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Optimal Reinsurance and Dividend Strategies with Capital Injections in Cramér-Lundberg Approximation Model

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Abstract. In this paper, we consider a diffusion approximation to a classical risk process with the possibility of quota-share and excess-of-loss reinsurance, while in addition the company controls the amount of dividends paid out to the shareholders as well as the capital injections. The objective is to maximize the cumulative expected discounted dividends minus the penalized discounted capital injections until the time of bankruptcy. We show that the optimal combinational reinsurance strategy must be pure excess-of-loss reinsurance. The control problem is solved by constructing some suboptimal model which allows no bankruptcy by capital injection. Then we obtain the analytical expressions for the value function and the optimal strategies and it is concluded that they are the same as those in the case of no bankruptcy.

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1. Introduction

A traditional method to measure the risk of an insurance company is to calculate the ruin probabilities, (see Asmussen [2], Browne [5]). However many practitioners seek some other measures of risk such as the expected discounted value of its future dividends payments proposed by De Finetti [7]. Recently, there has been an upsurge on optimizing dividends payout with (re)-insurance setting in diffusion models (see Asmussen and Taksar [1]; Paulsen and Gjessing [14]; Asmussen *et al.* [3]; Højgaard and Taksar [10,11]; Paulsen [15]). They showed that the optimal dividend strategy is the barrier strategy.

However, a problem with the optimal strategy of De Finetti's problem is that ruin will occur almost surely. Therefore , Sethe and Taksar [16] suggested a model that can control its risk by injecting capitals whenever the surplus becomes negative. The value of the company is associated with the expected present value of the net dividends payout minus the injected capitals until the ruin time. We refer to three papers in which sufficient deposit must be made to make the reserve process nonnegative: Shreve *et al.* [17] in the general diffusion model, Avram *et al.* [4] in a Lévy setting, and Kulenko and Schmidli [12] in the classical

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risk model. Here we mention another paper Løkka and Zervos [13] in which the model is a particular case in Shreve *et al.* [17], however, there was no constraint on the capital injection or the bankruptcy, i.e., when there is deficit, the insurer can choose inject capitals or not (see also He and Liang [9], which incorporate a fixed cost for each deposit).

Most of the papers dealing with reinsurance only consider pure quota-share or excess-of-loss reinsurance, however, the insurer has the choice of a combination of the two in reality. Løkka and Zervos [13] considered the optimal dividend and injecting strategies in the diffusion model without reinsurance, which concluded whenever there is deficit it is optimal to inject capitals to guarantee no bankruptcy when the costs of collecting capitals are relatively low, otherwise it should not inject capitals and let it go bankrupt. Asmussen *et al.* [3] studies the optimal excess-of-loss reinsurance and dividend policies in the diffusion model without capital injection. Inspired by the ideas above, we consider the combination of proportional and excess-of-loss reinsurance with the possibility of capital injection in the diffusion model.

The paper is organized as follows. In Section 2 we give a rigorous mathematical formulation of the problem. We show that the optimal combinational reinsurance must be the pure excess-of-loss reinsurance in Section 3. Section 4 is devoted to the associated suboptimal problem, which doesn't allow bankruptcy ($V_c(x)$), and we solve it analytically. Section 5 is concerned with the solution to the general control problem that involves no constraints on the capital injection or the bankruptcy. We prove the value function and the optimal strategies are the same as those in the case of no bankruptcy $V_c(x)$, which is different from the results in Løkka and Zervos [13].

2. Formulation of the control problem

We make some notations which will be used in the following:
$\mathbb{E}X$: the expectation of r.v. X;
$\mathbb{D}X$: the variance of r.v. X;
C^2: the set of all the twice continuously differentiable functions;
$\mathbb{I}$: indicator function;
$x \wedge y := \min\{x, y\}$;
$x^+ := \max\{x, 0\}$.

Our results will be formulated within the framework of controlled diffusion approximation models. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a filtered probability space supporting a standard Brownian motion $\{B_t\}$. For convenience we start from the classical risk process:

$$X_t^0 = x + pt - \sum_{i=1}^{N_t} Y_i,$$

where $x \geq 0$ is the initial reserve; $\{N_t\}$, representing the claim times up to time t, is a Poisson process with intensity $\eta > 0$; $\{Y_i, i = 1, 2, \cdots\}$, independent of $\{N_t\}$, is an i.i.d. sequence of positive random variables representing the successive individual claim amounts and having cumulative distribution function (c.d.f.) $F(x)$ with finite first and second moments μ_∞, σ_∞^2, respectively. In this paper we assume the premium is calculated via expected value principle, then we have

$$p = (1+\theta)\eta\mu_\infty,$$

where $\theta > 0$ is the relative safety loading of the insurer.

Assume the insurer takes a combination of quota-share and excess-of-loss reinsurance in the way of Centeno [6]: Firstly, the insurer chooses a quota-share retention level k ($0 \le k \le 1$), i.e., the insurer's aggregate claims, net of quota-share reinsurance, are kY. Secondly, the insurer chooses excess-of-loss reinsurance retention level $a \in [0,A]$, where $A := \sup\{a : F(a) < 1\}$, i.e., the insurer's aggregate claims, net of quota-share and excess-of-loss reinsurance, are $kY \wedge a$. Although non-cheap reinsurance (the reinsurer uses a higher relative safety loading than the insurer's) is more realistic, we still consider the cheap reinsurance (the reinsurer uses the same relative safety loading as the insurer's) in this paper since the problem becomes too difficult to solve in the case of non-cheap reinsurance. Then the surplus process is given by

$$X_t^{(k,a)} = x + p^{(k,a)}t - \sum_{i=1}^{N_t}(kY_i \wedge a),$$

where the premium rate is

$$p^{(k,a)} = (1+\theta)\eta\mathbb{E}[kY_i \wedge a].$$

Since the stochastic process $\{X_t^{(k,a)} - x\}$ has stationary independent increments and

$$\mathbb{E}[X_t^{(k,a)} - x] = \eta\theta kt\mathbb{E}[Y_i \wedge \frac{a}{k}]$$

and

$$\mathbb{D}[X_t^{(k,a)} - x] = \eta k^2 t\mathbb{E}[Y_i \wedge \frac{a}{k}]^2,$$

then the diffusion approximation to the surplus process $X_t^{(k,a)}$ is given by

$$dX_t^{(k,a)} = \eta\theta k\mu(\frac{a}{k})dt + \sqrt{\eta}k\sigma(\frac{a}{k})dB_t,$$

where

$$\mu(a) = \mathbb{E}[Y_i \wedge a] = \int_0^a \bar{F}(x)dx, \tag{2.1}$$

$$\sigma(a) = \sqrt{\mathbb{E}[Y_i \wedge a]^2} = \sqrt{\int_0^a 2x\bar{F}(x)dx.} \tag{2.2}$$

We use $\{(k(t),a(t))\}$ to describe a dynamic reinsurance strategy. In addition to purchasing reinsurance, the insurance company pays dividends to its shareholders and allows for capital injections when necessary. We denote by $D(t)$ and $Z(t)$, which are increasing and *càdlàg* with $D_{0_-} = 0$, $Z_{0_-} = 0$, the accumulated amount of dividends and injected capitals up to time t, respectively. A control policy $\{(k(t),a(t);D(t),Z(t))\}$, denoted by $(k,a;D,Z)$ for simplicity, is said to be admissible if it is a four-dimensional $(\mathscr{F}_t)$-adapted stochastic process and satisfies $\Delta D \le X$, because otherwise, we can realize arbitrary high payoffs by making arbitrary high dividend payment at time 0, which is unrealistic. Without loss of generality, we assume $\eta = 1$, then the controlled surplus process $\{X_t^{(k,a;D,Z)}\}$ becomes

$$dX_t^{(k,a;D,Z)} = \theta k_t\mu(\frac{a_t}{k_t})dt + k_t\sigma(\frac{a_t}{k_t})dB_t - dD(t) + dZ(t). \tag{2.3}$$

We denote by $\varphi(x)$ the set of all admissible control policies with initial reserve x. For any $\pi=(k,a;D,Z)\in\varphi(x)$, the corresponding performance is defined as

$$V^{(k,a;D,Z)}(x)=\mathbb{E}_x\left[\beta_1\int_{0_-}^{\tau^\pi}e^{-\delta t}dD(t)-\beta_2\int_{0_-}^{\tau^\pi}e^{-\delta t}dZ(t)\right], \tag{2.4}$$

where δ is the discounted rate, $\beta_2>1$ and $\beta_1<1$. We interpret $1-\beta_1$ as the tax rate for dividend and β_2-1 as the proportional costs rate for capital injection. τ^π is the corresponding ruin time defined by $\tau^\pi=\inf\left\{t\geq 0: X_t^{(\pi)}<0\right\}$. The objective is to find the value function which is defined as

$$V(x)=\sup_{\pi\in\varphi(x)}V^\pi(x) \tag{2.5}$$

and the optimal policy $\pi^*=(k^*,a^*;D^*,Z^*)$ such that $V(x)=V^{\pi^*}(x)$.

Inspired by the idea in Løkka and Zervos [13], we get the following arguments: in view of the Markovian structure of the above problem, once the model parameters are fixed, we can expect that the optimal strategy should either allow for the surplus process to hit $(-\infty,0)$ by no injection at any time, or should keep the company never bankrupt by the means of capital injection, which corresponds to the subpoptimal model $V_c(x)$ in Section 4.

In the following, we first show that the optimal combinational reinsurance must be the pure excess-of-loss, i.e. $k^*(t)\equiv 1$. Then we solve the suboptimal problem $V_c(x)$. Finally from the properties of $V_c(x)$ it is concluded that, whatever the model parameters are, the optimal choice is to guarantee no bankruptcy, that is, the optimal strategies and the value function are the same as those in the model $V_c(x)$.

3. The optimal reinsurance

In this section we will show the optimal combinational reinsurance is always the pure excess-of-loss reinsurance.

Lemma 3.1. *Let*

$$\mathscr{R}(a)=\frac{\sigma^2(a)}{[\mu(a)]^2}, \tag{3.1}$$

then $\mathscr{R}(a)$ is an increasing function of a for $a\geq 0$.

Proof. It is proved in Proposition 3.1 in Asmussen *et al.* [3]. ∎

Proposition 3.1. *For any fixed $(k,a;D,Z)\in\varphi(x)$, there exists $(1,\tilde{a};\tilde{D},\tilde{Z})\in\varphi(x)$ such that*

$$V^{(k,a;D,Z)}(x)\leq V^{(1,\tilde{a};\tilde{D},\tilde{Z})}(x). \tag{3.2}$$

Proof. For any fixed $(k(t),a(t);D(t),Z(t))\in\varphi(x)$, there exists $\tilde{a}(t)$ such that

$$k_t\sigma\left(\frac{a_t}{k_t}\right)=\sigma(\tilde{a}_t). \tag{3.3}$$

Easy to see $\tilde{a}_t\leq\frac{a_t}{k_t}$, so in view of Lemma 3.1, we have

$$\frac{\sigma^2(\tilde{a}_t)}{[\mu(\tilde{a}_t)]^2}\leq\frac{\sigma^2(a_t/k_t)}{[\mu(a_t/k_t)]^2},$$

which implies

$$k_t\mu\left(\frac{a_t}{k_t}\right)\leq\mu(\tilde{a}_t).$$

Let $\tilde{D}(t):=D(t)+\theta\int_0^t\left[\mu(\tilde{a}_s)-k_s\mu\left(\frac{a_s}{k_s}\right)\right]ds\geq D(t)$, $\tilde{Z}(t):=Z(t)$, then $\tilde{D}(t)$ and $\tilde{Z}(t)$ are both increasing and

$$\begin{aligned}dX_t^{(1,\tilde{a};\tilde{D},\tilde{Z})}&=\theta\mu(\tilde{a}_t)dt+\sigma(\tilde{a}_t)dB_t-d\tilde{D}(t)+d\tilde{Z}(t)\\&=\theta k_t\mu(\frac{a_t}{k_t})dt+k_t\sigma(\frac{a_t}{k_t})dB_t-dD(t)+dZ(t).\end{aligned}$$

Hence we get $\tau_x^{(k,a;D,Z)}=\tau_x^{(1,\tilde{a};\tilde{D},\tilde{Z})}$, while $\tilde{D}(t)\geq D(t)$ and $\tilde{Z}(t)=Z(t)$, so we get

$$V^{(k,a;D,Z)}(x)\leq V^{(1,\tilde{a};\tilde{D},\tilde{Z})}(x).$$ ∎

The following corollary is a direct consequence of Proposition 3.1.

Corollary 3.1.

$$V(x)=\sup_{(1,a;D,Z)\in\varphi(x)}V^{(1,a;D,Z)}(x). \tag{3.4}$$

Remark 3.1. For simplicity we write (a,D,Z) for $(1,a;D,Z)$ in the following.

4. The solution to the suboptimal problem

4.1. The associated suboptimal problem

We consider an associated suboptimal problem corresponding to the maximum of the performance index over a set of appropriate admissible strategies.

Definition 4.1. *(The company never bankrupt)*
Given an initial reserve $x\geq 0$, Let $\varphi_c(x)=\{(a,D,Z)\in\varphi(x)|X(t)\geq 0, for\, all\ t\geq 0\}$. We define the associated value function $V_c(x)$ by

$$V_c(x)=\sup_{(a,D,Z)\in\varphi_c(x)}V^{(a,D,Z)}(x). \tag{4.1}$$

Through the above definition we can easily get the relationship

$$V(x)\geq V_c(x),\ for\ \ all\ x\geq 0. \tag{4.2}$$

Lemma 4.1. *$\mu\circ\sigma^{-1}$ is an concave function.*

Proof. From (2.1) and (2.2), it is obvious that $\mu(a)$ and $\sigma(a)$ is strictly increasing on $[0,A]$, thus the inverses of $\mu(\cdot)$ and $\sigma(\cdot)$ exist, which are denoted by $\mu^{-1}(\cdot)$ and $\sigma^{-1}(\cdot)$, respectively.

According to Asmussen *et al.* [3], Let $\rho=\mu^{-1}$ and $\phi=\sigma^2\circ\rho$, then we have $\phi'(u)=2\rho(u)$, which implies

$$\frac{d}{du}\sqrt{\phi(u)}=\frac{a}{\sigma(a)}|_{a=\rho(u)}.$$

Now differentiating $\frac{a}{\sigma(a)}$ w.r.t a yields

$$\frac{d}{da}\left(\frac{a}{\sigma(a)}\right) = \frac{\sigma^2(a)-a^2\bar{F}(a)}{\sigma^3(a)} = \frac{-\int_0^a x^2 d\bar{F}(x)}{\sigma^3(a)} \geq 0, \quad a \in [0,A],$$

where we use (2.2) in the first equality and integration by part is applied in the second equality. Together with $\rho'(u) > 0$, we come to the following

$$\frac{d^2}{du^2}\sqrt{\phi(u)} = \frac{d}{da}\frac{a}{\sigma(a)}|_{a=\rho(u)} \cdot \rho'(u) \geq 0,$$

that is, $\sqrt{\phi(u)} = \sigma \circ \mu^{-1}(u)$ is convex, from which we get the concavity of $\mu \circ \sigma^{-1}$. ∎

In the model of no bankruptcy $V_c(x)$, it is clear that it cannot be optimal to make capital injections before they are really necessary because of the discounting, so we deduce that it is optimal to inject capitals only when the surplus becomes negative, therefore we need only to choose $(a(t),D(t))$, such that the corresponding injection process becomes

$$Z^{(a,D)}(t) := -\inf_{s\leq t}\left(\left[X^{(a)}(s) - D(s)\right] \wedge 0\right), \tag{4.3}$$

where $X^{(a)}$ is the controlled surplus process connected to the strategy $(1,a,0,0)$. We will sometimes use the abbreviated notation $X^{(a,D)}$ and $V^{(a,D)}$ for the controlled surplus process and the performance index connected to the strategy $\left\{a(t),D(t),Z^{(a,D)}(t)\right\}$ in the following. Thus the formula (4.1) can be rewritten as

$$V_c(x) = \sup_{(a,D)\in\varphi_c(x)} V^{(a,D)}(x). \tag{4.4}$$

Proposition 4.1. *$V_c(x), x \geq 0$ is a nonnegative, increasing and concave function.*

Proof. The monotonicity of $V_c(x)$ is obvious.

Next we show that $V_c(x) \geq 0$ for all $x \geq 0$. For any fixed $x \geq 0$, it is easy to see that the strategy (a^0,D^0,Z^0) with $a^0(t) = D^0(t) = Z^0(t) = 0$ for all $t \geq 0$ is an admissible strategy in $\varphi_c(x)$. Therefore, we have

$$V_c(x) \geq V^{(a^0,D^0,Z^0)}(x) = 0.$$

Lastly we show its concavity.

Let x_1, x_2 be two initial values and $\left(a_1,D_1,Z_1^{(a_1,D_1)}\right)$, $\left(a_2,D_2,Z_2^{(a_2,D_2)}\right)$ be two admissible control strategies for x_1 and x_2 respectively. Let $x_3 = \lambda x_1 + (1-\lambda)x_2$, $0 \leq \lambda \leq 1$. We can construct an admissible strategy for x_3 as follows:
Firstly, there existing $\{a(t)\}$ such that

$$\sigma(a(t)) = \lambda\sigma(a_1(t)) + (1-\lambda)\sigma(a_2(t)),$$

so by Lemma 4.1, we get

$$\mu(a(t)) \geq \lambda\mu(a_1(t)) + (1-\lambda)\mu(a_2(t)).$$

Define

$$Z(t) := \lambda Z_1^{(a_1,D_1)}(t) + (1-\lambda)Z_2^{(a_2,D_2)}(t) \tag{4.5}$$

and

$$D(t) := \lambda D_1(t) + (1-\lambda)D_2(t) + \int_0^t \Big[\mu(a(s)) - \lambda\mu(a_1(s)) - (1-\lambda)\mu(a_2(s))\Big]ds \tag{4.6}$$
$$\geq \lambda D_1(t) + (1-\lambda)D_2(t).$$

Noting that

$$\begin{aligned} Z^{(a,D)}(t) &= -\inf_{s\leq t}\Big(\Big[X^{(a)}(s) - D(s)\Big]\wedge 0\Big) \\ &= -\inf_{s\leq t}\Big(\Big[\lambda(X^{(a_1)}(s) - D_1(s)) + (1-\lambda)(X^{(a_2)}(s) - D_2(s))\Big]\wedge 0\Big) \\ &\leq \lambda Z^{(a_1,D_1)}(t) + (1-\lambda)Z^{(a_2,D_2)}(t) = Z(t), \end{aligned}$$

we have

$$X^{(a,D,Z)}(t) = X^{(a,D)}(t) + Z(t) - Z^{(a,D)}(t) \geq X^{(a,D)}(t) \geq 0,$$

which shows that the strategy (a,D,Z) is admissible for the initial value x_3, then from (4.5) and (4.6) we conclude that

$$\begin{aligned} V_c(x_3) \geq V_c^{(a,D,Z)}(x_3) &= \mathbb{E}\left[\beta_1\int_{0_-}^{\infty} e^{-\delta t}dD(t) - \beta_2\int_{0_-}^{\infty} e^{-\delta t}dZ(t)\right] \\ &\geq \lambda V^{(a_1,D_1)}(x_1) + (1-\lambda)V^{(a_2,D_2)}(x_2). \end{aligned}$$

Thus

$$V_c(x_3) \geq \lambda V_c(x_1) + (1-\lambda)V_c(x_2),$$

from which the concavity of $V_c(x)$ is derived. ∎

4.2. The solution to the problem $V_c(x)$

For any fixed $a\in\mathbb{R}$, we establish an operator $\mathscr{L}^a$ on the space C^2 which is frequently used in the following and is defined by:

$$\mathscr{L}^a f(x) = \frac{1}{2}\sigma^2(a)f''(x) + \theta\mu(a)f'(x) - \delta f(x)$$

for any $f\in C^2$.

The following proposition is well-known from the dynamic programming principle in Fleming and Soner [8] (It is actually the combination of Theorem 5.1 in Asmussen *et al.* [3] and (4.1) and (4.2) in Løkka and Zervos [13]):

Proposition 4.2. *If the function $V_c(x)\in C^2$, then it satisfies the following HJB equation*

$$\max\left(\sup_{a\in[0,A]}\mathscr{L}^a V_c(x), -V_c'(x)+\beta_1, V_c'(x)-\beta_2\right) = 0, \tag{4.7}$$

with the boundary condition

$$V_c'(0) = \beta_2. \tag{4.8}$$

Proposition 4.3. *Assume there exists $g\in C^2$ such that it is an increasing and concave solution to the HJB equation (4.7) with the boundary condition (4.8). Then*

(i) *The function g coincides with V_c. That is*

$$V_c(x) = g(x),\ x \geq 0.$$

In addition, $b_1 := \inf\{x \geq 0, V_c'(x) \leq \beta_1\} > 0$ exists, and $g(x)$ (or $V_c(x)$) satisfies the following equations

$$\sup_{a\in[0,A]} \left\{ \frac{1}{2}\sigma^2(a)g''(x) + \theta\mu(a)g'(x) - \delta g(x) \right\} = 0, \quad \textit{for} \quad x \in [0,b_1], \tag{4.9}$$

$$g(x) = \beta_1(x-b_1) + g(b_1), \quad \textit{for} \quad x \geq b_1, \tag{4.10}$$

$$g'(0) = \beta_2,. \tag{4.11}$$

(ii) *Further, pick $a^*(\cdot)$ be such that*

$$\sup_{a\in[0,A]} \mathscr{L}^a g(x) = \mathscr{L}^{a^*(x)} g(x), \tag{4.12}$$

holds for all $x \geq 0$, then $\pi^* := (a^*(X_t^*), D^*(t), Z^{(a^*,D^*)}(t))$ is the optimal strategy, where X_t^* is the surplus process under the optimal strategy,

$$D^*(t) := (x-x^*)^+ \mathbb{I}_{\{t=0\}} + \int_{(0,t]} \mathbb{I}_{\{X_s^*=b_1\}} dD^*(s) \tag{4.13}$$

and $Z^{(a^*,D^*)}(t)$ is defined in the same way as (4.3).

Proof. The proof of "$V_c(x) = g(x)$" in (i) is similar to those of Proposition 3.2 and Proposition 3.3 in Højgaard and Taksar [10] (we can refer to Theorem 5.2 in Asmussen *et al.* [3] and Theorem 4.1 in Løkka and Zervos [13], which also directly give the results without showing the details). It is based on a slightly modified standard verification procedure for the mixed singular/regular control.

For the rest part of **i)**, from the properties that $g(x)$ satisfies, it is obvious that $b_1 = \inf\{x : g'(x) \leq \beta_1\} > 0$ exists, moreover, we have $g'(x) > \beta_1$ for $x < b_1$ and $g'(x) = \beta_1$ for $x \geq b_1$. Thus $g(x)$ (or $V_c(x)$) satisfies the equations (4.9)-(4.11).

For (ii), we only need to show $g(x) = V^{\pi^*}(x)$.

Let π^* be as in the statement. In fact, according to the theory of Skorohod's equation, there exists a unique stochastic process $(D^*(t), Z^*(t))$ such that the following conditions hold:
(a) $X^*(t) := X^{(a^*,0,0)}(t) - D^*(t) + Z^*(t) \in [0,b_1]$, for all $t \geq 0$;
(b) $D^*(0) = (x-b_1)^+$, $Z^*(0) = 0$, and $D^*(t)$, $Z^*(t)$ are both nondecreasing on t;
(c) $D^*(t)$ and $Z^*(t)$ are flat off $\{t \geq 0, X^*(t) = b_1\}$ and $\{t \geq 0, X^*(t) = 0\}$, respectively.

Thus $D^*(t)$ can be expressed as the form in (4.13), $Z^*(t)$ is obviously the same as $Z^{(a^*,D^*)}(t)$ defined in (4.3) and $X^*(t)$ is the controlled process under the strategy $(a^*(X_t^*), D^*(t), Z^{(a^*,D^*)}(t))$. Actually $D^*(t)$ and $Z^*(t)$ are the local times of the reflected process $X^*(t)$ at the boundaries b_1 and 0, respectively. Obviously, the bankruptcy time under this strategy is $\tau^* = \infty$.

From (4.12), it follows that

$$\max\left(\mathscr{L}^{a^*} g(x), -g'(x) + \beta_1, g'(x) - \beta_2 \right) = 0. \tag{4.14}$$

For any stochastic process D, let D^c denote its continuous part. We consider g as in the statement and obtain

$$\begin{aligned} & e^{-\delta t} g(X_t^*) - g(x) \\ & = \int_0^t e^{-\delta s} \mathscr{L}^{a^*} g(X_s^*) ds + \int_0^t e^{-\delta s} \theta \mu(a_s^*) g'(X_s^*) dB(s) \\ & \quad - \int_0^t e^{-\delta s} g'(X_s^*) dD_s^{*c} + \int_0^t e^{-\delta s} g'(X_s^*) d(Z^{(a^*,D^*)})_s^c + [g(b_1) - g(x)] \mathbb{I}_{\{x>b_1\}} \\ & = \int_0^t e^{-\delta s} \theta \mu(a_s^*) g'(X_s^*) dB(s) - \beta_1 \int_0^t e^{-\delta s} dD_s^{*c} \\ & \quad + \beta_2 \int_0^t e^{-\delta s} dZ_s^{(a^*,D^*)} - \beta_1 (x - b_1)^+, \end{aligned} \tag{4.15}$$

where we use the generalized Itô's formula in the first equality; the second equality holds true for the following reason: In view of (4.14), the first term on the r.h.s. of the first equality is equal to 0; Since, by definition, the dividend process D^* continuously increases only at the boundary b_1, apart from a possible jump $(x-b_1)^+$ at time 0, and the injection process $Z^{(a^*,D^*)}$ continuously increases only at the boundary 0 without any jump, together with the conditions $g'(x) = \beta_1$ for all $x \geq b_1$ and $g'(0) = \beta_2$, we easily deduce that second equality holds true.

In view of the boundness of $g'(\cdot)$ on $[0, b_1]$ and the fact that X^* is a reflected process at the boundaries 0 and b_1, we conclude that $M = \{M_t\}_{t \geq 0}$ is a uniformly integrable martingale, where $M_t := \int_0^t e^{-\delta s} \theta \mu(a_s^*) g'(X_s^*) dB(s)$. On the other hand, $g(\cdot)$ is obviously bounded on $[0, b_1]$. Therefore, taking expectation on both sides of (4.15) and letting $t \to \infty$, we can deduce that

$$g(x) = \mathbb{E}_x\left[\int_{0-}^{\infty} e^{-\delta t} dD^*(t) - \beta_2 \int_{0-}^{\infty} e^{-\delta t} dZ^{(a^*,D^*)}\right] = V^{\pi^*}(x).$$

Thus π^* is the optimal strategy. ∎

From Proposition 4.3, we can see that what we need to do is to construct an increasing and concave solution $g(x) \in C^2$ to (4.7)–(4.8). Suppose such a function $g(x)$ exists, then $g(x) = V_c(x)$.

We first establish a lemma which will be required in constructing the function $g(x)$.

Define

$$d_\pm := \frac{-\theta\mu_\infty \pm \sqrt{\theta^2 \mu_\infty^2 + 2\delta\sigma_\infty^2}}{\sigma_\infty^2}, \tag{4.16}$$

$$m^* := \frac{1}{d_+ - d_-} \ln \frac{d_-(\theta + d_+ A)}{d_+(\theta + d_- A)}, \tag{4.17}$$

and

$$H(x) := \int_0^x \frac{\sigma^2(y)}{2y\left[-\theta\frac{\sigma^2(y)}{2y} + \theta\mu(y) + \frac{\delta y}{\theta}\right]} dy, \; x \geq 0, \tag{4.18}$$

then from the expressions (2.1) and (2.2) we easily conclude that $H(x), x \geq 0$ is a non-negative and strictly increasing function, so the inverse of $H(\cdot)$ exist, which is denoted by

$H^{-1}(\cdot)$. Thus we can define

$$p(x) := \beta_2 e^{-\int_0^{H(A)-H(x)} \frac{\theta}{H^{-1}(y+H(x))}dy}, \; x \in [0,A].$$

Lemma 4.2. *If and only if*

$$\beta_2 \geq \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*}, \tag{4.19}$$

there exists a unique solution $x \in [0,A]$ to the equation

$$p(x) = \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*}.$$

Proof. By simple differentiation operations, we deduce that

$$p'(x) = p(x)\theta H'(x)\left[1/A + \int_0^{H(A)-H(x)} \frac{1}{[H^{-1}(y+H(x))]^2 h\Big(H^{-1}(y+H(x))\Big)}dy\right] > 0,$$

where

$$h(y) = \frac{\sigma^2(y)}{2y\left[-\theta\frac{\sigma^2(y)}{2y} + \theta\mu(y) + \frac{\delta y}{\theta}\right]} > 0,$$

which implies that $p(x)$ is increasing on $[0,\infty)$.

By applying L'Hôspital, we deduce from (4.18), (2.1) and (2.2) that

$$\lim_{x\downarrow 0} H'(x) = \lim_{x\downarrow 0} \frac{\sigma^2(x)}{2x\left[-\theta\frac{\sigma^2(x)}{2x} + \theta\mu(x) + \frac{\delta x}{\theta}\right]} = \frac{\theta}{\theta^2 + 2\delta} > 0,$$

thus we have

$$e^{\int_0^{H(A)} \frac{\theta}{H^{-1}(y)}dy} = \int_0^A \frac{\theta}{x} H'(x)dx = \infty. \tag{4.20}$$

Therefore we can conclude that

$$p(0) = \beta_2 e^{-\int_0^{H(A)} \frac{\theta}{H^{-1}(y)}dy} = 0 < \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*}.$$

It is easy to see $p(A) = \beta_2$, therefore there exists a unique solution $x \in [0,A]$ if and only if (4.19) holds. ∎

To construct an increasing, concave function $g(x) \in C^2$ to the equations (4.7)-(4.8), we first conjecture its expression according to the conditions which $g(x)$ must satisfy. We have conjectured the corresponding expressions for $g(x)$ under two different parameter relationships in the **Appendix**. In the following theorem we will show that they indeed satisfy the conditions in Proposition 4.3, which implies $V_c(x) = g(x)$:

Theorem 4.1. **(i)** *If*

$$\beta_2 \geq \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*}, \tag{4.21}$$

then the value function $V_c(x)$ is given by

$$V_c(x) = \begin{cases} c_0 + \beta_2 \int_0^x e^{-\int_0^z \frac{\theta}{H^{-1}(y+H(a^*(0)))} dy} dz & \text{if } 0 \le x \le x_1, \\ c_1 e^{d_+ x} + c_2 e^{d_- x} & \text{if } x_1 \le x \le b_1, \\ \beta_1 (x - b_1) + c_3 & \text{if } x \ge b_1, \end{cases} \tag{4.22}$$

where $d_\pm, m^*, H(\cdot)$ are given by (4.16)–(4.18) and all the other parameters are determined by (5.12)–(5.20) and the optimal strategies (a^*, D^*, Z^*) are given as follows:
The optimal excess-of-loss retention level $a^*(t) = a^*\left(X^{(a^*,D^*,Z^*)}(t)\right)$, where $a^*(x)$ is determined by (5.12) when $x \le x_1$ and $a^*(x) = A$ for $x \ge x_1$; The optimal dividend and capital injection strategies (D^*, Z^*) reflect the surplus at the endpoints of the interval $[0, b_1]$, that is, D^* and Z^* are the corresponding local times of the reflected process $\{X^{(a^*,D^*,Z^*)}\}$ at the boundary b_1 and 0, respectively, where $X^{(a^*,D^*,Z^*)}$ is the surplus process under the optimal strategies.

(ii) *If*

$$\beta_2 < \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*}, \tag{4.23}$$

then the value function $V_c(x)$ is given by

$$V_c(x) = \begin{cases} F_1 e^{d_+ x} + F_2 e^{d_- x} & \text{if } 0 \le x \le b_2, \\ \beta_1 (x - b_2) + V_c(b_2) & \text{if } x \ge b_2; \end{cases} \tag{4.24}$$

where F_1, F_2 and b_2 are given by (5.22)-(5.23) and the optimal strategies are given as follows:
$a^* \equiv A$, i.e., it is optimal to buy no reinsurance at all; The optimal dividend and capital injection strategies (D^*, Z^*) reflect the surplus at the endpoints of the interval $[0, b_2]$.

Proof. First $g(x)$ given by (5.18) and (5.21) are obviously twice continuously differentiable from their construction. We will verify that $g(x)$ given by (5.18) and (5.21) satisfy the other conditions in Proposition 4.3 under the two different cases, respectively.

(i) In the case of

$$\beta_2 \ge \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*},$$

from the construction of $g(x)$ given by (5.18), it suffices to prove the following conditions:

$$\begin{cases} g'(x) > 0, \quad g''(x) \le 0, \ x \in [0, \infty), \\ \beta_1 \le g'(x) \le \beta_2, \qquad x \in [0, b_1], \\ \sup\limits_{a \in [0,A]} \left\{ \frac{1}{2}\sigma^2(a) g''(x) + \theta\mu(a) g'(x) - \delta g(x) \right\} = 0, \quad x \in [x_1, b_1], \\ \sup\limits_{a \in [0,A]} \left\{ \frac{1}{2}\sigma^2(a) g''(x) + \theta\mu(a) g'(x) - \delta g(x) \right\} \le 0, \quad x \in [b_1, \infty). \end{cases} \tag{4.25}$$

Firstly, it is obvious that $g(x)$ is increasing on $[0,\infty)$ and concave on $[0,x_1]$ and $[b_1,\infty)$. Moreover, from (5.16) we obtain

$$e^{(d_+-d_-)x} \le e^{(d_+-d_-)b_1} = \frac{-c_2 d_-^2}{c_1 d_+^2}, \text{ for } x \in [x_1, b_1],$$

which implies $g(x)$ is concave on $[x_1,b_1]$. Together with the continuity of $g'(x)$ at x_1 and b_1, we deduce that $g(x)$ is concave on $[0,\infty)$. In addition, $g'(b_1)=\beta_1$, hence the first and the second condition in (4.25) hold.

Secondly, for any fixed $x \in [x_0,b_0]$, we view $\mathscr{L}^a g(x)$ as a function of $a \in [0,A]$, then $\frac{d}{da}\mathscr{L}^a g(x) = \bar{F}(a)[ag''(x)+\theta g'(x)]$. If we can prove

$$ag''(x)+\theta g'(x) \ge 0,\ a \in [0,A], \tag{4.26}$$

then by the construction of $g(x)$ we have

$$\sup_{a\in[0,A]} \mathscr{L}^a g(x) = \mathscr{L}^A g(x) = 0, \text{ for } x \in [x_1,b_1],$$

which implies that the third condition in (4.25) is satisfied.

From (5.15) and (5.16), (4.26) is equivalent to

$$e^{(d_+-d_-)(x-b_1)} \ge \frac{d_+(\theta+d_-a)}{d_-(\theta+d_+a)}.$$

So we only need to show

$$e^{(d_+-d_-)(x_1-b_1)} \ge \frac{d_+(\theta+d_-a)}{d_-(\theta+d_+a)}.$$

From (5.17), the above inequality can be reduced to

$$\frac{A+\theta/d_-}{A+\theta/d_+} \ge \frac{a+\theta/d_-}{a+\theta/d_+},$$

which obviously holds.

Lastly, since $g''(x)\equiv 0$ and $g'(x)\equiv \beta_1$ for all $x \ge b_1$, we have

$$\begin{aligned}
&\sup_{a\in[0,A]} \left\{ \frac{1}{2}\sigma^2(a)g''(x)+\theta\mu(a)g'(x)-\delta g(x) \right\} \\
&= \sup_{a\in[0,A]} \{\theta\beta_1\mu(a)-\delta g(x)\} = \theta\beta_1\mu_\infty - \delta g(x) \\
&\le \theta\beta_1\mu_\infty - \delta g(b_1) = \sup_{a\in[0,A]} \mathscr{L}^a g(b_1) = 0.
\end{aligned}$$

Thus the last one in (4.25) holds true.

On the other hand, by the construction of $g(x)$ it is easy to see that $a^*(x)$ in the statement (i) satisfies (4.12). From Proposition 4.3 (ii), we deduce that the strategies (a^*,D^*,Z^*) stated in (i) are the optimal strategies under (4.21).

(ii) In the case of $\beta_2 < \frac{-\beta_1 d_-}{d_+-d_-}e^{d_+m^*} + \frac{\beta_1 d_+}{d_+-d_-}e^{d_-m^*}$, it was showed in Løkka and Zervos [13] that $g(x)$ given by (5.21) is an increasing, concave function and satisfies the following HJB equation

$$\max\left\{\mathscr{L}^A g(x), -g'(x)+\beta_1, g'(x)-\beta_2\right\} = 0,$$

with the boundary condition $g'(0) = \beta_2$. So it suffices to prove the maximum of $\sup_{a\in[0,A]} \Big\{\frac{1}{2}\sigma^2(a)$ $g''(x) + \theta\mu(a)g'(x) - \delta g(x)\Big\}$, where $g(x)$ is given by (5.21), is such that $a^*(x) \equiv A$ for $x \in [0, b_2]$:

For any fixed $x \in [0, b_2]$, differentiating $\mathscr{L}^a g(x)$ w.r.t a leads to

$$\frac{d}{da}\mathscr{L}^a g(x) = \bar{F}(a)[ag''(x) + \theta g'(x)],\ a \in [0, A].$$

Due to $g'(x) > 0$ and $g''(x) < 0$, it is easy to see that the maximal point of $\mathscr{L}^a g(x)$ on $a \in [0, \infty)$ is equal to $\frac{-\theta g'(x)}{g''(x)}$. So we only need to prove $\frac{-\theta g'(x)}{g''(x)} \geq A$ for all $x \in [0, b_2]$, from the expression (5.21), which is equivalent to show

$$e^{(d_+ - d_-)(x - b_2)} \geq \frac{d_+(\theta + d_- A)}{d_-(\theta + d_+ A)} = e^{(d_+ - d_-)(x_1 - b_1)},\ x \in [0, b_2],$$

where the last equality is deduced by (5.17). Therefore it is sufficient to show

$$b_2 \leq b_1 - x_1,$$

due to the fact that b_2 is the unique solution to (5.23) and $\beta_1(d_+ e^{-d_- x} - d_- e^{-d^+ x})$ is increasing on x, which obviously holds when

$$\beta_2 < \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+(x_1 - b_1)} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_-(x_1 - b_1)}.$$

Similar to the previous case, the strategies (a^*, D^*, Z^*) stated in (ii) are indeed the optimal strategies under (4.23). ∎

Remark 4.1. Since when $\beta_2 = \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*}$ we have $x_1 = 0$, thus the value function also has the form of (4.24) and the optimal dividend barrier b_1 is the unique solution to (5.23). Therefore it doesn't matter to change " $<$ " into " $\leq$ " in (4.23).

Remark 4.2. In view of (5.17), when $A = \infty$, i.e., the claim size distribution has unbounded support, we get $x_1 = b_1$, which means that the insurer begins to pay dividends out as soon as the reinsurance stops.

Remark 4.3. From Theorem 4.1, in the case $V_c(x)$ (no bankruptcy by capital injection), we can see that whether reinsurance is needed depends on the model parameters. Specifically speaking, the company needs reinsurance when the costs of collecting capitals are relatively high ((4.21)) and needn't when the costs are low ((4.23)). It is financially intuitive. Because in the case of low costs the effect of dividend revenue is much stronger than that of capital outflow, no reinsurance is purchased to prevent from reducing the potential profits; while with high costs, the effects of capital outflow is stronger than that of dividend revenue, reinsurance should be taken to reduce the cumulative amounts of injected capitals.

5. The solution to the control problem

Lemma 5.1. *(Verification Lemma) If $h(x)$ satisfies*

$$\max\left(\sup_{a\in[0,A]} \mathscr{L}^a h(x), -h'(x) + \beta_1, h'(x) - \beta_2\right) \leq 0, \tag{5.1}$$

$$h(0) \geq 0, \tag{5.2}$$

where

$$\mathscr{L}^a h(x) = \frac{1}{2}\sigma^2(a)h''(x) + \theta\mu(a)h'(x) - \delta h(x), \tag{5.3}$$

then

$$h(x) \geq V(x). \tag{5.4}$$

Proof. For any fixed initial value $x \geq 0$ and any admissible strategy $\pi = (a, D, Z) \in \varphi(x)$, denote the corresponding ruin time by τ^π (sometimes τ for simplicity). We denote by D^c and Z^c the continuous part of the processes D, Z respectively, and ΔD and ΔZ be the jump part of the processes D, Z respectively. Using the generalized $It\hat{o}'s$ formula, we deduce that

$$\begin{aligned}
&e^{-\delta(t\wedge\tau)}h(X^\pi_{t\wedge\tau}) - h(x) \quad (5.5)\\
&= \int_0^{t\wedge\tau} e^{-\delta s}\left[-\delta h(X^\pi_s)ds + h'(X^\pi_s)d(X^\pi_s)^c + \frac{1}{2}h''(X^\pi_s)d\langle X^\pi\rangle^c_s\right]\\
&\quad + \sum_{0\leq s\leq t\wedge\tau}(h(X^\pi_s) - h(X^\pi_{s-}))\\
&= \int_0^{t\wedge\tau} e^{-\delta s}[-\delta h(X^\pi_s) + \theta\mu(a_s)h'(X^\pi_s) + \frac{1}{2}\sigma^2(a_s)h''(X^\pi_s)]ds\\
&\quad + \int_0^{t\wedge\tau} e^{-\delta s}\theta\mu(a_s)h'(X^\pi_s)dB(s) - \int_0^{t\wedge\tau} e^{-\delta s}h'(X^\pi_s)dD^c(s) \quad (5.6)\\
&\quad + \int_0^{t\wedge\tau} e^{-\delta s}h'(X^\pi_s)dZ^c(s) + \sum_{0\leq s\leq t\wedge\tau} e^{-\delta s}\int_0^{\triangle D_s}[-h'(X^\pi_s - z)]dz\\
&\quad + \sum_{0\leq s\leq t\wedge\tau} e^{-\delta s}\int_0^{\triangle Z_s} h'(X^\pi_s - z)dz,\\
&\leq \int_0^{t\wedge\tau} e^{-\delta s}\mathscr{L}^a h(X^\pi_s)ds + \int_0^{t\wedge\tau} e^{-\delta s}\theta\mu(a_s)h'(X^\pi_s)dB(s)\\
&\quad - \beta_1\int_0^{t\wedge\tau} e^{-\delta s}h'(X^\pi_s)dD(s) + \beta_2\int_0^{t\wedge\tau} e^{-\delta s}h'(X^\pi_s)dZ(s). \quad (5.7)
\end{aligned}$$

In view of (5.1), taking expectations on both sides of (5.7), we obtain

$$E\left[e^{-\delta(t\wedge\tau)}h(X^\pi_{t\wedge\tau})\right] - h(x) \leq -\beta_1 E_x\Big[\int_0^{t\wedge\tau} e^{-\delta s}dD(s)\Big] + \beta_2 E_x\Big[\int_0^{t\wedge\tau} e^{-\delta s}dZ(s)\Big]. \tag{5.8}$$

By the definition of τ and the boundary condition (5.2), we can prove

$$\liminf_{t\to\infty} E[e^{-\delta(t\wedge\tau)}h(X^\pi_{t\wedge\tau})] = e^{-\delta\tau}h(0)I_{(\tau<\infty)} + \liminf_{t\to\infty} E[e^{-\delta t}h(X^\pi_t)I_{(\tau=\infty)}] \geq 0,$$

together with (5.8), which implies

$$h(x) \geq E_x[\beta_1\int_{0_-}^{\tau} e^{-\delta t}dD(t) - \beta_2\int_{0_-}^{\tau} e^{-\delta t}dZ(t)] = V^{(\pi)}(x), \text{ for any } \pi \in \varphi(x).$$

Therefore, we get $h(x) \geq V(x)$. ∎

The following theorem is the main result of this paper.

Theorem 5.1. $V(x) = V_c(x)$ *and the optimal strategies are the same as those in* $V_c(x)$.

Proof. On the one hand, since $V_c(0) \geq 0$, which is proved in Proposition 4.1, together with Proposition 4.3, which implies that $V_c(x)$ satisfies the equations (5.1)-(5.2), then by Lemma 5.1 we get

$$V_c(x) \geq V(x).$$

On the other hand, from (4.2), we have the inverse inequality $V_c(x) \leq V(x)$. So the conclusion holds. ∎

Remark 5.1. In Løkka and Zervos [13] in which there is no reinsurance, whether the company should inject capitals when there is deficit depends on the relationships between the parameters in the model. They have either $V(x) = V_c(x)$ or $V(x) = V_d(x)$ under different parameter conditions, see (5.3) and (5.4) in that paper, in which $V_d(x)$ corresponds to the model of no capital injection and it is defined in the following way:

$$V_d(x) := \sup_{(a,D,0)\in\varphi(x)} V^{(a,D,0)}(x)),$$

that is, $V_c(x)$ may have better or worse performance than $V_d(x)$ under different conditions. It is concluded in that paper that the company should not inject capitals when the costs of collecting capitals are relatively high, otherwise it should inject capitals to guarantee no bankruptcy. However, from Theorem 5.1 we find that once we add cheap excess-of-loss reinsurance in the model, $V(x) = V_c(x)$ whatever relationship the model parameters have.

Appendix

Suppose there exists an increasing and concave function $g(x) \in C^2$ to the HJB equation (4.7) with the boundary condition (4.8), then (4.9)-(4.11) in Proposition 4.3 holds, and the function $g(x)$ can be conjectured as follows:

For $x \leq b_1$, Since $g''(x) < 0$ and $g'(x) > 0$, by differentiation we find the maximum $a^*(x)$ satisfying

$$a^*(x) = -\theta \frac{g'(x)}{g''(x)}. \tag{5.9}$$

Substituting (5.9) into (4.9) yields

$$[-\theta \frac{\sigma^2(a^*)}{2a^*} + \theta\mu(a^*)]g'(x) - \delta g(x) = 0 \tag{5.10}$$

with $a^* = a^*(x)$. Differentiating w.r.t. x in (5.10) and then using (5.9), eventually we get

$$(a^*(x))' = \frac{2a^*}{\sigma^2(a^*)}\left[-\theta\frac{\sigma^2(a^*)}{2a^*} + \theta\mu(a^*) + \frac{\delta}{\theta}a^*\right] > 0. \tag{5.11}$$

Then we can deduce that $a^*(x)$ is an increasing function and

$$a^*(x) = H^{-1}\Big(x + H(a^*(0))\Big), \tag{5.12}$$

where the initial value $a^*(0)$ is to be determined. Suppose there exists $x_1 \in [0, b_1]$ such that $a^*(x_1) = A$, which implies that $a^*(x) \leq A$ for all $x \leq x_1$ and

$$x_1 = H(A) - H(a^*(0)), \tag{5.13}$$

From (4.11) and (5.9), we conclude that the solution of $g(x)$ on $[0,x_1]$ takes the form of

$$g(x) = c_0 + \beta_2 \int_0^x e^{-\theta \int_0^z \frac{1}{a^*(y)} dy} dz,\ x \in [0,x_1]$$

with $c_0 = g(0)$. Taking $x = 0$ in (5.10), we get

$$c_0 = \frac{\theta\beta_2}{\delta}\Big[\mu(a^*(0)) - \frac{\sigma^2(a^*(0))}{2a^*(0)}\Big]. \tag{5.14}$$

Suppose $a^*(x) = A$ for $x \in [x_1,b_1]$, then from (4.9) we deduce

$$g(x) = c_1 e^{d_+x} + c_2 e^{d_-x},\ x \in [x_1,b_1], \tag{5.15}$$

where c_1, c_2 can be obtained as follows by the continuity of $g'(x)$, $g''(x)$ at $x = b_1$

$$c_1 = \frac{-\beta_1 d_-}{d_+(d_+ - d_-)} e^{-d_+b_1} > 0, \qquad c_2 = \frac{\beta_1 d_+}{d_-(d_+ - d_-)} e^{-d_-b_1} < 0. \tag{5.16}$$

Using (5.9), (5.15) and the continuity of $g'(x)$, $g''(x)$ at $x = x_1$, we get

$$\frac{c_1 d_+ e^{d_+x_1} + c_2 d_- e^{d_-x_1}}{c_1 d_+^2 e^{d_+x_1} + c_2 d_-^2 e^{d_-x_1}} = \frac{g'(x_1)}{g''(x_1)} = -\frac{a^*(x_1)}{\theta} = -\frac{A}{\theta},$$

together with (5.16), which implies

$$e^{(d_+ - d_-)(x_1 - b_1)} = \frac{d_+(\theta + d_-A)}{d_-(\theta + d_+A)} \in (0,1),$$

Then it follows that

$$b_1 = x_1 + \frac{1}{d_+ - d_-} \ln \frac{d_-(\theta_2 + d_+A)}{d_+(\theta_2 + d_-A)} = x_1 + m^*, \tag{5.17}$$

which confirms the supposition $x_1 \le b_1$.

To summarize, we can construct $g(x)$ as follows:

$$g(x) = \begin{cases} c_0 + \beta_2 \int_0^x e^{-\theta \int_0^z \frac{1}{a^*(y)} dy} dz & \text{if } 0 \le x \le x_1, \\ c_1 e^{d_+x} + c_2 e^{d_-x} & \text{if } x_1 \le x \le b_1, \\ \beta_1(x - b_1) + c_3 & \text{if } x \ge b_1, \end{cases} \tag{5.18}$$

where

$$c_3 = \frac{\theta\mu_\infty\beta_1}{\delta} \tag{5.19}$$

is obtained from (4.9) and the continuity of $g'(x)$ and $g''(x)$ at b_1.

From the statements above, all the parameters will be determined if $a^*(0)$ is worked out, which satisfies the following by the continuity of $g'(x)$ at $x_1 = H(A) - H(a^*(0)$:

$$\beta_2 e^{-\int_0^{H(A)-H(a^*(0))} \frac{\theta}{H^{-1}(y+H(a^*(0)))} dy} = \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_-m^*}. \tag{5.20}$$

From Lemma 4.2, we deduce that if and only if the model parameters satisfy (4.19), the equation (5.20) has a unique solution $a^*(0) \in [0,A]$. So the function $g(x)$ will be constructed under two different cases depending on the model parameters:

(i) In the case of

$$\beta_2 \geq \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*},$$

we have constructed the function $g(x)$ in the form of (5.18) with all the unknown parameters determined in the statements above.

(ii) In the case of

$$\beta_2 < \frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*},$$

from Lemma 4.2 we can see that there exists no solution $a^*(0) \in [0, A]$ to the equation (5.20). It is obvious that when β_2 decreases to $\frac{-\beta_1 d_-}{d_+ - d_-} e^{d_+ m^*} + \frac{\beta_1 d_+}{d_+ - d_-} e^{d_- m^*}$, the solution $a^*(0)$ increases to A, so we can suppose $a^*(x) \equiv A$ for all $x \geq 0$ in this case. Then the equations (4.9)-(4.11) become

$$\begin{cases} \frac{1}{2}\sigma_\infty^2 g''(x) + \theta\mu_\infty g'(x) - \delta g(x) = 0, \quad x \in [0, b_1], \\ g(x) = \beta_1(x - b_1) + g(b_1), \quad x \geq b_1, \\ g'(0) = \beta_2, \end{cases}$$

which is actually the case in Løkka and Zervos [13] (involving no reinsurance) and the function $g(x)$ has the following form:

(5.21)
$$g(x) = \begin{cases} F_1 e^{d_+ x} + F_2 e^{d_- x} & \text{if } 0 \leq x \leq b_2, \\ \beta_1(x - b_2) + V_c(b_2) & \text{if } x \geq b_2, \end{cases}$$

where

(5.22)
$$F_1 = \frac{-\beta_1 d_-}{d_+(d_+ - d_-)} e^{-d_+ b_2} > 0, \qquad F_2 = \frac{\beta_1 d_+}{d_-(d_+ - d_-)} e^{-d_- b_2} < 0$$

and b_2 is the unique solution to the following equation on $x \in [0, \infty)$

(5.23)
$$\beta_1(d_+ e^{-d_- x} - d_- e^{-d^+ x}) = \beta_2(d_+ - d_-).$$

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