BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(1) (2013), 211–219*

Unbounded Weighted Radon Measures and Dual of Certain Function Spaces with Strict Topology

[1]S. MAGHSOUDI AND [2]A. REJALI
[1]Department of Mathematics, University of Zanjan, Zanjan, 45195-313, Iran
[2]Department of Mathematics, University of Isfahan, Isfahan, Iran
[1]s_maghsodi@znu.ac.ir, [2]rejali@sci.ui.ac.ir

Abstract. Let X be a C-distinguished topological space, and let ω be a weight function on X. Denote by $C_b(X,\omega)$ the space of all real-valued functions f with $f/\omega \in C_b(X)$, and by $\widetilde{C}_b(X,\omega)$ the space of all real-valued continuous functions f such that f/ω is bounded. We introduce certain locally convex topologies on $C_b(X,\omega)$ and $\widetilde{C}_b(X,\omega)$, and as our main results we determine their duals.

2010 Mathematics Subject Classification: Primary 46E10, 46E27 Secondary: 46A03, 46A40

Keywords and phrases: C-distinguished space, strict topology, strong dual, weighted function space, weighted Radon measure.

1. Introduction

Let X be a C-distinguished topological space; that is, a Hausdorff topological space X such that the space of real-valued bounded continuous functions on X, $C_b(X)$, separates points of X. By a weight function on X we mean a Borel measurable function $\omega : X \to (0,\infty)$ such that ω^{-1} is bounded on the compact subsets of X. Let $C_b(X,\omega)$ be the space of all real-valued functions f with $f/\omega \in C_b(X)$, and let $\widetilde{C}_b(X,\omega)$ be the space of all real-valued continuous functions f such that f/ω is bounded. It is easy to see that, with the usual pointwise operations and the norm $\|.\|_\omega$ defined by $\|f\|_\omega := \|f/\omega\|$ the spaces $C_b(X,\omega)$ and $\widetilde{C}_b(X,\omega)$ are Banach spaces. In the special case $\omega = 1$ the space $C_b(X,\omega)$ is the usual space $C_b(X)$.

In his seminal work, Buck [5] defined the strict topology β on $C_b(X)$ for a locally compact space X. Later, Giles [6] extended the result of Buck to the completely regular spaces. The strict topology β is the locally convex topology defined by the seminorms $\mathscr{P}_\varphi$, where

$$\mathscr{P}_\varphi(g) = \sup\{\, \varphi(x)\,|g(x)| : x \in X \,\} \quad (g \in C_b(X)),$$

and φ varies through the set of all positive bounded Borel measurable functions vanishing at infinity. An interesting and important property of the strict topology is that the dual space of $(C_b(X),\beta)$ can be identified with the space of all finite regular Borel measures on X; see [6]

Communicated by Rosihan M. Ali, Dato'.
Received: December 3, 2010; *Revised:* March 10, 2011.

and [7] for more details. Many authors have studied the so-called strict topologies; see for example [1, 3, 8–10, 14–16].

Our aim in this paper is to introduce and study locally convex topologies $\beta(X,\omega)$ and $\widetilde{\beta}(X,\omega)$ on $C_b(X,\omega)$ and $\widetilde{C}_b(X,\omega)$, respectively. We will show that the dual of these spaces can be identified in a natural way with a Banach space of weighted Radon measures on X.

2. Space of weighted Radon measures

Let us recall some basic concepts and results from measure theory as given in [4]. The σ-algebra generated by the open subsets of X is called the σ-algebra of Borel sets and denoted $\mathscr{B}=\mathscr{B}(X)$. A (positive) Radon measure is a Borel measure $\mu:\mathscr{B}(X)\longrightarrow[0,\infty]$ such that $\mu(C)$ is finite for $C\in\mathscr{K}(X)$, and μ is inner-regular; that is,

$$\mu(B)=\sup\{\mu(C):C\subseteq B, C\in\mathscr{K}(X)\}\quad (B\in\mathscr{B}(X)),$$

where $\mathscr{K}=\mathscr{K}(X)$ denotes the family of all compact subsets of X. The set of all positive Radon measures on X is denoted by $M^+(X)$. By $M_b^+(X)$ we denote the set of all positive bounded Radon measures on X.

We need the following simple lemma.

Lemma 2.1. *Let $\varphi:X\to[0,\infty]$ be a Borel measurable function on the Hausdorff space X such that is bounded on the compact subsets of X. If $\mu\in M^+(X)$ then $\mu\varphi\in M^+(X)$, where*

$$\mu\varphi(B)=\int_B\varphi\,d\mu\quad (B\in\mathscr{B}).$$

Proof. It is clear that $\mu\varphi$ is a positive measure which is bounded on compact subsets. For each compact subset K of X, let $\mu_K=\mu\chi_K$. It is clear that μ_K is a Radon measure. Directing the family $\mathscr{K}$ by inclusion, we get $\mu_K\nearrow\mu$. Hence, by [4, Exercise 1.29],

$$\begin{aligned}\sup\left\{\int_K\varphi\,d\mu:K\subseteq B, K\in\mathscr{K}\right\}&=\sup\left\{\int_K\varphi\,d\mu_K:K\subseteq B, K\in\mathscr{K}\right\}\\&=\int_X\varphi\,d(\mu\chi_B)=\int_B\varphi\,d\mu,\end{aligned}$$

for each Borel subset B. This proves the lemma. ∎

The following lemma is an easy application of Lemma 2.1.

Lemma 2.2. *Let X be a Hausdorff space, $\varphi:X\to(0,\infty)$ be a Borel measurable function, and μ be a positive Borel measure. Then*

(i) $(\mu\varphi)\varphi^{-1}=(\mu\varphi^{-1})\varphi=\mu$.

(ii) *If φ^{-1} is bounded on the compact subsets and $\mu\varphi\in M^+(X)$, then $\mu\in M^+(X)$.*

Let X be a Hausdorff space, and let $\omega:X\longrightarrow(0,\infty)$ be a weight function on X. Let $M_b^+(X,\omega)$ be the set of all positive Radon measures μ on X such that $\mu\omega\in M_b^+(X)$. Define the equivalence relation '$\sim$' on $M_b^+(X,\omega)\times M_b^+(X,\omega)$ by

$$(\mu,\nu)\sim(\mu',\nu')\quad\text{if and only if}\quad \mu+\nu'=\mu'+\nu.$$

Let $[\mu,\nu]$ be the equivalence class of $(\mu,\nu)\in M_b^+(X,\omega)\times M_b^+(X,\omega)$. Now we define

$$M_b(X,\omega)=\left\{[\mu,\nu]:\mu,\nu\in M_b^+(X,\omega)\right\}.$$

Then $M_b(X,\omega)$ with the operations defined by

$$[\mu,\nu]+[\mu',\nu']=[\mu+\mu',\nu+\nu']$$

and

$$\lambda[\mu,\nu]=\begin{cases}[\lambda\mu,\lambda\nu] & \text{if } \lambda\geq 0,\\ [\lambda\nu,\lambda\mu] & \text{otherwise,}\end{cases}$$

for $[\mu,\nu],[\mu',\nu']\in M(X)$ and $\lambda\in\mathbb{R}$, the norm $\|.\|_\omega$ defined by

$$\|[\mu,\nu]\|_\omega=\|\mu\omega-\nu\omega\|,$$

for $[\mu,\nu]\in M_b(X,\omega)$, is a real normed space. Recall that $\|.\|$ denotes the usual norm of a bounded Radon measure.

The following proposition states the elementary properties of $M_b^+(X,\omega)$.

Proposition 2.1. *Let X be a Hausdorff space, and let $\omega: X\longrightarrow(0,\infty)$ be a weight function on X. Then*

(i) *For each $\eta\in M_b(X,\omega)$ there exist unique measures $\eta^+,\eta^-\in M_b^+(X,\omega)$ such that $\eta=[\eta^+,\eta^-]$ and $\eta^+\perp\eta^-$.*

(ii) *$(M_b(X,\omega),\|.\|_\omega)$ is a Banach space.*

Proof. To prove (i), let $\eta=[\mu,\nu]$ for some $\mu,\nu\in M_b^+(X,\omega)$. Then $\theta:=\mu\omega-\nu\omega\in M_b(X)$. By the Hahn decomposition theorem, there exist unique measures $\theta^+,\theta^-\in M_b^+(X)$ such that $\theta=\theta^+-\theta^-$ and $\theta^+\perp\theta^-$. Now set $\eta^+=\theta^+\omega^{-1}$ and $\eta^-=\theta^-\omega^{-1}$. Thus $\mu+\eta^-=\nu+\eta^+$, and so $\eta=[\eta^+,\eta^-]$.

(ii) That $(M_b(X,\omega),\|.\|_\omega)$ is a normed space is straightforward. Now, we prove that $(M_b(X,\omega),\|.\|_\omega)$ is complete. To prove this, let $(\eta_n)_n$ be a Cauchy sequence in $M_b(X,\omega)$ and note that $\|\eta_n\|_\omega=\|(\mu_n\omega-\nu_n\omega)^+\|+\|(\mu_n\omega-\nu_n\omega)^-\|$. Then, it is clear that $(\mu_n\omega-\nu_n\omega)_n$ is a Cauchy sequence in $M_b(X)$, and so converges to, say θ, in $M_b(X)$. Put $\eta_1=\theta^+\omega^{-1}$, $\eta_2=\theta^-\omega^{-1}$ and $\eta=[\eta_1,\eta_2]$. Then $\eta_n\to\eta$; this follows from the fact that

$$\|\eta_n-\eta\|_\omega=\|(\eta_n^+\omega-\eta_2\omega)-(\mu_n\omega-\eta_1\omega)^-\|\leq\|\eta_n^+\omega-\theta^+\|+\|\eta_n^-\omega-\theta^-\|. \quad ∎$$

For $\eta\in M_b(X,\omega)$, by the standard representation of η we mean the expression $[\eta^+,\eta^-]$ stated in Proposition 2.1.

3. The dual space of $C_b(X,\omega)$ with $\beta(X,\omega)$ topology

We call a function $\varphi: X\longrightarrow[0,\infty)$ an ω-hood, if $\omega\varphi$ is a bounded Borel measurable function such that for each $\varepsilon>0$ there exists a compact subset $K\subseteq X$ such that $\varphi(x)<\varepsilon\omega^{-1}(x)$ for all $x\in X\setminus K$. The set of all ω-hoods on X is denoted by $Hd(X,\omega)$. Then, ω-strict topology $\beta(X,\omega)$ on $C_b(X,\omega)$ is the locally convex topology generated by the seminorms $\{P_\varphi:\varphi\in Hd(X,\omega)\}$, where

$$P_\varphi(g)=\sup\{\varphi(x)|g(x)|: x\in X\}$$

for $g\in C_b(X,\omega)$. In the case $\omega=1$, $\beta(X,\omega)$ coincides with the strict topology on $C_b(X)$ defined in [5] and [6]. Note that $\beta(X,\omega)$ topology is weaker than $\|.\|_\omega$ topology on $C_b(X,\omega)$. So, $(C_b(X,\omega),\beta(X,\omega))^*$ by the norm

$$\|F\|=\sup\{|F(g)|: g\in C_b(X,\omega),\ ||g||_\omega\leq 1\},$$

is a normed space. The following result is a generalization of [7, Theorem 4.6] to the more general setting of the weighted space of functions.

Proposition 3.1. *Let X be a C-distinguished space, and let $\omega : X \longrightarrow (0,\infty)$ be a weight function on X. Then the map $\eta \mapsto I_{\eta^+} - I_{\eta^-}$ from $M_b(X,\omega)$ onto $(C_b(X,\omega),\beta(X,\omega))^*$ is an isometric isomorphism, where $\eta = [\eta^+,\eta^-]$ is the standard representation of η and*

$$I_\mu(f) = \int_X f\,d\mu$$

for $f \in C_b(X,\omega)$ and $\mu \in M_b^+(X,\omega)$.

Proof. Define $T_\omega : (C_b(X),\beta)^* \to (C_b(X,\omega),\beta(X,\omega))^*$ by $T_\omega(g) = g\omega$, where $(g\omega)(\varphi) = g(\omega\varphi)$ for all $\varphi \in C_b(X)$. It is easy to see that T_ω is a well-defined linear isometric isomorphism. Also, the map $\eta \mapsto \eta^+\omega - \eta^-\omega$ is a linear isometric isomorphism from $M_b(X,\omega)$ onto $M_b(X)$, where $\eta = [\eta^+,\eta^-] \in M_b(X,\omega)$. Now we only need to recall from [7, Theorem 4.6] that $(C_b(X),\beta(X,1))^*$ is isometrically isomorphism to $M_b(X)$. ∎

The following example shows that the elements of $(C_b(X,\omega),\beta(X,\omega))^*$ cannot, in general, be represented by a signed measure on X.

Example 3.1. Let m be the Lebesgue measure on $\mathbb{R}$, and let $\mu = m\chi_{[0,\infty)}$, $\nu = m\chi_{(-\infty,0)}$ and $\eta = [\mu,\nu]$. Also, let $\omega : \mathbb{R} \to (0,\infty)$ be defined by

$$\omega(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \\ 1/x^2 & \text{otherwise.} \end{cases}$$

Define $I(f) = \int_{\mathbb{R}} f\,d\mu - \int_{\mathbb{R}} f\,d\nu$ for all $f \in C_b(\mathbb{R},\omega)$. Then $\eta \in M_b(\mathbb{R},\omega)$ and $I \in (C_b(\mathbb{R},\omega),\beta(\mathbb{R},\omega))^*$, but there is no signed measure ζ on $\mathbb{R}$ such that $I(f) = \int_{\mathbb{R}} f\,d\zeta$ for all $f \in C_b(\mathbb{R},\omega)$.

4. The dual space of $\widetilde{C}_b(X,\omega)$ with $\widetilde{\beta}(X,\omega)$ topology

In this section we define locally convex topology $\widetilde{\beta}(X,\omega)$ on the space $\widetilde{C}_b(X,\omega)$ and determine its dual space. Let X be a Hausdorff space and ω be a weight function on X. Recall that $\widetilde{C}_b(X,\omega)$ is the set of all real-valued continuous functions f on X such that f/ω is bounded and $\|f\|_\omega = \|f/\omega\|$ for $f \in \widetilde{C}_b(X,\omega)$. The set of all positive functions in $\widetilde{C}_b(X,\omega)$ is denoted by $\widetilde{C}_b^+(X,\omega)$. We use the usual notation $f \wedge g = \min\{f,g\}$ for real-valued functions f and g. We start with the following lemma which will be needed in the sequel.

Lemma 4.1. *Let either X be completely regular and ω be lower semi-continuous, or X be locally compact and ω^{-1} be bounded on each compact subset of X. Then the following hold:*

(i) *If $x \neq y$, then there is a function $f \in \widetilde{C}_b^+(X,\omega)$ such that $f(x) = 0$ and $f(y) = 1$.*
(ii) *For each compact subset K, there is a $g \in \widetilde{C}_b^+(X,\omega)$ such that $g \geq \chi_K$.*

Proof. (i) Suppose first that ω is lower semi-continuous, so it is clear that $\omega = \sup\{f \in C_b^+(X) : f \leq \omega\}$. Thus, there is an $f_0 \in C_b^+(X)$ such that $f_0(y) > 0$ and $f_0/\omega \leq 1$. Also there is $k \in C_b^+(X)$ such that $k(x) = 0$ and $k(y) = 1$. Put $f = k \wedge 1/(f_0(y))f_0$, whence f satisfies (i).

Next assume that X is locally compact and ω^{-1} is locally bounded, then $C_c(X) \subseteq \widetilde{C}_b^+(X,\omega)$, where $C_c(X)$ denotes the space of all real-valued continuous functions with compact support. In this case there is an $f_0 \in C_c(X)$ such that $f_0(y) = 0$ and $f_0/\omega \leq 1$. This completes the proof of (i).

(ii) Let $y \in K$, then there is $f_y \in \widetilde{C}_b^+(X,\omega)$ such that $f_y(y) > \omega(y)/2$. If $g_y := 2/(f_y(y))f_y \in \widetilde{C}_b^+(X,\omega)$ and $U_y := \{x \in X : g_y(x) > 1\}$, then $K \subseteq \bigcup_{i=1}^n U_{y_i}$, say, for some $y_i \in X$. Put $g := \sum_{i=1}^n g_{y_i}$, so clearly g satisfies (ii). ∎

The locally convex topology on $\widetilde{C}_b(X,\omega)$ generated by the seminorms $\mathscr{P}_\varphi$, where

$$\mathscr{P}_\varphi(g) = \{\,\varphi(x)|g(x)| : x \in X\,\}$$

and $\varphi \in Hd(X,\omega)$, is denoted by $\widetilde{\beta}(X,\omega)$. The following result shows that any positive $\widetilde{\beta}(X,\omega)$-continuous functional on $\widetilde{C}_b(X,\omega)$ can be represented by a locally bounded Radon measure.

Proposition 4.1. *Let either X be completely regular and ω be lower semi-continuous, or X be locally compact and ω^{-1} be bounded on each compact subset of X. Furthermore, suppose that $I \in (\widetilde{C}_b(X,\omega),\widetilde{\beta}(X,\omega))^*$ be a positive functional. Then there exists a unique measure $\mu \in M^+(X)$ such that $I(g) = \int_X g\,d\mu$ for $g \in \widetilde{C}_b^+(X,\omega)$. Moreover, μ is locally bounded and*

$$\mu(K) = \inf\left\{I(g) : g \in \widetilde{C}_b(X,\omega),\, g \geq \chi_K\right\}$$

for all compact subsets K.

Proof. Let I be a positive functional in $(\widetilde{C}_b(X,\omega),\widetilde{\beta}(X,\omega))^*$. Then, by [11, Theorem 3], we only required to show that the following conditions satisfy:

(i) $I(g) = \sup\{I(g \wedge n) : n \in \mathbb{N}\}$ for all $g \in \widetilde{C}_b(X,\omega)$.

(ii) If $(K_\alpha)_\alpha$ is a decreasing net of compact subsets such that $\cap_\alpha K_\alpha = \emptyset$, then

$$\inf\{I(h) : h \in \widetilde{C}_b(X,\omega),\, h \geq \chi_{K_\alpha} \text{ for some } \alpha\} = 0.$$

(iii) The family $\mathscr{K}(X)$ exhausts I.

To prove (i), first note that if $(h_\alpha)_\alpha$ is a decreasing net in $\widetilde{C}_b^+(X,\omega)$ such that converges pointwise to zero with $\|h_\alpha\| < M$, for some positive number M, then $I(h_\alpha) \to 0$. This is because that given $\varepsilon > 0$ and $\varphi \in Hd(X,\omega)$, we can choose compact subset K so that $|(\varphi\omega)(x)| < \varepsilon/M$ for all $x \in X \setminus K$. By the Dini theorem $h_\alpha \to 0$ uniformly on compact set K. Thus there is α_0 such that

$$\|h_\alpha\| \leq \frac{\varepsilon}{\|\varphi\omega\|\,\|\omega^{-1}\|_K}$$

for $\alpha \geq \alpha_0$, and hence $\|\varphi h_\alpha\| \leq \varepsilon$ for $\alpha \geq \alpha_0$. It follows that $h_\alpha \to 0$ in the $\widetilde{\beta}(X,\omega)$ topology. Using the fact that I is $\widetilde{\beta}(X,\omega)$-continuous, we conclude that $I(h_\alpha) \to 0$. Now, for any $g \in \widetilde{C}_b^+(X,\omega)$, note that the decreasing sequence $(g - g \wedge n)_n$ converges pointwise to zero and $\|g - g \wedge n\|_\omega \leq \|g\|_\omega$. Thus $I(g - g \wedge n) \to 0$, and we find that

$$I(g) = \sup_{n \in \mathbb{N}} I(g \wedge n).$$

For proving (ii) it is only need to observe that for any decreasing net $(K_\alpha)_\alpha$ of compact subsets with $\cap_\alpha K_\alpha = \emptyset$, there is α_0 such that $K_{\alpha_0} = \emptyset$.

Finally, we show that (iii) holds. Let $I \in (\widetilde{C}_b(X,\omega),\widetilde{\beta}(X,\omega))^*$. The continuity of I implies that there is $\varphi \in Hd(X,\omega)$ such that $|I(g)| \leq \mathscr{P}_\varphi(g)$ for all $g \in \widetilde{C}_b(X,\omega)$. Given $\varepsilon > 0$ and $f \in \widetilde{C}_b^+(X,\omega)$. Then there is $K \in \mathscr{K}$ such that

$$(\varphi\omega)(x) < \varepsilon/(||f||_\omega + 1)$$

for $x \in X \setminus K$. Suppose that $g \in \widetilde{C}_b^+(X,\omega)$ with $g \leq f$ and $g = 0$ on compact set K. We obtain

$$|I(g)| \leq \sup\{\varphi(x)|g(x)| : x \in X \setminus K\} = \sup\left\{(\varphi\omega)(x)|(g\omega^{-1})(x)| : x \in X \setminus K\right\}$$
$$\leq \varepsilon/(||f||_\omega + 1)\sup\left\{|\frac{g}{\omega}(x)| \, : \, x \in X \setminus K\right\} \leq \varepsilon.$$

This shows that $\mathscr{K}$ exhausts I. Now the assertions of the proposition easily follow. ∎

Proposition 4.2. *Let X be a C-distinguished space, and let ω be a weight function on X. Then a subset of $\widetilde{C}_b(X,\omega)$ is $\|.\|_\omega$-bounded if and only if it is $\widetilde{\beta}(X,\omega)$-bounded.*

Proof. Let B be a $\widetilde{\beta}(X,\omega)$-bounded set in $\widetilde{C}_b(X,\omega)$, and suppose that B is not $\|.\|_\omega$-bounded. Then there is a sequence $(g_n) \subseteq B$ such that $\|g_n\|_\omega > n$ for all $n \geq 1$. For each $n \geq 1$, choose x_n in X such that $g(x_n) \geq n\,\omega(x_n)$. Let

$$\varphi = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{g_n(x_n)} \chi_{x_n}$$

and note that $\varphi \in Hd(X,\omega)$. Since B is $\widetilde{\beta}(X,\omega)$-bounded, there is a constant $s > 0$ such that $B \subseteq s\,\{g : P_\varphi(g) < 1\}$. We therefore have

$$\sqrt{n} \leq \|\varphi\, g_n\| < \frac{1}{s}$$

which is a contradiction. The converse is clear. ∎

Let us recall that the strong topology on $(\widetilde{C}_b(X,\omega), \widetilde{\beta}(X,\omega))^*$ is the topology of uniform convergence on the bounded subsets of $\widetilde{C}_b(X,\omega)$ with respect to the weak topology $\sigma(\widetilde{C}_b(X,\omega), (\widetilde{C}_b(X,\omega), \widetilde{\beta}(X,\omega))^*)$. By the norm topology on $(\widetilde{C}_b(X,\omega), \widetilde{\beta}(X,\omega))^*$ we mean the topology given on $(\widetilde{C}_b(X,\omega), \widetilde{\beta}(X,\omega))^*$ by the norm

$$\|F\| = \sup\{|F(g)| : g \in C_b(X,\omega),\ ||g||_\omega \leq 1\}.$$

As an immediate consequence of Proposition 4.2 we obtain the following result.

Corollary 4.1. *Let X be a C-distinguished space, and let ω be a weight function on X. Then the strong topology and the norm topology are equivalent on $(\widetilde{C}_b(X,\omega), \widetilde{\beta}(X,\omega))^*$.*

We need the following lemma to prove our main result.

Lemma 4.2. *Let X be a completely regular space, and let $\omega : X \to (0,\infty)$ be a lower semicontinuous function on X. For any $\eta = [\eta^+, \eta^-] \in M_b(X,\omega)$ the functional Φ defined as $\Phi(\eta)(g) = \int_X g\,d\eta^+ - \int_X g\,d\eta^-$, is $\widetilde{\beta}(X,\omega)$-continuous on $\widetilde{C}_b(X,\omega)$ and $\|\Phi\| = \|\eta\|_\omega$*

Proof. Let us first show that the linear map $\Phi : M_b(X,\omega) \to (\widetilde{C}_b(X,\omega), \widetilde{\beta}(X,\omega))^*$ is well-defined. For this end, suppose that $\mu \in M^+(X,\omega)$ be arbitrary. Then $\mu\omega \in M_b^+(X)$, so that there is an increasing sequence $(K_n)_n$ of compact subsets such that $\mu\omega(X) = \mu\omega(K_0)$, where $K_0 = \bigcup_{n=1}^{\infty} K_n$ and $\mu\omega(K_{n+1} \setminus K_n) < 2^{-n}$ for each natural number n. Put $\phi_0 := \sum_{n=1}^{\infty} 2^{-n}\chi_{K_{n+1}\setminus K_n}$. Then

$$\int_X \varphi_0^{-1}\, d(\mu\omega) = \sum_{n=1}^{\infty} 2^{-n}(\mu\omega)(K_{n+1} \setminus K_n) < \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Setting $\phi := \phi_0\omega^{-1}$, then $\varphi \in Hd(X,\omega)$ and , for each $g \in \widetilde{C}_b(X,\omega)$, we obtain

$$\int_X g\,d\mu = \int_{K_0} \frac{g}{\omega}\,d(\mu\omega) = \int_{K_0} (g\varphi)\varphi_0^{-1}\,d(\mu\omega) \leq \mathscr{P}_\varphi(g)\int_X \varphi_0^{-1}\,d(\mu\omega) \leq \mathscr{P}_\varphi(g).$$

From the above, we deduce that I_μ is in $(\widetilde{C}_b(X,\omega),\widetilde{\beta}(X,\omega))^*$, which means Φ is well-defined.

It remains to show that Φ is isometry. Let us first assume that μ be a positive measure in $M_b(X,\omega)$. On the one hand, we have

$$\begin{aligned}\|\mu\|_\omega = \int_X \omega\,d\mu &= \sup\left\{\int_X f\,d\mu : f \leq \omega, f \in C_b^+(X)\right\}\\ &\leq \sup\left\{|\int_X g\,d\mu| : \|g\|_\omega \leq 1, g \in \widetilde{C}_b(X,\omega)\right\} = \|I_\mu\|.\end{aligned}$$

On the other hand, for all $g \in \widetilde{C}_b(X,\omega)$,

$$|I_\mu(g)| = \left|\int_X g\,d\mu\right| \leq \|g\|_\omega\|\mu\|_\omega,$$

and therefore, $\|I_\mu\| \leq \|\mu\|_\omega$. Thus we find $\|I_\mu\| = \|\mu\|_\omega$. Now suppose that $\eta = [\eta^+,\eta^-] \in M_b(X,\omega)$ and noting that $I_{\eta^+} \wedge I_{\eta^-} = 0$. Thus

$$\|I_{\eta^+} - I_{\eta^-}\| = \|I_{\eta^+}\| + \|I_{\eta^-}\| = \|\eta^+\|_\omega + \|\eta^-\|_\omega = \|\eta\|_\omega,$$

which completes the proof. ∎

In the sequel we need the following lemma. For a convenient account of locally convex vector lattices, see [1].

Lemma 4.3. *Let V be a locally convex topological vector lattice of the real-valued functions on X. Then for any $I \in V^*$ there exist unique positive functionals $I^+, I^- \in V^*$ such that $I = I^+ - I^-$ and $I^+ \wedge I^- = 0$.*

Proof. By hypothesis there is a family of seminorms $\{\mathscr{P}_\alpha : \alpha \in \Lambda\}$ that generates the topology on V such that if $|f| \leq |g|$, for $f,g \in V$, then $\mathscr{P}_\alpha(f) \leq \mathscr{P}_\alpha(g)$ for all $\alpha \in \Lambda$; see [2, Theorem 6.1]. Also, by [15, Section 3.1], there exist $\alpha_i,\ 1 \leq i \leq n$, and $\lambda \in \mathbb{R}^+$ such that

$$|I(g)| \leq \lambda \sup\{\mathscr{P}_{\alpha_i}(g) : 1 \leq i \leq n\}$$

for $g \in V$. Thus for each positive function $f \in V$ the set $\{|I(g)| : g \in V, |g| \leq f\}$ is bounded in $\mathbb{R}$. Now, by applying [13, Corollary 2.4.2], the result is immediate. ∎

We are now ready to prove the main theorem of this section.

Theorem 4.1. *Let X be a completely regular space, and let $\omega : X \to (0,\infty)$ be a lower semi-continuous function on X. Then the map $\Phi : M_b(X,\omega) \to (\widetilde{C}_b(X,\omega),\widetilde{\beta}(X,\omega))^*$, defined as $\Phi(\eta) = I_{\eta^+} - I_{\eta^-}$, is an isometric isomorphism, where*

$$I_\mu(g) = \int_X g\,d\mu$$

for $g \in \widetilde{C}_b(X,\omega)$ and $\mu \in M^+(X)$.

Proof. In view of Lemma 4.2, we only required to show that Φ is bijective. To verifying that Φ is surjective, suppose that $I \in (\widetilde{C}_b(X,\omega), \widetilde{\beta}(X,\omega))^*$ be an arbitrary element. By Lemma 4.3, we can write $I = I^+ - I^-$, where I^+ and I^- are positive functionals in $(\widetilde{C}_b(X,\omega), \widetilde{\beta}(X,\omega))^*$ such that $I^+ \wedge I^- = 0$. Also Theorem 4.1 implies that for some measures $\mu_1, \mu_2 \in M^+(X)$,

$$I_{\mu_1}(g) = \int_X g\, d\mu_1, \quad I_{\mu_2}(g) = \int_X g\, d\mu_2$$

for $g \in \widetilde{C}_b(X,\omega)$. Recalling that I^+ and I^- are $\widetilde{\beta}(X,\omega)$-continuous, we can find $\varphi_1, \varphi_2 \in Hd(X,\omega)$ such that

$$|I_{\mu_1}(g)| \leq \mathscr{P}_{\varphi_1}(g), \quad |I_{\mu_2}(g)| \leq \mathscr{P}_{\varphi_2}(g)$$

for $g \in \widetilde{C}_b(X,\omega)$. Using the lower semi-continuity of ω, it can be verified that $\omega = \sup\{f : f \in C_b^+(X), f \leq \omega\}$, and thus we obtain, for $i = 1, 2$,

$$\begin{aligned} \|\mu_i\|_\omega &= \sup\left\{\langle \mu_i, f\rangle : f \leq \omega, f \in C_b^+(X)\right\} \leq \sup\left\{\mathscr{P}_{\varphi_i}(f) : f \leq \omega, f \in C_b^+(X)\right\} \\ &\leq \|\varphi_i \omega\| < \infty, \end{aligned}$$

where we write $\langle \mu, f\rangle$ for $\int_X f\, d\mu$. Thus $\mu_i \in M_b^+(X,\omega)$ for $i = 1, 2$. Now set $\eta := ((\mu_1\omega - \mu_2\omega)^+\omega^{-1}, (\mu_1\omega - \mu_2\omega)^-\omega^{-1}) \in M_b(X,\omega)$ and note that $\eta^+ = (\mu_1\omega - \mu_2\omega)^+\omega^{-1}$ and $\eta^- = (\mu_1\omega - \mu_2\omega)^-\omega^{-1}$. Hence

$$\begin{aligned} I(g) &= I^+(g) - I^-(g) = \langle \mu_1, g\rangle - \langle \mu_2, g\rangle \\ &= \langle (\mu_1\omega - \mu_2\omega)^+\omega^{-1}, g\rangle - \langle (\mu_1\omega - \mu_2\omega)^-\omega^{-1}, g\rangle \\ &= I_{\eta^+}(g) - I_{\eta^-}(g) \end{aligned}$$

for $g \in \widetilde{C}_b(X,\omega)$, which means that Φ is surjective.

Next we are going to prove that Φ is injective. Suppose, towards a contradiction, that $[\mu, \nu], [\mu', \nu'] \in M_b(X,\omega)$ be such that

$$I_\mu - I_\nu = I_{\mu'} - I_{\nu'}.$$

Then

$$\langle \mu + \nu', g\rangle = \langle \mu' + \nu, g\rangle$$

for all $g \in \widetilde{C}_b(X,\omega)$. Now let U be an open subset in X. Since the function $\omega\chi_U$ is lower semi-continuous, so $\omega\chi_U = \sup\{f : f \in C_b^+(X), f \leq \omega\chi_U\}$. This together with [4, Theorem 1.5] imply that

$$\begin{aligned} \left\langle \mu + \nu', \omega\chi_U \right\rangle &= \sup\left\{\langle \mu + \nu', f\rangle : f \in C_b^+(X), f \leq \omega\chi_U\right\} \\ &= \sup\left\{\langle \mu' + \nu, f\rangle : f \in C_b^+(X), f \leq \omega\chi_U\right\} \\ &= \left\langle \mu' + \nu, \omega\chi_U \right\rangle. \end{aligned}$$

Thus $(\mu\omega + \nu'\omega)(U) = (\mu'\omega + \nu\omega)(U)$, and thanks to the inner regularity, we find that $\mu\omega + \nu'\omega = \mu'\omega + \nu\omega$. Invoking Lemma 2.2, we conclude that

$$\mu + \nu' = ((\mu + \nu')\omega)\omega^{-1} = ((\mu' + \nu)\omega)\omega^{-1} = \mu' + \nu.$$

That is $[\mu, \nu] = [\mu', \nu']$. This completes the proof. ∎

Let us remark that Theorem 4.1 states that the Banach space $M_b(X,\omega)$ can be viewed as the dual of certain subspaces of continuous functions when X is completely regular and ω is lower semi-continuous on X.

We conclude this work with the following example which shows that the conclusion of Theorem 4.1 does not hold in general.

Example 4.1. For each $n\in\mathbb{N}$, let $X_n := [2n+1/2, 2n+3/2]$. Put $X=\bigcup_{n=1}^{\infty} X_n$, and let λ be the Lebesgue measure on $\mathbb{R}$ and $\mu_n = (1/n)\lambda$ on X_n. Also let $A_n = \mathbb{Q}\cap X_n$, where $\mathbb{Q}$ is the set of all rational numbers, and, for $n\in\mathbb{N}$, $\omega_n(x) = 1/n$ if $x\in A_n$ and $\omega_n(x)=1$ otherwise. Furthermore, define $\omega(x)=\omega_n(x)$ for $x\in X_n$, and $\nu=\sum_{n=1}^{\infty}\mu_n$. Define G to be open in X if and only if $G\cap X_n$ is open in X_n for each $n\in\mathbb{N}$. Then X is a locally compact Hausdorff space such that ω and ω^{-1} are locally bounded Borel measurable functions on X. Define $I(g)=\int_X g\,d\nu$, it is easily verified that $I(g)\le ||g||_\omega(\sum_{n=1}^{\infty} 1/n^2)<\infty$ for $g\in\widetilde{C}_b(X,\omega)$ and also $I\in(\widetilde{C}_b(X,\omega),\widetilde{\beta}(X,\omega))^*$. But, note that $\int_X \omega d\nu = \sum_{n=1}^{\infty} 1/n = \infty$, which shows that $\nu\notin M_b(X,\omega)$.

References

[1] J. Aguayo-Garrido, Strict topologies on spaces of continuous functions and u-additive measure spaces, *J. Math. Anal. Appl.* **220** (1998), no. 1, 77–89.

[2] C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics*, second edition, Mathematical Surveys and Monographs, 105, Amer. Math. Soc., Providence, RI, 2003.

[3] J. Arhippainen and J. Kauppi, Generalization of the topological algebra $(C_b(X),\beta)$, *Studia Math.* **191** (2009), no. 3, 247–262.

[4] C. Berg, J. P. R. Christensen and P. Ressel, *Harmonic Analysis on Semigroups*, Graduate Texts in Mathematics, 100, Springer, New York, 1984.

[5] R. C. Buck, Bounded continuous functions on a locally compact space, *Michigan Math. J.* **5** (1958), 95–104.

[6] R. Giles, A generalization of the strict topology, *Trans. Amer. Math. Soc.* **161** (1971), 467–474.

[7] R. A. Hirschfeld, Riots, *Nieuw Arch. Wisk. (3)* **22** (1974), 1–43.

[8] A. K. Katsaras, On the strict topology in the nonlocally convex setting. II, *Acta Math. Hungar.* **41** (1983), no. 1–2, 77–88.

[9] S. S. Khurana, Strict topologies as topological algebras, *Czechoslovak Math. J.* **51(126)** (2001), no. 2, 433–437.

[10] S. E. Mosiman and R. F. Wheeler, The strict topology in a completely regular setting: relations to topological measure theory, *Canad. J. Math.* **24** (1972), 873–890.

[11] D. Pollard and F. Topsøe, A unified approach to Riesz type representation theorems, *Studia Math.* **54** (1975), no. 2, 173–190.

[12] H. H. Schaefer, *Topological Vector Spaces*, Springer, New York, 1971.

[13] H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer, New York, 1974.

[14] F. D. Sentilles and D. C. Taylor, Factorization in Banach algebras and the general strict topology, *Trans. Amer. Math. Soc.* **142** (1969), 141–152.

[15] W. H. Summers, A representation theorem for biequicontinuous completed tensor products of weighted spaces, *Trans. Amer. Math. Soc.* **146** (1969), 121–131.

[16] E. Wolf, A note on spectra of weighted composition operators on weighted Banach spaces of holomorphic functions, *Bull. Malays. Math. Sci. Soc. (2)* **31** (2008), no. 2, 145–152.