BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(1) (2013), 221–227*

On a Conjecture Concerning Some Nonlinear Difference Equations

[1]CHANG-WEN PENG AND [2]ZONG-XUAN CHEN

[1,2]School of Mathematical Sciences, South China Normal University, Guangzhou 510631, P. R.China
[1]pengcw716@126.com, [2]chzx@vip.sina.com

Abstract. In this paper, we mainly study non-existence of infinite order entire solutions of the nonlinear difference equation of the form

$$f(z)^n + q(z)f(z+1) = c\sin bz,$$

where $n(\geq 2)$ is an integer, $q(z)$ is a non-constant polynomial, which concerns a conjecture raised by Yang and Laine.

2010 Mathematics Subject Classification: Primary 30D35, 39A10

Keywords and phrases: Difference equation, entire solution, hyper-order.

1. Introduction

A function $f(z)$ is called meromorphic if it is analytic in the complex plane $\mathbb{C}$ except isolated poles. In what follows, we assume that the reader is familiar with the standard notations and results of Nevanlinna's value distribution theory as the proximity function $m(r,f)$, the integrated counting function $N(r,f)$, the characteristic function $T(r,f)$, see e.g. [12, 14, 16, 23]. Partial latest results concerning meromorphic functions are obtained in [2, 7, 8, 15, 18–20]. We also use notations $\sigma(f)$, $\mu(f)$, $\lambda(f)$ for the order, the lower order, the exponent of convergence of zeros of meromorphic function f, respectively.

Recently, meromorphic solutions to difference equations in the complex plane have been investigated in several papers, see e.g. [1, 4–6, 9–11, 13, 17, 21]. The background for these studies is in the difference variant of the Nevanlinna theory, initiated by Halburd and Korhonen in [9]. Here they proved a difference analogue to the logarithmic derivative lemma, see [9, Theorem 2.1 and Corollary 2.2]. Independently, Chiang and Feng obtained similar results in [6], including, in addition, pointwise estimates for $f(z+\eta)/f(z)$, see [6, Corollary 2.5 and Theorem 8.2]. Later on, Halburd, Korhonen and Tohge proposed a difference analogue to the logarithmic derivative lemma for meromorphic functions of hyper-order less than one:

Communicated by V. Ravichandran.
Received: January 29, 2011; *Revised:* June 7, 2011.

Theorem 1.1. [11, Theorem 5.1] *Let $f(z)$ be a non-constant meromorphic function and $c \in \mathbb{C}$. If f is of finite order, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O\left(\frac{\log r}{r} T(r,f)\right)$$

for all r outside of a set satisfying

$$\limsup_{r\to\infty} \frac{\int_{E\cap[1,r)} dt/t}{\log r} = 0,$$

i.e., outside of a set E of zero logarithmic density. If $\sigma_2(f) = \sigma_2 < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\sigma_2-\varepsilon}}\right)$$

for all r outside of a set of finite logarithmic measure, where $\sigma_2(f)$ denotes the hyper-order of $f(z)$, defined as

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In what follows, we also make use of the notion of lower hyper-order, defined as

$$\mu_2(f) := \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

For a more complete presentation of the difference Nevanlinna theory, including a difference variant of the second main theorem, see [10].

As to the applications of difference Nevanlinna theory to difference equations in the complex plane, we recall [21], and in particular the following two theorems therein:

Theorem 1.2. *A nonlinear difference equation*

$$f(z)^3 + q(z)f(z+1) = c\sin bz, \tag{1.1}$$

where $q(z)$ is a non-constant polynomial and $b,\ c \in \mathbb{C}$ are nonzero constants, does not admit entire solutions of finite order. If $q(z) = q$ is a constant, then Equation (1.1) *possesses three distinct entire solutions of finite order, provided $b = 3n\pi$ and $q^3 = (-1)^{n+1}\frac{27}{4}c^2$ for a nonzero integer n.*

Theorem 1.3. *Let $n \geq 4$ be an integer, $Q(z,f)$ be a linear differential difference polynomial of f, not vanishing identically, and h be a meromorphic function of finite order. Then the differential difference equation*

$$f(z)^n + Q(z,f) = h(z) \tag{1.2}$$

possesses at most one admissible transcendental entire solution of finite order such that all coefficients of $Q(z,f)$ are small functions of f. If such a solution f exists, then f is of the same order as h.

In [21], Yang and Laine also posed the following conjecture:

Conjecture 1.1. *There exists no entire function of infinite order that satisfies the difference equation of the type*

$$f^n(z) + q(z)f(z+1) = c\sin bz, \tag{1.3}$$

where $q(z)$ is a non-constant polynomial, b, c are nonzero constants and $n \geq 2$ is an integer.

In this paper, we mainly study this conjecture and partially answer the question.

2. Main results

In this paper, we obtain the following theorems.

Theorem 2.1. *Consider the nonlinear difference equation of the form*

$$f^n(z) + q(z)f(z+1) = c\sin bz, \tag{2.1}$$

where $q(z)$ is a non-constant polynomial, b, c are nonzero constants and $n \geq 2$ is an integer. Suppose that an entire function $f(z)$ satisfies any one of the following three conditions:

(i) $\lambda(f) < \sigma(f) = \infty$;
(ii) $\lambda_2(f) < \sigma_2(f)$;
(iii) $\mu_2(f) < 1$.

Then $f(z)$ cannot be a solution of Equation (2.1).

Theorem 2.2. *Let a polynomial $q(z)$ not vanishing identically, b, c be nonzero constants and $n \geq 2$ be an integer. If the nonlinear difference Equation* (2.1) *has an entire solution f of hyper-order $\sigma_2(f) < 1$, then $\sigma(f) = 1$.*

Our methods of proofs are different from the methods applied in [21].

3. Proofs of the theorems

We need the following lemmas to prove our main results.

Lemma 3.1. [22] *Let $f_j(z)(j = 1,...,n)(n \geq 2)$ be meromorphic functions, $g_j(z)(j = 1,...,n)$ be entire functions, and satisfy*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(ii) *when* $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ *is not a constant;*
(iii) *when* $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_k})), (r \to \infty, r \notin E)$, *where* $E \subset (1,\infty)$ *is of finite linear measure or finite logarithmic measure.*

Then $f_j(z) \equiv 0 (j = 1,...,n)$.

Lemma 3.2. [3, 14] *Let f be a transcendental entire function of infinite order and $\sigma_2(f) = \alpha < \infty$. Then f can be represented as*

$$f(z) = U(z)e^{V(z)},$$

where U and V are entire functions such that

$$\lambda(f) = \lambda(U) = \sigma(U), \lambda_2(f) = \lambda_2(U) = \sigma_2(U),$$
$$\sigma_2(f) = \max\left\{\sigma_2(U), \sigma_2(e^V)\right\},$$

where the notation $\lambda_2(f)$ denotes the hyper exponent of convergence of zeros of entire function f by

$$\lambda_2(f) = \limsup_{r\to\infty} \frac{\log\log N\left(r, \frac{1}{f}\right)}{\log r}.$$

Proof of the Theorem 2.1. (i) Let f be an entire solution to Equation (2.1), and satisfy $\lambda(f) < \sigma(f) = \infty$. Thus, by Lemma 3.2, $f(z)$ can be rewritten as $f(z) = Q(z)e^{g(z)}$, where $Q(z)$ is an entire function, $g(z)$ is a transcendental entire function, such that $\sigma(Q) = \lambda(Q) = \lambda(f) < \infty$. Substituting $f(z) = Q(z)e^{g(z)}$ into (2.1), we obtain that

$$Q(z)^n e^{ng(z)} + q(z)Q(z+1)e^{g(z+1)} = c\sin bz. \tag{3.1}$$

Set $H(z) = g(z+1) - ng(z)$. Then

$$Q(z)^n + q(z)Q(z+1)e^{H(z)} = ce^{-ng(z)} \sin bz. \tag{3.2}$$

If $H(z)$ is a polynomial, then

$$\sigma(Q(z)^n + q(z)Q(z+1)e^{H(z)}) < \infty,$$

and

$$\sigma(ce^{-ng(z)} \sin bz) = \infty.$$

This is a contradiction.

If $H(z)$ is a transcendental entire function, then (3.1) can be rewritten as

$$Q(z)^n e^{ng(z)} + q(z)Q(z+1)e^{g(z+1)} - ce^{h(z)} \sin bz = 0, \tag{3.3}$$

where $h(z) \equiv 0$. Since $G_1(z) = e^{g(z+1)-ng(z)}$, $G_2(z) = e^{g(z+1)-h(z)}$, $G_3(z) = e^{ng(z)-h(z)}$ are infinite order entire functions of regular growth, we see that for $j = 1,2,3$,

$$\begin{cases} T(r, Q(z)^n) = o\{T(r, G_j)\}, \\ T(r, -c \sin bz) = o\{T(r, G_j)\}, \\ T(r, q(z)Q(z+1)) = o\{T(r, G_j)\}. \end{cases} \tag{3.4}$$

Thus, by Lemma 3.1 and (3.3), we have

$$Q(z)^n \equiv 0,\ q(z)Q(z+1) \equiv 0,\ -c \sin bz \equiv 0, \tag{3.5}$$

which is a contradiction.

(ii) Suppose that f is an entire solution to Equation (2.1), and satisfies $\lambda_2(f) < \sigma_2(f)$. By Lemma 3.2, we may rewrite $f(z)$ as $f(z) = Q(z)e^{g(z)}$, where $Q(z)$ is an entire function, $g(z)$ is a transcendental entire function such that

$$\lambda_2(Q) = \sigma_2(Q) = \lambda_2(f) < \sigma_2(e^g) = \sigma(g).$$

Substituting $f(z) = Q(z)e^{g(z)}$ into (2.1), we get (3.1) and (3.2), where $H(z) = g(z+1) - ng(z)$.

If $\sigma(H) < \sigma(g)$, then

$$\sigma_2(Q(z)^n + q(z)Q(z+1)e^{H(z)}) \leq \max\{\sigma_2(Q), \sigma(H)\} < \sigma(g) = \sigma_2(ce^{-ng(z)} \sin bz).$$

This contradicts (3.2).

If $\sigma(H) = \sigma(g)$, then we can get (3.3). Set $G_1(z) = e^{g(z+1)-ng(z)}$, $G_2(z) = e^{g(z+1)-h(z)}$, $G_3(z) = e^{ng(z)-h(z)}$. Using the same method as in the proof of (i), we see that (3.4) and (3.5) hold. This is a contradiction.

(iii) Assume that f is an entire solution to Equation (2.1) and $\mu_2(f) < 1$. By Equation (2.1), we conclude that

$$|f(z)^n| \leq |q(z)||f(z+1)| + |c \sin bz|.$$

Set $\deg q = k$. Then $|q(z)| \leq r^{k+1}$. Since $|c \sin bz| = |c(e^{ibz} - e^{-ibz})/(2i)| \leq |c/2| \cdot 2e^{|b|r}$, we have

$$|f(z)^n| \leq r^{k+1} M(r, f(z+1)) + |c|e^{|b|r}. \tag{3.6}$$

Without loss of generality, we may assume that $|c| = |b| = 1$, and we assume $k+1 = P$. By (3.6), we have

$$M(r, f)^n \leq r^P M(r+1, f) + e^r.$$

Moreover

$$n\log M(r,f)\le \log M(r+1,f)+P\log r+r,$$

that is

$$\log M(r+1,f)\ge n\log M(r,f)-(P\log r+r)\ge n\log M(r,f)-2r.$$

Similarly we have

$$\begin{aligned}\log M(r+2,f)&\ge n\log M(r+1,f)-2(r+1)\ge n(n\log M(r,f)-2r)-2(r+1)\\&=n^2\log M(r,f)-[2nr+2(r+1)].\end{aligned}$$

By an inductive argument, we get

$$\begin{aligned}\log M(r+s,f)\ge n^s\log M(r,f)-2[n^{s-1}r+n^{s-2}(r+1)+\cdots\\+n(r+s-2)+(r+s-1)].\end{aligned}\tag{3.7}$$

Set

$$H_s(r)=2[n^{s-1}r+n^{s-2}(r+1)+\cdots+n(r+s-2)+(r+s-1)].$$

Thus

$$\begin{aligned}H_s(r)&=2[n^{s-1}r+n^{s-2}(r+1)+\cdots+n(r+s-2)+(r+s-1)]\\&=2n^{s-1}\left[r+\frac{r+1}{n}+\frac{r+2}{n^2}+\cdots+\frac{r+s-1}{n^{s-1}}\right].\end{aligned}$$

Set

$$I=\sum_{s=1}^{\infty}a_s=\sum_{s=1}^{\infty}\frac{r+s-1}{n^{s-1}}.$$

Since

$$\lim_{s\to\infty}\frac{a_{s+1}}{a_s}=\lim_{s\to\infty}\frac{\frac{r+s}{n^s}}{\frac{r+s-1}{n^{s-1}}}=\lim_{s\to\infty}\frac{r+s}{n(r+s-1)}=\frac{1}{n}\le\frac{1}{2}<1,$$

we see that the series I is convergent.

Suppose that the series I converges to the number J. So that, we obtain

$$\log M(r+s,f)\ge n^s\log M(r,f)-2n^{s-1}J=n^s\left[\log M(r,f)-\frac{2}{n}J\right].$$

Thus, we have

$$\begin{aligned}\log\log M(r+s,f)&\ge s\log n+\log\left[\log M(r,f)-\frac{2}{n}J\right]\\&=s\log n\left[1+\frac{\log(\log M(r,f)-\frac{2}{n}J)}{s\log n}\right].\end{aligned}\tag{3.8}$$

From (3.8), we have

$$\frac{\log\log\log M(r+s,f)}{\log(r+s)}\ge\frac{\log s+\log\log n+\log\left[1+\frac{\log(\log M(r,f)-\frac{2}{n}J)}{s\log n}\right]}{\log(r+s)}.\tag{3.9}$$

When $s\to\infty$, we have

$$\liminf_{s\to\infty}\frac{\log s}{\log(r+s)}=\liminf_{s\to\infty}\frac{\log s}{\log s(1+\frac{r}{s})}=1.$$

When r takes all values on $[r_0, r_0+1]$ and s takes all values on $\{1,2,...\}$, we see that $r+s$ gets all values on $[r_0,\infty)$. Hence by (3.9), we get $\mu_2(f)\geq 1$. This contradicts our assumption. ∎

Proof of Theorem 2.2. Suppose that f is an entire solution to Equation (2.1), and satisfies hyper-order $\sigma_2(f)=\sigma_2<1$. By Theorem 1.1, we may choose ε such that $\varepsilon<1-\sigma_2$, so $T(r,f)/r^{1-\sigma_2-\varepsilon}<T(r,f)$, hence we have $m(r,(f(z+\eta))/(f(z)))=o(T(r,f))$. By (2.1), we have

$$f^n(z)=-q(z)\frac{f(z+1)}{f(z)}f(z)+c\sin bz. \tag{3.10}$$

From (3.10) and Theorem 1.1, we conclude that

$$\begin{aligned}nT(r,f(z))&=nm(r,f(z))=m(r,f^n(z))\\ &\leq m(r,-q(z))+m\left(r,\frac{f(z+1)}{f(z)}\right)+m(r,f(z))+m(r,c\sin bz)\\ &\leq o(T(r,f))+T(r,f(z))+m(r,c\sin bz).\end{aligned} \tag{3.11}$$

Hence

$$(n-1)T(r,f)\leq o(T(r,f))+m(r,c\sin bz). \tag{3.12}$$

From (3.12), we immediately conclude that f has to be of finite order, and $\sigma(f)\leq 1$.

Now we show that $\sigma(f)=1$. Suppose to the contrary that $\sigma(f)<1$. Then $\sigma(f^n(z)+q(z)f(z+1))<1$ and $\sigma(c\sin bz)=1$. This is a contradiction by Equation (2.1). ∎

Acknowledgement. The authors Chen Zongxuan (corresponding author) *et al.* are very grateful to the referees for many valuable comments and suggestions which greatly improved the presentation of this paper. The project was supported by National Natural Science Foundation of China (No. 11171119 and No. 10871076).

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