BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(1) (2013), 229–241*

A Degenerate and Singular Parabolic System Coupled Through Boundary Conditions

[1]YONG-SHENG MI, [2]CHUN-LAI MU AND [3]SHOU-MING ZHOU
[1,2,3]College of Mathematics and Statistics, Chongqing University, Chongqing 400044, P. R. China
[1]College of Mathematics and Computer Sciences, Yangtze Normal University, Fuling 408100, Chongqing, P. R. China
[1]miyongshen@163.com, [2]chunlaimu@yahoo.com.cn, [3]zhoushouming76@163.com

Abstract. The paper deals with the global existence and nonexistence for degenerate and singular parabolic system with nonlinear boundary condition. By using the comparison principle and constructing the self-similar super-solution and sub-solution, we obtain the critical global existence curve. The critical curve of Fujita type is conjectured with the aid of some new results. An interesting feature of our results is that the critical global existence curve and the critical Fujita curve are determined by a matrix and by the solution of a linear algebraic system, respectively.

2010 Mathematics Subject Classification: 35K55, 35K65, 35B40

Keywords and phrases: Critical global existence curve, critical Fujita curve, nonlinear boundary flux, blow-up.

1. Introduction and main results

In this paper, we investigate the following parabolic equations

$$u_{it} = (|u_{ix}|^{p_i}(u_i^{m_i})_x)_x, \quad (i = 1,2,...,k),\ x > 0,\ 0 < t < T, \tag{1.1}$$

subject to nonlinear boundary conditions

$$-|u_{ix}|^{p_i}(u_i^{m_i})_x(0,t) = u_{i+1}^{q_i}(0,t), \quad (i = 1,2,...,k),\ u_{k+1} := u_1,\ 0 < t < T, \tag{1.2}$$

with initial data

$$u_i(x,0) = u_{i0}(x), \quad (i = 1,2,...,k),\ x > 0, \tag{1.3}$$

where parameters $0 < m_i < 1,\ -1 < p_i < 1-m_i, q_i > 0, k \geq 2, (i = 1,2,...,k)$ and $u_{i0}(i = 1,2,...,k)$ are nonnegative continuous functions with compact support in R_+. Let the initial data be appropriately smooth functions and satisfy the compatibility condition.

Nonlinear parabolic equations (1.1) come from the theory of turbulent diffusion (see [3,7] and references therein) and appear in population dynamics, chemical reactions, heat transfer, and so on. The equations (1.1) include both the porous medium operator (with $p_i = 0$)

Communicated by Norhashidah M. Ali.
Received: December 29, 2010; *Revised:* April 17 2011.

and the gradient-diffusivity the $p-$Laplacian operator ($m_i = 1$, this case is not under consideration due to the imposed restriction $m_i < 1$), which have been the subject of intensive study (see [3,4,7,8,10,12,14,20,26,28] and references therein).

The problems on blow-up and global existence conditions, blow-up rates to nonlinear parabolic equations have been intensively studied (see [2–4,8,10,17,18,20,23–27,29,30] and references therein). In particular, many paper have been devoted to study critical exponents of (1.1)–(1.3) in the slow diffusion case (see [8,18,25,28,30]). Recently, many authors transfer their attention to the fast diffusion case(see [4,14,24,27]), and many important results about critical exponents have been obtained. The concept of critical Fujita exponents was proposed by Fujita in the 1960s during discussion of the heat conduction equation with a nonlinear source (see [6]).

In [7], Galaktionov and Levine studied the following single equation

$$\begin{aligned} &u_t = \nabla(|\nabla u|^{\sigma}\nabla u^m) + u^p, \quad x \in R^N,\ t > 0, \\ &u(x,0) = u_0(x), \quad x \in R^N, \end{aligned}$$

where $\sigma > 0,\ m > 1,\ p > 1$ and $u_0(x)$ is a bounded positive continuous function. They shown that the critical exponent is $p_c = m + \sigma + (\sigma+2)/N$.

Recently, Mi, Mu and Chen [14] studied the following problem

$$\begin{aligned} u_t &= (|u_x|^p(u^m)_x)_x, \quad x > 0, 0 < t < T, \\ -|u_x|^p(u^m)_x(0,t) &= u^q(0,t), \quad 0 < t < T, \\ u(x,0) &= u_0(x), \quad x > 0, \end{aligned} \tag{1.4}$$

where $0 < m < 1, -1 < p < 1-m, q > 0$. They obtained the critical global existence exponent $q_0 = (2p+m+1)/(p+2)$ and the critical Fujita exponent $q_c = 2p+m+1$.

There are some works on the blow-up properties for a general semilinear diffusion system coupled through nonlinear boundary conditions

$$\begin{aligned} &u_{it} = \Delta u_i, \quad (i = 1,2,...,k),\ (x,t) \in \Omega \times (0,+\infty), \\ &\frac{\partial u}{\partial n} = u_{i+1}^{p_i}, \quad (i = 1,2,...,k),\ u_{k+1} := u_1, (x,t) \in \partial\Omega \times (0,+\infty), \\ &u_i(x,0) = u_{i0}(x), \quad x \in \Omega, \end{aligned}$$

where $\Omega \in R^N$ is a bounded domain or $\Omega = R_+^N$ (see [15, 16, 22]), or through nonlinear reaction terms

$$u_{it} = \Delta u_i + u_{i+1}^{p_i},\ (i = 1,2,...,k),\ u_{k+1} := u_1,\ (x,t) \in \Omega \times (0,+\infty),$$

where $\Omega \in R^N$ or $\Omega = R^N$ (see [5,21] and references therein).

Motivated by the references cited above, in this paper, we focus on system (1.1)–(1.3) with parameters $0 < m_i < 1, -1 < p_i < 1-m_i, q_i > 0 (i = 1,2,...,k)$ and $u_{i0} (i = 1,2,...,k)$ are continuous, nonnegative functions with compact support in R_+. We will construct various kinds of self-similar supersolution and subsolutions to obtain the critical global existence curve of system (1.1)–(1.3). The critical curve of Fujita type is conjectured with the aid of some new results. A interesting feature of our results is that the critical global existence curve and the critical Fujita curve are determined by a matrix and by the solution of a linear algebraic system, respectively.

We remark the main difference between $p_i \geq 0, m_i > 1$ and our current settings $-1 < p_i < 1-m_i, 0 < m_i < 1$, we take $p_i = 0$ for example. For the former, Equation (1.1) having

$m_i > 1$ are the well-known porous medium equations, while for the latter, Equation (1.1) having $0 < m_i < 1$ are the so-called fast diffusion equations. The porous medium equations have finite speed of propagation property, that is, solutions with compactly supported initial data stay compactly supported, which makes comparison with global supersolutions easier when one is restricted to compactly supported initial data. However, the solutions of the fast diffusion equations shall become instantaneously positive everywhere for any nontrivial nonnegative initial data, and hence we have to take care of the decay of the solutions.

To state our results, we need to introduce some useful symbols. Set

$$A = \begin{pmatrix} 1 & -\frac{p_1+2}{2p_1+m_1+1}q_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -\frac{p_2+2}{2p_2+m_2+1}q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\frac{p_{k-1}+2}{2p_{k-1}+m_{k-1}+1}q_{k-1} \\ -\frac{p_k+2}{2p_k+m_k+1}q_k & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

and by a series of standard computations, we have $\det A = 1 - \prod_{i=1}^{k}((p_i+2)/(2p_i+m_i+1)q_i)$. We shall see that $\det A$ is the critical global existence curve. Next, let $(\alpha_1, \alpha_2, ..., \alpha_{k-1}, \alpha_k)^T$ be the solution of the following linear algebraic system

$$A(\alpha_1, \alpha_2, ..., \alpha_{k-1}, \alpha_k)^T = (-1, -1, ..., -1, -1)^T.$$

Thus a direct computation also shows that $\alpha_i > 0 (i = 1, 2, ..., k)$ if and only if $\det A < 0$. We further define $l_i = (\alpha_i(1 - p_i - m_i) + 1)/(p_i + 2)$ and $\alpha_{k+1} = \alpha_1$, then have

$$\alpha_i + 1 = p_i\alpha_i + p_i l_i + m_i\alpha_i + 2l_i, p_i\alpha_i + p_i l_i + m_i\alpha_i + l_i = q_i\alpha_{i+1}, \quad (i = 1, 2, ..., k). \tag{1.5}$$

Our main results in this paper are stated as follows.

Theorem 1.1.

(1) *If* $\prod_{i=1}^{k}((p_i+2)/(2p_i+m_i+1)q_i) \le 1$ *(i.e.* $\det A \ge 0$*), then every nonnegative solution of the system* (1.1)-(1.3) *is global in time.*

(2) *If* $\prod_{i=1}^{k}((p_i+2)/(2p_i+m_i+1)q_i) > 1$ *(i.e.* $\det A < 0$*), then the system* (1.1)–(1.3) *has a solution that blows up.*

Theorem 1.2. *Assume* $\prod_{i=1}^{k}((p_i+2)/(2p_i+m_i+1)q_i) > 1$ *(i.e.,* $\det A < 0$*).*

(1) *If* $\min_i\{l_i - \alpha_i\} > 0$*, then there exists a global solution to the system* (1.1)–(1.3).

(2) *If* $\max_i\{l_i - \alpha_i\} < 0$*, then every nonnegative nontrivial solution of the system* (1.1)–(1.3) *blows up in finite time.*

Remark 1.1. Theorem 1.1 show that the critical global existence curve of (1.1)–(1.3) is $\prod_{i=1}^{k}((p_i+2)/(2p_i+m_i+1)q_i) = 1$ (i.e. $\det A = 0$), the restriction $\max\{l_i - \alpha_i\} < 0$ in the Theorem 1.2 (2) is rather technical. It comes from the construction of the so-called Zel'dovich-Kompaneetz-Barenblatt profile. We believe that the critical Fujita curve is $\min_i\{l_i - \alpha_i\} = 0$.

Remark 1.2. Unfortunately, we cannot obtain some results concerning some missing cases for $\det A < 0$, (for instance, the case $\min_i\{l_i - \alpha_i\} \le 0 \le \max_i\{l_i - \alpha_i\}$). We expect to answer this question in near future.

The rest of this paper is organized as follows. Some preliminaries will be given in Section 2. In Section 3, we consider the critical global existence curve and prove Theorem 1.1. The proof of Theorem 1.2 is shown in Section 4.

2. Preliminaries

Let T be the maximal existence time of a solution $(u_1, u_2, ..., u_k)$, which may be finite or infinite. If $T < \infty$, then $\| u_1 \|_\infty + \| u_2 \|_\infty + ... + \| u_k \|_\infty$ becomes unbounded in finite time and we say that the solution blows up. If $T = \infty$, we say that the solution is global.

As it is well known that degenerate and singular equations need not possess classical solutions, we give a precise definition of a weak solution to (1.1)–(1.3).

Definition 2.1. *let* $T > 0, m = (m_1, m_2, ..., m_k), q = (q_1, q_2, ..., q_k), u = (u_1, u_2, ..., u_k), u_0 = (u_{01}, u_{02}, ..., u_{0k}), u^q = (u_2^{q_1}, u_3^{q_2}, ..., u_k^{q_{k-1}}, u_1^{q_k}), \omega = (\omega_1, \omega_1, ..., \omega_k)$ $Q_T = (0, +\infty) \times (0, T]$. *A vector function* $u(x,t)$ *is called an upper (lower) solution of* (1.1)–(1.3) *in* Q_T *with nonlinear flux* u^q *if:*

1° $u, \in L^\infty(0,T; W^{1,\infty}(\Omega)) \cap W^{1,2}(0,T; L^2(Q_T)), u(x,0) \geq (\leq) u_0$*; and,*

2° *for any positive function* $\omega(0,T; W^{1,2}(\Omega)) \cap L^2(Q_T)$*, we have*

$$\int\int_{Q_T} \left(u_t \omega - |\frac{\partial u}{\partial x}|^p \frac{\partial u^m}{\partial x} \frac{\partial \omega}{\partial x} \right) dxdt \geq (\leq) \int_0^T \int_{Q_T} \omega u^q ds_x dt \tag{2.1}$$

$u(x,t)$ *is called a weak solution of* (1.1)–(1.3) *if it is both a weak upper and a lower solution.*

Next we give a preliminary proposition.

Proposition 2.1. *Assume that* $u_0 = (u_{01}, u_{02}, ..., u_{0k})$ *is positive* C^1 *functions and* $u = (u_1, u_2, ..., u_k)$ *is any weak solution of* (1.1)-(1.3). *Also assume that* $\underline{u} = (\underline{u_1}, \underline{u_2}, ..., \underline{u_k}) \geq \delta_0 = (\delta_0, \delta_0, ..., \delta_0) > 0$ *and* $\overline{u} = (\overline{u_1}, \overline{u_2}, ..., \overline{u_k})$ *are a lower and an upper solution of* (1.1)-(1.3) *in* Q_T *, respectively, with nonlinear boundary flux* $\underline{\lambda u^q} = \underline{\lambda}(\underline{u_2^{q_1}}, \underline{u_3^{q_2}}, ..., \underline{u_k^{q_{k-1}}}, \underline{u_1^{q_k}})$ *and* $\overline{\lambda}\overline{u^q} = \overline{\lambda}(\overline{u_2^{q_1}}, \overline{u_3^{q_2}}, ..., \overline{u_k^{q_{k-1}}}, \overline{u_1^{q_k}})$*, where* $0 < \underline{\lambda} < 1 < \overline{\lambda}$. *Then we have* $\overline{u} \geq u \geq \underline{u}$ *in* Q_T .

Proof. For small σ, letting $\psi_\sigma = \min\{1, \max\{z/\sigma, 0\}\}, z \in R$, and setting $\omega_1 = \psi_\sigma(\underline{u}_1 - u_1)$, according to the definition of upper and lower solutions, we have

$$\int\int_{Q_T} \left((\underline{u_1} - u_1)_t \psi_\sigma(\underline{u}_1 - u_1) - \left(\left|\frac{\partial \underline{u}}{\partial x}\right|^{p_1} \frac{\partial \underline{u}^{m_1}}{\partial x} - \left|\frac{\partial u}{\partial x}\right|^{p_1} \frac{\partial u^{m_1}}{\partial x} \right) \frac{\partial (\psi_\sigma(\underline{u}_1 - u_1))}{\partial x} \right) dxdt$$

$$\leq \int_0^T \int_{Q_T} \psi_\sigma(\underline{u}_1 - u_1)(\underline{\lambda u_2^{q_1}} - u_2^{q_1}) ds_x dt. \tag{2.2}$$

Define

$$\chi(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{2.3}$$

As in [1],by letting $\sigma \to 0$ we get

$$\int\int_{Q_\tau} ((\underline{u_1} - u_1)_t \chi(\underline{u_1} - u_1) \leq \int_0^\tau \int_{Q_\tau} \chi(\underline{u}_1 - u_1) \left(\underline{\lambda u_2^{q_1}} - u_2^{q_1} \right) ds_x dt, \tag{2.4}$$

that is,

$$\int\int_{Q_\tau} ((\underline{u_1} - u_1)_+|_{t=\tau} \leq \int_0^\tau \int_{Q_\tau} \left(\underline{\lambda u_2^{q_1}} - u_2^{q_1} \right)_+ ds_x dt, \tag{2.5}$$

where $W_+ = \max\{W,0\}$. Similarly, we have

$$\int\int_{Q_\tau}((\underline{u_i}-u_i)_+|_{t=T} \le \int_0^\tau \int_{Q_\tau} \left(\underline{\lambda u}_{i+1}^{q_i} - u_{i+1}^{q_i}\right)_+ ds_x dt, \quad i=2,...,k,\ u_{k+1}=u_1. \tag{2.6}$$

Since $\underline{\lambda} < 1, u \ge \delta_0 > 0$ and $\underline{u}(x,0) \le u_0(x)$, it follows from the continuity of $\underline{u}$ and u that there exists a $\tau > 0$ sufficiently small such that $\underline{\lambda u^p} \le u^p$ for $(x,t) \in Q_\tau$ It follows from (2.5) and (2.6) that $u \ge \underline{u}$ in Q_τ.

Denote $\tau^* = \sup\{\tau \in [0,T] : \underline{u}(x,t) \le u(x,t)\ for\ all\ (x,t) \in Q_\tau\}$. We claim that $\tau^* = t$. Otherwise, from the continuity of u, u there exists $\varepsilon > 0$ such that $\underline{u} \le u$ in $Q_{\tau^*+\varepsilon}$ which contradicts the definition of τ^*, Hence $\underline{u} \le u$ for al $(x,t) \in Q_T$.

Obviously, $\delta = \min_{i=1,2,...,k}\{\min_{(0,+\infty)} u_{i0}\} > 0$ is a lower solution of (1.1)-(1.3) in Q_T. Therefore, $u \ge \delta > 0$ in Q_T. Using this fact, as in the above proof we can prove that $\overline{u} \ge u$ in Q_T. ∎

3. Critical global existence curve

In this section, we characterize when the solutions to the problem (1.1)–(1.3) are global in time for any initial data or they may blow up for large initial values. The basic idea of the proof is to compare from above with global in time supersolutions or from below with blowing up subsolutions.

Proof of Theorem 1.1

(1). In order to prove that the solution $(u_1, u_2, ..., u_k)$ of (1.1)–(1.3) is global, we look for a globally defined in time supersolution of the self-similar form

$$\overline{u}_i(x,t) = e^{\chi_{2i-1}t}(M + e^{-L_i x e^{\chi_{2i}t}})^{\frac{1}{m_i}}, \quad (i=1,2,...,k),\ x \ge 0, t \ge 0,$$

where $M = \max_{i\in\{1,2,...,k\}}\{\| u_{0i} \|_\infty^{m_i} + 1, (1-p_i-m_i)/((p_i+2)m_i e)\}$, L_i are to be chosen. Obviously, we have $\overline{u}_i(x,0) \ge u_{0_i}(x)$ $(i=1,2,...,k)$, for $x \ge 0$. Since $-ye^{-y} \ge -e^{-1}$ for $y > 0$, after a direct computation, we obtain

$$\begin{aligned}
\overline{u}_{it} &= \chi_{2i-1} e^{\chi_{2i-1}t}\left(M + e^{-L_i x e^{\chi_{2i}t}}\right)^{\frac{1}{m_i}} - \frac{\chi_{2i}}{m_i} L_i x e^{\chi_{2i}t} e^{-L_i x e^{\chi_{2i}t}} \left(M + e^{-L_i x e^{\chi_{2i}t}}\right)^{\frac{1}{m_i}-1} e^{\chi_{2i-1}t} \\
&\ge \chi_{2i-1} e^{\chi_{2i-1}t}\left(M + e^{-L_i x e^{\chi_{2i}t}}\right)^{\frac{1}{m_i}} - \frac{\chi_{2i}}{m_i} e^{-1} \left(M + e^{-L_i x e^{\chi_{2i}t}}\right)^{\frac{1}{m_i}-1} e^{\chi_{2i-1}t} \\
&\ge \left(\chi_{2i-1} - \frac{\chi_{2i}}{m_i M e}\right) M^{\frac{1}{m_i}} e^{\chi_{2i-1}t} = \chi_{2i-1}\left(1 - \frac{1-p_i-m_i}{(p_i+2)m_i M e}\right) M^{\frac{1}{m_i}} e^{\chi_{2i-1}t},
\end{aligned}$$

$$(|\overline{u}_{ix}|^{p_i}(\overline{u}_i^{m_i})_x = -\frac{L_i^{p_i+1}}{m_i^{p_i}} e^{p_i(\chi_{2i-1}+\chi_{2i})t + (m_i\chi_{2i-1}+\chi_{2i})t} e^{-(L_i x + p_i L_i x)e^{\chi_{2i}t}} \left(M + e^{-L_i x e^{\chi_{2i}t}}\right)^{p_i(\frac{1}{m_i}-1)},$$

$$(|\overline{u}_{ix}|^{p_i}(\overline{u}_i^{m_i})_x)_x \le (p_i+1)\frac{L_i^{p_i+2}}{m_i^{p_i}} e^{p_i(\chi_{2i-1}+\chi_{2i})t + (m_i\chi_{2i-1}+2\chi_{2i})t} M^{p_i(\frac{1}{m_i}-1)}$$

in $R_+ \times R_+, i = 1,2,...,k$. On the other hand, on the boundary we have

$$-|\overline{u}_{ix}|^{p_i}(\overline{u}_i^m)_x(0,t) = \frac{L_i^{p_i+1}}{m_i^{p_i}} e^{p_i(\chi_{2i-1}+\chi_{2i})t + (m_i\chi_{2i-1}+\chi_{2i})t}(M+1)^{p_i(\frac{1}{m_i}-1)},$$

$$\overline{u}_{i+1}^{q_i}(0,t) = e^{q_i\chi_{2i+1}t}(M+1)^{\frac{q_i}{m_{i+1}}}, \overline{u}_{k+1} = \overline{u}_1, \chi_{2k+1} = \chi_1, \quad (i=1,2,...,k).$$

Therefore, we can see that $(\overline{u}_1, \overline{u}_2, ..., \overline{u}_k)$ is a supersolution of problem (1.1)–(1.3) provided that

$$\chi_{2i-1}\left(1-\frac{1-p_i-m_i}{(p_i+2)m_iMe}\right)M^{\frac{1}{m_i}}e^{\chi_{2i-1}t} \geq (p_i+1)\frac{L_i^{p_i+2}}{m_i^{p_i}}e^{p_i(\chi_{2i-1}+\chi_{2i})t+(m_i\chi_{2i-1}+2\chi_{2i})t}M^{p_i(\frac{1}{m_i}-1)},$$

and

$$\frac{L_i^{p_i+1}}{m_i^{p_i}}e^{p_i(\chi_{2i-1}+\chi_{2i})t+(m_i\chi_{2i-1}+\chi_{2i})t}(M+1)^{p_i(\frac{1}{m_i}-1)} \geq e^{q_i\chi_{2i+1}t}(M+1)^{\frac{q_i}{m_{i+1}}},$$

where $\chi_{2k+1} := \chi_1, m_{k+1} := m_1$. In order to verify the above inequalities, we only need impose

$$\chi_{2i-1} \geq p_i(\chi_{2i-1}+\chi_{2i})+m_i\chi_{2i-1}+2\chi_{2i}, \quad (i=1,2,...,k), \tag{3.1}$$

$$p_i(\chi_{2i-1}+\chi_{2i})+m_i\chi_{2i-1}+\chi_{2i} \geq q_i\chi_{2i+1}, \quad (i=1,2,...,k), \tag{3.2}$$

and

$$\chi_{2i-1}\left(1-\frac{1-p_i-m_i}{(p_i+2)m_iMe}\right)M^{\frac{1}{m_i}} \geq (p_i+1)\frac{L_i^{p_i+2}}{m_i^{p_i}}M^{p_i(\frac{1}{m_i}-1)}, \quad (i=1,2,...,k), \tag{3.3}$$

$$\frac{L_i^{p_i+1}}{m_i^{p_i}}(M+1)^{p_i(\frac{1}{m_i}-1)} \geq (M+1)^{\frac{q_i}{m_{i+1}}}, \quad (i=1,2,...,k). \tag{3.4}$$

Now we show that such choice in (3.1)–(3.4) is valid. Firstly, by taking

$$L_i = m_i^{\frac{p_i}{p_i+1}}(M+1)^{\frac{q_i}{(p_i+1)m_{i+1}}-\frac{p_i-m_ip_i}{m_i(p_i+1)}}, \quad (i=1,2,...,k),$$

we see that (3.4) holds. Secondly, to obtain (3.1), we take $\chi_{2i-1} = p_i(\chi_{2i-1}+\chi_{2i}) + m\chi_{2i-1}+2\chi_{2i}, (i=1,2,...,k)$, that is

$$\chi 2i = \frac{1-p_i-m_i}{p_i+2}\chi_{2i-1}, \quad (i=1,2,...,k). \tag{3.5}$$

Meanwhile, we must ensure that such choice is suitable for (3.2). To this end, we substitute (3.5) into (3.2) and then (3.2) becomes

$$\chi_{2i+1} \leq \frac{2p_i+m_i+1}{q_i(p_i+2)}\chi_{2i-1}, \quad (i=1,2,...,k). \tag{3.6}$$

Therefore, if we further take $\chi_{2i+1} = (2p_i+m_i+1)/(q_i(p_i+2))l_{2i-1}$, $(i=1,2,...,k-1)$, then we only need to show for the case $i=k$,

$$\chi_1 = \chi_{2k+1} \leq \frac{2p_k+m_k+1}{q_k(p_k+2)}l_{2k-1} = \chi_1\prod_{i=1}^{k}\frac{2p_i+m_i+1}{q_i(p_i+2)}.$$

Clearly, this is true under the assumption $\prod_{i=1}^k(2p_i+m_i+1)/(q_i(p_i+2)) \geq 1$ (i.e., $\det A \geq 0$). Finally, we can choose χ_1, and then $\chi_2, ..., \chi_k$, large enough such that (3.3) holds.

Therefore, we have proved that $(\overline{u}_1, \overline{u}_2, ..., \overline{u}_k)$ is a global supersolution of system (1.1)–(1.3). Hence the comparison principle gives $(\overline{u}_1, \overline{u}_2, ..., \overline{u}_k) \geq (u_1, u_2, ..., u_k)$ and we conclude that $(u_1, u_2, ..., u_k)$ is global.

(2). To prove the non-existence of global solutions, we construct a blow-up self-similar subsolution of the system (1.1)–(1.3). Construct the functions

$$\underline{u}_i(x,t) = (T-t)^{-\alpha_i}f_i(\xi_i), \quad \xi_i = x(T-t)^{-l_i}, \ (i=1,2,...,k), \tag{3.7}$$

where $\alpha_i, l_i (i=1,2,...,k), \alpha_{k+1}=k_1$, were defined as before, T is a positive constant and $f_i \geq 0 (i=1,2,...,k),\ f_{k+1}=f_1$, which are to be determined.

After some computations, we have

$$\underline{u}_{it} = (T-t)^{-(\alpha_i+1)}(\alpha_i f_i(\xi_i) + l_i \xi_i f_i{}'(\xi_i)),$$
$$|\underline{u}_{ix}|^{p_i}(\underline{u}_i^{m_i})_x = (T-t)^{-p_i\alpha_i - p_i l_i - m_i\alpha_i - l_i}|f_i{}'|^{p_i}(f_i^{m_i})'(\xi_i),$$
$$(|\underline{u}_{ix}|^{p_i}(\underline{u}_i^{m_i})_x)_x = (T-t)^{-p_i\alpha_i - p_i l_i - m_i\alpha_i - 2l_i}(|f_i{}'|^{p_i}(f_i^{m_i})'(\xi_i))',$$

and

$$|\underline{u}_{ix}|^{p_i}(\underline{u}_i^{m_i})_x(0,t) = (T-t)^{-p_i\alpha_i - p_i l_i - m_i\alpha_i - l_i}|f_i{}'|^{p_i}(f_i^{m_i})'(0),$$
$$\underline{u}_{i+1}^{q_i}(0,t) = (T-t)^{-q_i\alpha_{i+1}} f_{i+1}^{q_i}(0), \underline{u}_{k+1} = \underline{u}_1, f_{k+1} = f_1.$$

Notice that

$$\alpha_i + 1 = p_i\alpha_i + p_i l_i + m_i\alpha_i + 2l_i, \quad p_i\alpha_i + p_i l_i + m_i\alpha_i + l_i = q_i\alpha_{i+1},$$

Thus, $(\underline{u}_1, \underline{u}_2, ..., \underline{u}_k)$ is subsolution of (1.1)–(1.3) provided that

$$(|f_i{}'|^{p_i}(f_i^{m_i})'(\xi_i))' \geq \alpha_i f_i(\xi_i) + l_i f_i'(\xi_i)\xi_i, \tag{3.8}$$

$$-|f_i{}'|^{p_i}(f_i^{m_i})'(0) \leq f_{i+1}{}^{q_i}(0). \tag{3.9}$$

$$f_i(\xi) = (A_i + B_i\xi_i)^{-\frac{p_i+2}{1-p_i-m_i}}, \quad (i=1,2,...,k), \tag{3.10}$$

where $A_i, B_i, (i=1,2,...,k)$ are positive constants to be determined. It is easy to see that

$$f_i'(\xi_i) = -B_i \frac{p_i+2}{1-p_i-m_i}(A_i + B_i\xi_i)^{-\frac{p_i+2}{1-p_i-m_i}-1}, \tag{3.11}$$

$$|f_i'|^{p_i}(f_i^{m_i})' = -m_i B_i^{p_i+1}\left(\frac{p_i+2}{1-p_i-m_i}\right)^{p_i+1}(A_i + B_i\xi_i)^{-\frac{2p_i+m_i+1}{1-p_i-m_i}}, \tag{3.12}$$

$$(|f_i'|^{p_i}(f_i^{m_i})')' = m_i B_i^{p_i+2}\left(\frac{2p_i+m_i+1}{1-p_i-m_i}\right)\left(\frac{p_i+2}{1-p_i-m_i}\right)^{p_i+1}(A_i + B_i\xi_i)^{-\frac{p_i+2}{1-p_i-m_i}}. \tag{3.13}$$

Substituting (3.10)–(3.13) into (3.8), then inequalities (3.8) are valid provided that

$$\alpha_i(A_i + B_i\xi_i)^{-\frac{p_i+2}{1-p_i-m_i}} - l_i\xi_i B_i \frac{p_i+2}{1-p_i-m_i}(A_i + B_i\xi_i)^{-\frac{p_i+2}{1-p_i-m_i}-1}$$
$$- m_i B_i^{p_i+2}\left(\frac{2p_i+m_i+1}{1-p_i-m_i}\right)\left(\frac{p_i+2}{1-p_i-m_i}\right)^{p_i+1}(A_i + B_i\xi_i)^{-\frac{p_i+2}{1-p_i-m_i}} \leq 0.$$

By taking B_i to satisfy

$$B_i \geq \left(\frac{\alpha_i(2p_i+m_i+1)}{m_i(1-p_i-m_i)}\left(\frac{1-p_i-m_i}{p_i+2}\right)^{p_i+1}\right)^{\frac{1}{p_i+2}},$$

noticing that $\prod_{i=1}^{k}((p_i+2)/(2p_i+m_i+1)q_i) < 1$(i.e. $\det A < 0$) imply $\alpha_i > 0\ (i=1,2,...,k)$. Therefore, we have shown that (3.8) is true.

On the other hand, the boundary conditions (3.9) are satisfied if we have

$$m_i B_i^{p_i+1}\left(\frac{p_i+2}{1-p_i-m_i}\right)^{p_i+1} A_i^{-\frac{2p_i+m_i+1}{1-p_i-m_i}} \leq A_{i+1}^{-\frac{q_i(p_{i+1}+2)}{1-p_{i+1}-m_{i+1}}}, \quad (i=1,2,...,k), \tag{3.14}$$

where $A_{k+1} := A_1, m_{k+1} := m_1, p_{k+1} := p_1$. In order to prove (3.14), we only need to show that there exist constants $A_i (i = 1,2,...,k)$ such that

$$\rho_i A_{i+1}^{\frac{q_i(p_{i+1}+2)}{1-p_{i+1}-m_{i+1}}} \leq A_i^{\frac{2p_i+m_i+1}{1-p_i-m_i}} \quad (i = 1,2,...,k), \tag{3.15}$$

with $\rho_i = m_i B_i^{p_i+1} \left((p_i+2)/(1-p_i-m_i)\right)^{p_i+1}$, $(i = 1,2,...k)$. To this purpose, we choose

$$A_{i+1} = \rho_i^{-\frac{1-p_{i+1}-m_{i+1}}{q_i(p_{i+1}+2)}} A_i^{\frac{(2p_i+m_i+1)(1-p_i-m_i)}{q_i(p_{i+1}+2)(1-p_i-m_i)}}, \quad \text{for } i = 1,2,...,k-1,$$

and then ensure that for the case $i = k$,

$$A_1 = A_{k+1} \leq \rho_k^{-\frac{1-p_{k+1}-m_{k+1}}{q_k(p_{k+1}+2)}} A_k^{\frac{(2p_k+m_k+1)(1-p_k-m_k)}{q_k(p_{k+1}+2)(1-p_k-m_k)}} = \rho_0 A_1^{\prod_1^k \frac{2p_i+m_i+1}{q_i(p_i+2)}}, \tag{3.16}$$

for some constant ρ_0 which is independent of $A_i (i = 1,2,...,k)$, clearly inequality (3.16) is true under the assumption $\prod_{i=1}^k (2p_i+m_i+1)/(q_i(p_i+2)) < 1$ and A_1 small enough. Thus the condition $\prod_{i=1}^k (2p_i+m_i+1)/(q_i(p_i+2)) < 1$ ensures that we can take A_i small enough such that inequalities (3.14) hold. Therefore, we have proved our claim. Then we have obtained (3.9).

Thus, $(\underline{u}_1, \underline{u}_2, ..., \underline{u}_k)$ given by (3.7) and (3.10) is a subsolution of system (1.1)–(1.3) with appropriately large initial data. By the comparison principle, which implies that the solution $(u_1, u_2, ..., u_k)$ of the system (1.1)–(1.3) with large initial data blow up in a finite time. The proof of Theorem 1.1 is complete. ∎

4. Critical Fujita curve

We devote this section to the proof of Theorem 1.2. That is, we shall show when all solutions of the system (1.1)–(1.3) blow up in a finite time or both global and non-global solutions exist.

Proof of Theorem 1.2

(1). We investigate the auxiliary functions

$$\overline{u}_i(x,t) = (\tau+t)^{-\alpha_i} F_i(\xi_i), \quad \xi_i = x(\tau+t)^{-l_i}, \ (i = 1,2,...,k) \tag{4.1}$$

where τ is a positive constant, $F_i(\xi_i)(i = 1,2,...,k)$ are to be determined later. By a direct computation, we obtain

$$\overline{u}_{it} = (\tau+t)^{-(\alpha_i+1)}(-\alpha_i F_i(\xi_i) - l_i \xi_i F_i'(\xi_i)),$$
$$|\overline{u}_{ix}|^{p_i}(\overline{u}_i^{m_i})_x = (\tau+t)^{-p_i\alpha_i - p_i l_i - m_i\alpha_i - l_i} |F_i'|^{p_i} (F_i^{m_i})'(\xi_i),$$
$$(|\overline{u}_x|^p(\overline{u}_i^{m_i})_x)_x = (\tau+t)^{-p_i\alpha_i - p_i l_i - m_i\alpha_i - 2l_i} (|F_i'|^{p_i}(F_i^{m_i})'(\xi_i))',$$

and

$$|\overline{u}_{ix}|^{p_i}(\overline{u}_i^{m_i})_x(0,t) = (\tau+t)^{-p_i\alpha_i - p_i l_i - m_i\alpha_i - l_i} |F_i'|^{p_i}(F_i^{m_i})'(0),$$
$$\overline{u}_{i+1}^{q_i}(0,t) = (\tau+t)^{-q_i\alpha_{i+1}} F_{i+1}{}^{q_i}(0).$$

It will be obtained from the above equalities and (1.9) that

$$\overline{u}_{it} \geq (|\overline{u}_x|^p(\overline{u}_i^{m_i})_x)_x, \quad x \geq 0, \ t > 0, \ (i = 1,2,...,k),$$
$$-|\overline{u}_{ix}|^{p_i}(\overline{u}_i^{m_i})_x \geq \overline{u}_{i+1}^{q_i}(0,t), \quad t > 0, \ (i = 1,2,...,k), \ u_{k+1} = u_k.$$

if the functions $F_i(\xi_i)(i=1,2,...,k)$ satisfy

$$(|F_i'|^{p_i}(F_i^{m_i})'(\xi_i))' + \alpha_i F_i(\xi_i) + l_i F'(\xi_i)\xi_i \le 0, \quad (i=1,2,...,k), \tag{4.2}$$

$$-|F_i'|^{p_i}(F_i^{m_i})'(0) \ge F_{i+1}{}^{q_i}(0), \quad t>0,\ (i=1,2,...,k),\ F_{k+1}=F_1. \tag{4.3}$$

Take

$$F_i(\xi_i) = H_i\left((a_ib_i)^{\frac{p_i+2}{p_i+1}} + (\xi_i+a_i)^{\frac{p_i+2}{p_i+1}}\right)^{-\frac{p_i+1}{1-p_i-m_i}}, \quad (i=1,2,...,k) \tag{4.4}$$

with $b_i>0, H_i>0, a_i>0, (i=1,2,...,k)$ to be determined. After a computation, we obtain

$$F_i'(\xi_i) = -H_i\frac{p_i+2}{1-p_i-m_i}\left((a_ib_i)^{\frac{p_i+2}{p_i+1}} + (\xi_i+a_i)^{\frac{p_i+2}{p_i+1}}\right)^{-\frac{p_i+1}{1-p_i-m_i}-1}(\xi_i+a)^{\frac{1}{p_i+1}},$$

$$|F_i'|^{p_i}(F_i^{m_i})' = -m_iH_i^{p_i+m_i}\left(\frac{p_i+2}{1-p_i-m_i}\right)^{p_i+1}\left((a_ib_i)^{\frac{p_i+2}{p_i+1}} + (\xi_i+a_i)^{\frac{p_i+2}{p_i+1}}\right)^{-\frac{p_i+1}{1-p_i-m_i}} \times(\xi_i+a_i),$$

$$(|F_i'|^{p_i}(F_i^{m_i})')' = -m_iH_i^{p_i+m_i}\left(\frac{p_i+2}{1-p_i-m_i}\right)^{p_i+1}\left((a_ib_i)^{\frac{p_i+2}{p_i+1}} + (\xi_i+a_i)^{\frac{p_i+2}{p_i+1}}\right)^{-\frac{p_i+1}{1-p_i-m_i}}$$
$$+m_iH_i^{p_i+m_i}\left(\frac{p_i+2}{1-p_i-m_i}\right)^{p_i+2}\left((a_ib_i)^{\frac{p_i+2}{p_i+1}} + (\xi_i+a_i)^{\frac{p_i+2}{p_i+1}}\right)^{-\frac{p_i+1}{1-p_i-m_i}-1} \times(\xi_i+a_i)^{\frac{p_i+2}{p_i+1}},$$

substituting above equalities into (4.2), let $y_i=\xi_i+a_i(i=1,2,...,k)$, then (4.2) can be transformed into the following inequality with respect y_i

$$G_i(y_i) = -e_{i1}y_i^{\frac{p_i+2}{p_i+1}} + e_{i2}a_iy_i^{\frac{1}{p_i+1}} - e_{i3}(a_ib_i)^{\frac{p_i+2}{p_i+1}} \le 0, \tag{4.5}$$

where

$$e_{i1} = m_iH_i^{p_i+m_i-1}\left(\frac{p_i+2}{1-p_i-m_i}\right)^{p_i+1} - H_i\alpha_i + l_iH_i\frac{p_i+2}{1-p_i-m_i} - m_iH_i^{p_i+m_i-1}\left(\frac{p_i+2}{1-p_i-m_i}\right)^{p_i+2},$$

$$e_{i2} = l_iH_i\frac{p_i+2}{1-p_i-m_i},$$

$$e_{i3} = m_iH_i^{p_i+m_i-1}\left(\frac{p_i\alpha_i+2}{1-p_i-m_i}\right)^{p_i+1} - H_i\alpha_i.$$

We only prove (4.5) for the case of $i=1$, and the others can be get in a similar way. Since $\min_i\{l_i-\alpha_i\}>0$ imply $l_1>\alpha_1>0$ we can choose a suitable constant $H_1>0$ such that $l_1 > m_1H_1^{p_1+m_1-1}((p_1+2)/(1-p_1-m_1))^{p_1+1} > \alpha_1 > 0$, for such H_1, it is easy to verify that $e_{11}>0, e_{12}>0, e_{13}>0$ and $G_1(y_1)$ is a concave function with respect to $y_1^{1/(p_1+1)}$, then $G_1(y_1)$ attains its maximum at $y_{1*}=(e_{12}a_1)/((p_1+2)e_{11})$. Therefore, the inequality (4.5) for $i=1$ is valid provided that

$$G_1(y_{1*}) = a_1^{\frac{p_1+2}{p_1+1}} \left(\frac{p_1+1}{p_1+2} \left(\frac{1}{e_{11}(p_1+2)} \right)^{\frac{1}{p_1+1}} e_{12}^{\frac{p_1+2}{p_1+1}} - e_{13} b_1^{\frac{p_1+2}{p_1+1}} \right) \le 0. \tag{4.6}$$

So, we only need to choose b_1 sufficiently large such that

$$b_1 \ge \left(\frac{(p_1+1)e_{12}}{(p_1+2)e_{13}} \right)^{\frac{p_1+1}{p_1+2}} \left(\frac{e_{12}}{(p_1+2)e_{11}} \right)^{\frac{1}{p_1+2}}.$$

Similarly, there exist $H_i > 0, b_i > 0$ $(i = 2,3,...,k)$ such that the inequalities (4.5) hold. Consequently, we have proved that inequalities (4.5) are true.

Finally, define

$$D_i = m_i H_i^{p_i+m_i} \left(\frac{p_i+2}{1-p_i-m_i} \right)^{p_i+1} \left(b_i^{\frac{p_i+2}{p_i+1}} + 1 \right)^{-\frac{p_i+1}{1-p_i-m_i}}, \quad (i = 1,2,...,k),$$

$$E_i = H_{i+1}^{q_i} \left(b_{i+1}^{\frac{p_{i+1}+2}{p_{i+1}+1}} + 1 \right)^{-\frac{q_i(p_{i+1}+1)}{1-p_{i+1}-m_{i+1}}}, \quad H_{k+1} = H_1,\ m_{k+1} = m_1,\ p_{k+1} = p_1,\ b_{k+1} = b_1.$$

And noting $\prod_{i=1}^{k}((p_i+2)/(2p_i+m_i+1)q_i) > 1$(i.e., $\det A < 0$), we choose a_i large enough such that

$$D_i a_i^{-\frac{2p_i+m_i+1}{1-p_i-m_i}} \ge E_i a_{i+1}^{-\frac{q_i(p_{i+1}+2)}{1-p_{i+1}-m_{i_1}}}, \quad (i = 1,2,...,k),$$
$$D_{k+1} = D_1, \quad E_{k+1} = E_1, \quad m_{k+1} = m_1, \quad p_{k+1} = p_1, \quad a_{k+1} = a_1$$

which implies that the inequalities (4.3) hold. Thus, for the case $\min_i\{l_i - \alpha_i\} > 0$, we have constructed a class of global self-similar supersolutions defined by (4.1) and (4.4). Owing to the comparison principle, the solution of the problem (1.1)–(1.3) is global if the initial datum $(u_{10}, u_{20}, ..., u_{k0})$ is small enough. ∎

Now we turn our attention to the blow-up results for any initial data, and begin with the space decay behavior of the solution to the system (1.1)–(1.3), which play an important role in the proof of Theorem 1.2 (2).

Lemma 4.1. *The positive solution of the problem* (1.1)–(1.3) *has, for each* $t \in (0,T)$,

$$\liminf_{x\to+\infty} x^{\frac{p_i+2}{1-m_i-p_i}} u_i(x.t) \ge \left(C_{m_i,p_i}^{-(p_i+1)} \right)^{\frac{1}{1-m_i-p_i}}, \quad (i = 1,2,...,k), \tag{4.7}$$

where T *is the maximal existence time for the solution, which may be finite or infinite, and*

$$C_{m_i,p_i} = \frac{1-m_i-p_i}{p_i+2} \left(\frac{1}{m(2p_i+m_i+1)} \right)^{\frac{1}{p_i+1}}, \quad (i = 1,2,...,k). \tag{4.8}$$

Proof. We only prove (4.7) for the case of $i=1$, and the others can be get in a similar way. Our idea is to show that any positive solution of the problem (1.1)–(1.3) is, for x large, bigger than the following similarity solution

$$U_\lambda(t,x) = \lambda^{\frac{p_1+2}{1-m_1-p_1}} U_1(t, \lambda x),$$

where

$$U_1(t,x) = t^{-\frac{1}{2p_1+m_1+1}}\left(1 + C_{m_1,p_1} x^{\frac{p_1+2}{p_1+1}} t^{-\frac{p_1+2}{(2p_1+m_1+1)(p_1+1)}}\right)^{\frac{p_1+1}{1-p_1-m_1}},$$

Let $0 < \tau < T_* < T$ and $S = [\tau, T_*] \times (1, +\infty)$, Since the positive solution $u_1(x,t)$ is continuous in $(0, T_*] \times [0, +\infty)$, there exists $\delta = \delta(\tau, T_*) > 0$ such that

$$\delta = min\, u_1(x,t), \quad \tau \le t \le T_*, \quad 0 \le x \le 1. \tag{4.9}$$

We now select $\gamma > 0$ such that

$$U_\lambda(t-\tau, x) \le \delta, \quad \tau \le t \le T_*, \quad x \ge \frac{1}{2}. \tag{4.10}$$

To this aim, according to the definition of $U_\lambda(t,x)$ we need

$$\lambda^{\frac{p_1+2}{1-m_1-p_1}}(t-\tau)^{-\frac{1}{2p_1+m_1+1}}\left(1 + C_{m_1,p_1}\lambda^{\frac{p_1+2}{p_1+1}} x^{\frac{p_1+2}{p_1+1}}(t-\tau)^{-\frac{p_1+2}{(2p_1+m_1+1)(p_1+1)}}\right)^{-\frac{p_1+1}{1-p_1-m_1}} \le \delta,$$

or

$$\delta^{\frac{m_1+p_1-1}{p_1+1}} \le \lambda^{-\frac{p_1+2}{p_1+1}}(t-\tau)^{\frac{1-p_1-m_1}{(2p_1+m_1+1)(p_1+1)}} + C_{m_1,p_1} x^{\frac{p_1+2}{p_1+1}}(t-\tau)^{-\frac{1}{p_1+1}},$$

for $\tau \le t \le T_*$ and $x \ge 1/2$, which is implied by

$$\delta^{\frac{m_1+p_1-1}{p_1+1}} \le \lambda^{-\frac{p_1+2}{p_1+1}}(t-\tau)^{\frac{1-m_1-p_1}{(2p_1+m_1+1)(p_1+1)}} + C_{m_1,p_1}\left(\frac{1}{2}\right)^{\frac{p_1+2}{p_1+1}}(t-\tau)^{-\frac{1}{p_1+1}}. \tag{4.11}$$

Since the right-hand side of (4.11) is bounded below by $\lambda^{-(2p_1+m_1+1)/(p_1+1)}c$, where $c = c(m_1, p_1) > 0$, the inequality (4.13) is satisfied if we choose λ such that $\lambda \le c\delta^{(1-m_1-p_1)/(2p_1+m_1+1)}$. Since $\partial U_\lambda/\partial t = \partial/\partial x\left(|\partial U_\lambda/\partial x|^{p_1}\,\partial U_\lambda^{m_1}/\partial x\right)$ in S and $U_\lambda(t-\tau) = 0$ for $t = \tau, x \ge 1$, by (4.9), (4.10) and the comparison principle we have

$$U_\lambda(t-\tau, x) \le u_1(x,t), \quad \tau < t < T_*, \quad x \ge 1.$$

Hence

$$\liminf_{x\to+\infty} x^{\frac{p_1+2}{1-m_1-p_1}} u_1(x,t) \ge \liminf_{x\to+\infty} x^{\frac{p_1+2}{1-m_1-p_1}} U_\lambda(t-\tau, x) = [C_{m_1,p_1}^{-(p_1+1)}(t-\tau)]^{\frac{1}{1-p_1-m_1}}, \tag{4.12}$$

since the right-hand side of (4.12) does not depend on λ, the estimate (4.7) holds by letting τ tend to 0 and T_* tend to T. ∎

Proof of Theorem 1.2

(2). Without loss of generality, we first assume that $u_i (i = 1, 2, ..., k)$ are nonincreasing in x, for if not we consider the (nonincreasing in x) solution $(\omega_1, \omega_2, ..., \omega_k)$ corresponding to the initial value $(\omega_{10}(x), \omega_{20}(x), ..., \omega_{k0}(x)), \omega_{i0}(x) = \inf\{u_{i0}(y), 0 \le y \le x\}, (i = 1, 2, ..., k)$ which are nonincreasing in x. If $(\omega_1, \omega_2, ..., \omega_k)$ blows up in finite time, so does $(u_1, u_2, ..., u_k)$. On the other hand, for every $\varepsilon > 0$ and $t_0 > 0$ fixed, by Lemma 4.1, there exists a constant $M > 0$ large enough that

$$u_i(x, t_0) \ge \left(\frac{(C_{m_i,p_i} + \varepsilon) x^{\frac{p_i+2}{p_i+1}}}{t_0^{\frac{1}{p_i+1}}}\right)^{-\frac{p_i+1}{1-m_i-p_i}}, \quad (i = 1, 2, ..., k), \text{ for } x \ge M,$$

and

$$u_i(x, t_0) \ge u_i(M, t_0), \quad (i = 1, 2, ..., k), \text{ for } 0 \le x \le M.$$

Now we construct the following well-known self-similar solution (the so-called Zel'dovich-Kompaneetz-Barenblatt profile [8, 19]) to (1.1)–(1.3) in the form

$$u_{iB}(x,t) = (\tau+t)^{-\frac{1}{m_i+2p_i+1}} h_i(\xi_i), \quad \xi_i = x(\tau+t)^{-\frac{1}{m_i+2p_i+1}}, \quad (i=1,2,...,k), \tag{4.13}$$

$$h_i(\xi_i) = \left(b_i^{\frac{p_i+2}{p_i+1}} + C_{m_i,p_i} \xi_i^{\frac{p_i+2}{p_i+1}} \right)^{-\frac{p_i+1}{1-p_i-m_i}}, \quad (i=1,2,...,k) \tag{4.14}$$

with $\tau > 0$, $b_i > 0$ and C_{m_i,p_i} is given in (4.8). It is not difficult to check that

$$(|h_i'|^{p_i}(h_i^{m_i})')'(\xi_i) + \frac{1}{m_i+2p_i+1}\xi_i h_i'(\xi_i) + \frac{1}{m_i+2p_i+1} h_i(\xi_i) = 0, \quad h_i'(0) = 0, \ (i=1,2,...,k).$$

combining with $h_i'(0) = 0$ $(i = 1,2,...,k)$, implies $(u_{iB})_x(0,t) = 0$, $(i = 1,2,...,k)$. Since $u_i(x,t)(i = 1,2,...,k)$ are nontrivial and nonnegative, we see that $u_i(0,t_0) > 0$ $(i = 1,2,...,k)$ for some $t_0 > 0$ (compare with a Barenblatt solution of the corresponding equations). Noticing that $u_i(x,t_0) > 0$ $(i = 1,2,...,k)$ are continuous (see [9, 26]), we can choose τ large enough and $b_i > 0$ small enough that

$$u_i(x,t_0) > u_{iB}(x,t_0), \quad (i=1,2,...,k) \text{ for } x > 0.$$

A direct calculation shows that $(u_{1B}, u_{2B}, ..., u_{kB})$ is a weak subsolution of (1.1)–(1.3) in $(0,+\infty) \times (t_0,+\infty)$ By the comparison principle, we obtain that

$$u_i(x,t) > u_{iB}(x,t), \quad (i=1,2,...,k) \text{ for } x > 0, t > t_0.$$

Since that $\max_i\{l_i - \alpha_i\} < 0$, we get $T^{l_i} \ll T^{\alpha_i}$ for large T. So there exists $t* \geq t_0$ satisfying

$$T^{l_i} \ll (\tau+t^*)^{\frac{1}{m_i+2p_i+1}} \ll T^{\alpha_i}, \qquad (i=1,2,...,k). \tag{4.15}$$

Let $\underline{u}_i (i = 1,2,...,k)$ be the functions given by (3.7) and (3.10). Then for any $x > 0$,

$$\underline{u}_i(x,0) \leq u_{iB}(x,t^*) \leq u_i(x,t^*), \quad (i=1,2,...,k).$$

It follows from the comparison principle that

$$\underline{u}_i(x,t) \leq u_i(x,t+t^*) \quad (i=1,2,...,k), \text{ for } x > 0, \quad t > 0.$$

As the proof of Theorem 1.1 (2), we see that $(\underline{u}_1, \underline{u}_2, ..., \underline{u}_k)$ blows up in a finite time T. Therefore, $(u_1, u_2, ..., u_k)$ blows up in a finite time which is not larger than $T+t^*$. Observing that (4.15) holds for general nontrivial $(u_{10}, u_{20}, ..., u_{k0})$, we know that every nonnegative, nontrivial solution of (1.1)–(1.3) blows up in finite time. The proof of Theorem 1.2 is complete. ∎

Acknowledgement. This work was partially supported by NSF of China (11071266), partially supported by Scholarship Award for Excellent Doctoral Student granted by Ministry of Education and partially supported by the Educational Science Foundation of Chongqing China (KJ121302).

References

[1] H. W. Alt and S. Luckhaus, Quasilinear elliptic-parabolic differential equations, *Math. Z.* **183** (1983), no. 3, 311–341.

[2] G. Astrita and G. Marrucci, *Principles of Non-Newtonian Fluid Mechanics*, McGraw-Hill, New York, 1974.

[3] K. Deng and H. A. Levine, The role of critical exponents in blow-up theorems: The sequel, *J. Math. Anal. Appl.* **243** (2000), no. 1, 85–126.

[4] R. Ferreira, A. de Pablo, F. Quiros and J. D. Rossi, The blow-up profile for a fast diffusion equation with a nonlinear boundary condition, *Rocky Mountain J. Math.* **33** (2003), no. 1, 123–146.
[5] M. Fila and P. Quittner, The blow-up rate for a semilinear parabolic system, *J. Math. Anal. Appl.* **238** (1999), no. 2, 468–476.
[6] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. I* **13** (1966), 109–124 (1966).
[7] V. A. Galaktionov and H. A. Levine, A general approach to critical Fujita exponents in nonlinear parabolic problems, *Nonlinear Anal.* **34** (1998), no. 7, 1005–1027.
[8] V. A. Galaktionov and H. A. Levine, On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary, *Israel J. Math.* **94** (1996), 125–146.
[9] A. V. Ivanov, Hölder estimates for quasilinear doubly degenerate parabolic equations, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **171** (1989), Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 20, 70–105, 185; translation in *J. Soviet Math.* **56** (1991), no. 2, 2320–2347.
[10] Z. X. Jiang and S. N. Zheng, *Doubly Degenerate Paralolic Equation with Nonlinear Inner Sources or Boundary Flux*, PhD Thesis, Dalian University of Tcchnology, In China, 2009.
[11] A. S. Kalashnikov, On a nonlinear equation appearing in the theory of non-stationary filtration, *Trudy Seminara I. G. Petrovski*, 1978 (in Russian).
[12] H. A. Levine, The role of critical exponents in blowup theorems, *SIAM Rev.* **32** (1990), no. 2, 262–288.
[13] Z. Li and C. Mu, Critical exponents for a fast diffusive polytropic filtration equation with nonlinear boundary flux, *J. Math. Anal. Appl.* **346** (2008), no. 1, 55–64.
[14] Y. S. Mi, C. L. Mu and B. T. Chen, Blow-up analysis for a fast diffusive parabolic equation with nonlinear boundary flux (preprint).
[15] Z. Lin, Blowup behaviors for diffusion system coupled through nonlinear boundary conditions in a half space, *Sci. China Ser. A* **47** (2004), no. 1, 72–82.
[16] M. Pedersen and Z. Lin, Blow-up analysis for a system of heat equations coupled through a nonlinear boundary condition, *Appl. Math. Lett.* **14** (2001), no. 2, 171–176.
[17] P. Quittner and P. Souplet, *Superlinear Parabolic Problems*, Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser, Basel, 2007.
[18] F. Quirós and J. D. Rossi, Blow-up sets and Fujita type curves for a degenerate parabolic system with nonlinear boundary conditions, *Indiana Univ. Math. J.* **50** (2001), no. 1, 629–654.
[19] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov, *Blow-Up in Quasilinear Parabolic Equations*, translated from the 1987 Russian original by Michael Grinfeld and revised by the authors, *de Gruyter Expositions in Mathematics*, **19**, de Gruyter, Berlin, 1995.
[20] J. L. Vazquez, *The Porous Medium Equations: Mathematical Theory*, Oxford University Press, Oxford, 2007.
[21] M. Wang, Blow-up rate for a semilinear reaction diffusion system, *Comput. Math. Appl.* **44** (2002), no. 5–6, 573–585.
[22] M. Wang, The blow-up rates for systems of heat equations with nonlinear boundary conditions, *Sci. China Ser. A* **46** (2003), no. 2, 169–175.
[23] S. Wang, C. Xie and M. Wang, Note on critical exponents for a system of heat equations coupled in the boundary conditions, *J. Math. Anal. Appl.* **218** (1998), no. 1, 313–324.
[24] S. Wang, C. Xie and M. Wang, The blow-up rate for a system of heat equations completely coupled in the boundary conditions, *Nonlinear Anal.* **35** (1999), no. 3, Ser. A: Theory Methods, 389–398.
[25] Z. Wang, J. Yin and C. Wang, Critical exponents of the non-Newtonian polytropic filtration equation with nonlinear boundary condition, *Appl. Math. Lett.* **20** (2007), no. 2, 142–147.
[26] Z. Q. Wu, J. N. Zhao, J. X. Yin and H. L. Li, *Nonlinear Diffusion Equations*, translated from the 1996 Chinese original and revised by the authors, World Sci. Publishing, River Edge, NJ, 2001.
[27] J. Yin and C. Jin, Critical extinction and blow-up exponents for fast diffusive p-Laplacian with sources, *Math. Methods Appl. Sci.* **30** (2007), no. 10, 1147–1167.
[28] S. Zheng, X. Song and Z. Jiang, Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux, *J. Math. Anal. Appl.* **298** (2004), no. 1, 308–324.
[29] J. Zhou and C. Mu, The critical curve for a non-Newtonian polytropic filtration system coupled via nonlinear boundary flux, *Nonlinear Anal.* **68** (2008), no. 1, 1–11.
[30] J. Zhou and C. Mu, On the critical Fujita exponent for a degenerate parabolic system coupled via nonlinear boundary flux, *Proc. Edinb. Math. Soc. (2)* **51** (2008), no. 3, 785–805.