $\mathscr{T}_{[A_l,A_1]}$ if and only if the restriction of f to A_1

$$f|_{A_1} : A_1 \to A_1$$

is an idempotent in $\mathscr{T}_{A_1}$ and $f([A_l, A_m]) = f(1)$.

(2) if $p = l' \ge 2$, then the restriction of f to $[A_l, A_{l'}]$

$$f|_{[A_l,A_{l'}]} : [A_l, A_{l'}] \to A_p$$

is an idempotent in $\mathscr{T}_{[A_l,A_{l'}]}$ if and only if the restriction of f to A_p

$$f|_{A_p} : A_p \to A_p$$

is an idempotent in $\mathscr{T}_{A_p}$ and $f([A_l, A_{p-1}]) = f((p-1)n+1)$.

To illustrate Lemma 3.4, let $m = 4$, $n = 2$ and $A_1 = \{1,2\}, A_2 = \{3,4\}, A_3 = \{5,6\}, A_4 = \{7,8\}$. Then $f_1 = (12111111) \in E(\mathscr{A}_2), f_2 = (12221111) \in E(\mathscr{A}_3)$ and $f_3 = (33344433) \in E(\mathscr{A}_4)$. Clearly, $f_1|_{A_1}$ is an idempotent in $\mathscr{T}_{A_1}, f_1([A_2, A_4]) = f_1(1)$, and $f_2|_{A_1}$ is an idempotent in $\mathscr{T}_{A_1}, f_2([A_3, A_4]) = f_2(1), f_2(A_2) = f_2(2)$, and $f_3|_{A_2}$ is an idempotent in $\mathscr{T}_{A_2}, f_3([A_4, A_1]) = f_3(3), f_3(A_3) = f_3(4)$.

Lemma 3.5. *For $2 \le l \le m$,*

$$|E(\mathscr{A}_l)| = \sum_{v=1}^{l-1} \sum_{j_1+j_2+\cdots+j_v=l-1} \prod_{w=1}^{v} j_w F_{2n} + \sum_{t=1}^{m-l+1} \sum_{k_1+k_2+\cdots+k_t=m} \left(\prod_{i=1}^{t-1} k_i F_{2n} \right) ((k_t - l + 1) F_{2n}),$$

where $(j_1, j_2, \cdots, j_v)$ is any positive integer solution to the equation $\sum_{w=1}^{v} j_w = l-1$, and $(k_1, k_2, \cdots, k_t)$ is any positive integer solution to the equation $\sum_{i=1}^{t} k_i = m$ and the final positive integer $k_t \ge l$.

Proof. Let $f \in E(\mathscr{A}_l)$. There are two cases to consider.

Case 1. $f(A_l) \subseteq A_p$ for $p \in \{1, 2, \cdots, l-1\}$. Since f is an idempotent, we can deduce that $f([A_l, A_m]) \subseteq A_p$. Let

$$v = |\{A \in X/E : A \cap f(X) \ne \emptyset\}|,$$

where $1 \leq v \leq l-1$. Suppose

$$f([A_l, A_{j_1}]) \subseteq A_{s_1},\ f([A_{j_1+1}, A_{j_1+j_2}]) \subseteq A_{s_2}, \cdots, \\ f([A_{j_1+j_2+\cdots+j_{v-1}+1}, A_{j_1+j_2+\cdots+j_{v-1}+j_v} = A_{l-1}]) \subseteq A_{s_v}, \tag{3.2}$$

where $(j_1, j_2, \cdots, j_v)$ is one positive integer solution to the equation $\sum_{w=1}^{v} j_w = l-1$, the subscript set $\{s_1, s_2, \cdots, s_v\} \subseteq \{1, 2, \cdots, l-1\}$ and

$$A_p = A_{s_1} < A_{s_2} < \cdots < A_{s_v} \leq A_{l-1}.$$

If $v = 1$, then f maps all the elements of X into A_p which has $l-1$ possible choices and so the number of f is $(l-1)F_{2n}$. Suppose that $v > 1$ and then, for each w $(1 \leq w \leq v)$, there are j_w possible choices for A_{s_w}. By Lemma 3.4, for the fixed positive integer solution $(j_1, j_2, \cdots, j_v)$ to the equation $\sum_{w=1}^{v} j_w = l-1$, the number of f satisfying (3.2) should be $\prod_{w=1}^{v} j_w F_{2n}$. So the number of all f satisfying (3.2) is $\sum_{j_1+j_2+\cdots+j_v=l-1} \prod_{w=1}^{v} j_w F_{2n}$. Taking the sum v from 2 to $l-1$, we obtain that the number of f satisfying (3.2) is $\sum_{v=2}^{l-1} \sum_{j_1+j_2+\cdots+j_v=l-1} \prod_{w=1}^{v} j_w F_{2n}$. Therefore, the number of f satisfying the condition that $f(A_l) \subseteq A_p$ for $p \in \{1, 2, \cdots, l-1\}$ is

$$(l-1)F_{2n} + \sum_{v=2}^{l-1} \sum_{j_1+j_2+\cdots+j_v=l-1} \prod_{w=1}^{v} j_w F_{2n} = \sum_{v=1}^{l-1} \sum_{j_1+j_2+\cdots+j_v=l-1} \prod_{w=1}^{v} j_w F_{2n}.$$

Case 2. $f(A_l) \subseteq A_p$ for $p \in \{l, l+1, \cdots, m\}$. Set

$$t = |\{A \in X/E : A \cap f(X) \neq \emptyset\}|,$$

where $1 \leq t \leq m-(l-1)$. Suppose

$$f([A_l, A_{l+k_1}]) \subseteq A_{s_1}, f([A_{l+k_1+1}, A_{l+k_1+k_2}]) \subseteq A_{s_2}, \cdots, \\ f([A_{l+k_1+k_2+\cdots+k_{t-1}+1}, A_{l+k_1+k_2+\cdots+k_{t-1}+k_t} = A_{l-1}]) \subseteq A_{s_t}, \tag{3.3}$$

where $(k_1, k_2, \cdots, k_t)$ is any integer solution to the equation $\sum_{i=1}^{t} k_i = m-1$ and $k_1 \geq 0$, $k_2 \geq 1, k_3 \geq 1, \cdots, k_{t-1} \geq 1, k_t \geq l$(since f maps at least E-classes $A_m, A_1, \cdots, A_{l-1}$ into A_{s_t}), the subscript set $\{s_1, s_2, \cdots, s_t\} \subseteq \{l, l+1, \cdots, m\}$ and

$$A_l \leq A_p = A_{s_1} < A_{s_2} < \cdots < A_{s_t} \leq A_m.$$

If $t = 1$, it is clear that the number of f is $(m-l+1)F_{2n}$. If $t = 2$, then there are (k_1+1) choices for A_{s_1} and $(k_2-l+1)(k_2 \geq l)$ choices for A_{s_2}. Thus the number of f is

$$\sum_{k_1+k_2=m-1} ((k_1+1)F_{2n})((k_2-l+1)F_{2n}).$$

If $3 \leq t \leq m-(l-1)$, there are (k_1+1) choices for A_{s_1}, and, for each i $(2 \leq i \leq t-1)$, k_i choices for A_{s_i}, and (k_t-l+1) choices for A_{s_t}. So, for the fixed integer solution $(k_1, k_2, \cdots, k_t)$ to the equation $\sum_{i=1}^{t} k_i = m-1$, the number of f satisfying (3.3) is $(k_1+1)F_{2n}\left(\prod_{i=2}^{t-1} k_i F_{2n}\right)(k_t-l+1)F_{2n}$. Thus, the number of all f satisfying (3.3) is

$$\sum_{k_1+k_2+\cdots+k_t=m-1} \left((k_1+1)F_{2n}(\prod_{i=2}^{t-1} k_i F_{2n})(k_t-l+1)F_{2n} \right).$$

Taking the sum t from 3 to $m-l+1$ yields

$$\sum_{t=3}^{m-l+1}\sum_{k_1+k_2+\cdots+k_t=m-1}\left((k_1+1)F_{2n}(\prod_{i=2}^{t-1}k_iF_{2n})(k_t-l+1)F_{2n}\right).$$

Therefore, the number of f satisfying the condition that $f(A_l)\subseteq A_p$ for $p\in\{l,l+1,\cdots,m\}$ is

$$\begin{aligned}&(m-l+1)F_{2n}+\sum_{k_1+k_2=m-1}((k_1+1)F_{2n})((k_2-l+1)F_{2n})\\&+\sum_{t=3}^{m-l+1}\sum_{k_1+k_2+\cdots+k_t=m-1}\left((k_1+1)F_{2n}(\prod_{i=2}^{t-1}k_iF_{2n})(k_t-l+1)F_{2n}\right)\\&=\sum_{t=1}^{m-l+1}\sum_{k_1+k_2+\cdots+k_t=m-1}\left((k_1+1)F_{2n}(\prod_{i=2}^{t-1}k_iF_{2n})(k_t-l+1)F_{2n}\right),\\&=\sum_{t=1}^{m-l+1}\sum_{k_1+k_2+\cdots+k_t=m}(\prod_{i=1}^{t-1}k_iF_{2n})\big((k_t-l+1)F_{2n}\big),\end{aligned}$$

where $(k_1,k_2,\cdots,k_t)$ is any positive integer solution to the equation $\sum_{i=1}^{t}k_i=m$ and the final positive integer $k_t\geq l$. Consequently,

$$|E(\mathscr{A}_l)|=\sum_{v=1}^{l-1}\sum_{j_1+j_2+\cdots+j_v=l-1}\prod_{w=1}^{v}j_wF_{2n}+\sum_{t=1}^{m-l+1}\sum_{k_1+k_2+\cdots+k_t=m}\left(\prod_{i=1}^{t-1}k_iF_{2n}\right)\big((k_t-l+1)F_{2n}\big).\quad\blacksquare$$

Remark 3.4. In Lemma 3.5, when $t=1$, we have

$$\sum_{k_1+k_2+\cdots+k_t=m}\left(\prod_{i=1}^{t-1}k_iF_{2n}\right)\big((k_t-l+1)F_{2n}\big)=(m-l+1)F_{2n}.$$

Example 3.1. By virtue of Lemma 3.5, we calculate $|E(\mathscr{A}_2)|,|E(\mathscr{A}_3)|,|E(\mathscr{A}_4)|$ for $m=4$, $n=3$ and have

$$|E(\mathscr{A}_2)|=F_6+(3F_6+2F_6F_6+F_6(2F_6)+F_6F_6F_6)=800,$$
$$|E(\mathscr{A}_3)|=(2F_6+F_6F_6)+(2F_6+F_6F_6)=160$$

and

$$|E(\mathscr{A}_4)|=(3F_6+F_6(2F_6)+2F_6F_6+F_6F_6F_6)+F_6=800.$$

Finally we consider the number of idempotents in $\bigcup_{t=1}^{n-1}\mathscr{B}_{s,t}(1\leq s\leq m)$. The following lemma comes from [2, Theorem 2.10].

Lemma 3.6. $|E(OP_n)|=F_{2n-1}+F_{2n+1}-(n^2-n+2)$.

Lemma 3.7. *Let $f\in\mathscr{B}_{s,t}$ with $1\leq s\leq m$ and $1\leq t\leq n-1$.*

(1) *If $f(X)\subseteq A_q$ for $q\neq s$, then $f:X\to A_q$ is an idempotent in $OP_E(X)$ if and only if $f|_{A_q}:A_q\to A_q$ is an idempotent in $\mathscr{T}_{A_q}$ and*

$$f([(s-1)n+t+1,A_{q-1}])=f(a),\quad f([A_{q+1},(s-1)n+t])=f(b),$$

where $a=minA_q=(q-1)n+1$ and $b=maxA_q=qn$.

(2) *If $f(X) \subseteq A_s$, then $f : X \to A_s$ is an idempotent in $OP_E(X)$ if and only if $f|_{A_s} : A_s \to A_s$ is an idempotent in $\mathscr{T}_{A_s}$, moreover, if $f((s-1)n+t+1) \le (s-1)n+t$, then $f([(s-1)n+t+1, f((s-1)n+t+1)]) = f((s-1)n+t+1)$, and if $f((s-1)n+t+1) > (s-1)n+t$, then $f([A_{s+1},(s-1)n+t] = f(sn)$.*

Proof. Here we only show (2). Since $f \in \mathscr{B}_{s,t}$, we have

$$f((s-1)n+t+1) \le f((s-1)n+t+2) \le \cdots \le f(mn) \le f(1) \le \cdots \le f((s-1)n+t).$$

We now suppose that $f : X \to A_s$ is an idempotent in $OP_E(X)$, then $f|_{A_s} : A_s \to A_s$ is also an idempotent in $\mathscr{T}_{A_s}$. Let $c = f((s-1)n+t+1)$ and $x \in [(s-1)n+t+1, c]$. If $c \le (s-1)n+t$, then $f(x) \le f(c) = c$ and $f(x) \ge f((s-1)n+t+1) = c$. Thus $f(x) = c$. If $c > (s-1)n+t$ and $x \in [A_{s+1},(s-1)n+t]$, then $f(sn) \le f(x)$ and we can assert that $f(sn) = f(x)$. Indeed, if $f(sn) < f(x)$. Noting that f maps X into A_s, we have $f(x) \le sn$ and $f(x) = f^2(x) \le f(sn)$, a contradiction. The sufficiency is clear and the proof is completed. ∎

Remark 3.5. In Lemma 3.7(1), we consider two special cases.

(1) If $q = 1$, then $f : X \to A_1$ is an idempotent in $OP_E(X)$ if and only if $f|_{A_1} : A_1 \to A_1$ is an idempotent in $\mathscr{T}_{A_1}$ and

$$f([(s-1)n+t+1, A_m]) = f(1),\ f([A_2,(s-1)n+t]) = f(n).$$

(2) If $q = m$, then $f : X \to A_m$ is an idempotent in $OP_E(X)$ if and only if $f|_{A_m} : A_m \to A_m$ is an idempotent in $\mathscr{T}_{A_m}$ and

$$f([(s-1)n+t+1, A_{m-1}]) = f((m-1)n+1),\ f([A_1,(s-1)n+t]) = f(mn).$$

To illustrate Lemma 3.7, let $m = 3, n = 5$ and $A_1 = \{1,2,3,4,5\}, A_2 = \{6,7,8,9,10\}$, $A_3 = \{11,12,13,14,15\}$. Let

$$g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 14 & 14 & 14 & 14 & 14 & 14 & 14 & 12 & 12 & 12 & 12 & 12 & 13 & 14 & 14 \end{pmatrix} \in E(\mathscr{B}_{2,2}),$$

$$g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 6 & 6 & 6 & 6 & 6 & 6 & 7 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \in E(\mathscr{B}_{2,2}),$$

and

$$g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 10 & 10 & 10 & 10 & 10 & 10 & 10 & 9 & 9 & 10 & 10 & 10 & 10 & 10 & 10 \end{pmatrix} \in E(\mathscr{B}_{2,2}).$$

Clearly, $g_1|_{A_3}$ is an idempotent in $\mathscr{T}_{A_3}$, $g_1([8,10]) = g_1(11), g_1([A_1,7]) = g_1(15)$, and $g_2|_{A_2}$ is an idempotent in $\mathscr{T}_{A_2}$, $g_2([8,g_2(8)]) = g_2(8)$, and $g_3|_{A_2}$ is an idempotent in $\mathscr{T}_{A_2}$, $g_3([A_3,7]) = g_3(10)$.

Lemma 3.8. *For $1 \le s \le m$,*

$$\sum_{t=1}^{n-1} |E(\mathscr{B}_{s,t})| = (n-1)(m-1)F_{2n} + 2(F_{2n-1} - 1).$$

Proof. Let $f \in E(\mathscr{B}_{s,t})$ for $1 \le t \le n-1$. Set

$$M_q^{\mathscr{B}_{s,t}} = \{f \in E(\mathscr{B}_{s,t}) : f(X) \subseteq A_q\}.$$

There are two cases to consider.

Case 1. $q \neq s$. Then, by Lemmas 3.1 and 3.7 (1), $|M_q^{\mathscr{B}_{s,t}}| = F_{2n}$ since f is order-preserving on the E-class A_q. Thus $|\cup_{q\neq s} M_q^{\mathscr{B}_{s,t}}| = (m-1)F_{2n}$.

Case 2. $q = s$. Then, by Lemma 3.7(2),
$|M_s^{\mathscr{B}_{s,t}}| = |\{f \in E(OP_n) : f(t+1) \le f(t+2) \le \cdots \le f(n-1) \le f(n) \le f(1) \le \cdots \le f(t)\}|$.
Noting that in OP_n, by Lemmas 3.6, the number of idempotents which are not order-preserving is $F_{2n-1} + F_{2n+1} - (n^2 - n + 2) - F_{2n}$, we have

$$\sum_{t=1}^{n-1} |M_s^{\mathscr{B}_{s,t}}| = F_{2n-1} + F_{2n+1} - (n^2 - n + 2) - F_{2n} + (n-1)n = 2(F_{2n-1} - 1).$$

Consequently,

$$\sum_{t=1}^{n-1} |E(\mathscr{B}_{s,t})| = (n-1)(m-1)F_{2n} + \sum_{t=1}^{n-1} |M_s^{\mathscr{B}_{s,t}}| = (n-1)(m-1)F_{2n} + 2(F_{2n-1} - 1). \quad \blacksquare$$

Observing that for $1 \le s, s' \le m$, $1 \le t, t' \le n-1$,

$$E(\mathscr{A}_s) \cap E(\mathscr{A}_{s'}) = E(\mathscr{B}_{s,t}) \cap E(\mathscr{B}_{s',t'}) = E(\mathscr{B}_{s,t}) \cap E(\mathscr{A}_s) = \{\langle 1\rangle, \langle 2\rangle, \cdots, \langle mn\rangle\},$$

and that the total number of idempotents $\langle 1\rangle, \langle 2\rangle, \cdots, \langle mn\rangle$ in $\mathscr{A}_2, \mathscr{A}_3, \cdots, \mathscr{A}_m, \bigcup_{t=1}^{n-1} \mathscr{B}_{s,t}$ $(1 \le s \le m)$ is $(m-1)mn + (n-1)m^2 n$, by Theorem 3.1, Lemma 3.5 and Lemma 3.8, we obtain the main result in this section.

Theorem 3.2.

$$\begin{aligned} |E(OP_E(X))| = &\sum_{t=1}^{m} \sum_{k_1+k_2+\cdots+k_t=m} \prod_{i=1}^{t} k_i F_{2n} + \sum_{l=2}^{m} \Bigg\{ \sum_{v=1}^{l-1} \sum_{j_1+j_2+\cdots+j_v=l-1} \prod_{w=1}^{v} j_w F_{2n} \\ &+ \sum_{t'=1}^{m-l+1} \sum_{k'_1+k'_2+\cdots+k'_{t'}=m} \left(\prod_{i=1}^{t'-1} k'_i F_{2n} \right) ((k'_{t'} - l + 1)F_{2n}) \Bigg\} \\ &+ m((n-1)(m-1)F_{2n} + 2(F_{2n-1} - 1)) - ((m-1)mn + (n-1)m^2 n), \end{aligned}$$

where $(k_1, k_2, \cdots, k_t)$ is any positive integer solution to the equation $\sum_{i=1}^{t} k_i = m$, and $(j_1, j_2, \cdots, j_v)$ is any positive integer solution to the equation $\sum_{w=1}^{v} j_w = l-1$, and $(k'_1, k'_2, \cdots, k'_{t'})$ is any positive integer solution to the equation $\sum_{i=1}^{t'} k'_i = m$ and the final positive integer $k'_{t'} \ge l$.

Example 3.2. Let $m = 4, n = 3$. By Theorem 3.1,

$$\begin{aligned} |E(\mathscr{A}_1)| &= 4F_6 + (F_6(3F_6) + (2F_6)(2F_6) + 3F_6(F_6)) \\ &\quad + (F_6F_6(2F_6) + F_6(2F_6)F_6 + (2F_6)F_6F_6) + F_6F_6F_6F_6 = 7840. \end{aligned}$$

From Example 3.1, we know $|E(\mathscr{A}_2)| = 800, |E(\mathscr{A}_3)| = 160$ and $|E(\mathscr{A}_4)| = 800$. It follows from Lemma 3.8 that $\sum_{t=1}^{2} |E(\mathscr{B}_{s,t})| = 6F_6 + 2(F_5 - 1) = 56$ for $1 \le s \le 4$. Thus,

$$|E(OP_E(X))| = 7840 + 800 + 160 + 800 + 56 \times 4 - (36 + 96) = 9692.$$

To conclude this section, we give the following Tables 3 and 4 providing the number of idempotents in $OP_E(X)$ and $O_E(X)$ for smaller m,n, respectively.

Table 3. The number of idempotents in $OP_E(X)$

$m\backslash n$	2	3	4	5	6
2	19	114	643	3727	22972
3	102	1016	12414	186328	3116238
4	513	9692	278337	10545529	454295384
5	2503	95198	6376621	600770505	66322745434
6	12066	941118	146363082	34233146606	9682664464596

Table 4. The number of idempotents in $O_E(X)$

$m\backslash n$	2	3	4	5	6
2	15	80	483	3135	21024
3	72	792	11088	178640	3069360
4	345	7840	254541	10179345	448105536
5	1653	77608	5843355	580044025	65420338896
6	7920	768240	134142624	33052330080	9550921373280

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