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Convergence Ball Analysis of a Modified Newton's Method Under Hölder Continuous Condition in Banach Space

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Abstract. A modified Newton's method which computes derivatives every other step is used to solve a nonlinear operator equation. An estimate of the radius of its convergence ball is obtained under Hölder continuous Fréchet derivatives in Banach space. An error analysis is given which matches its convergence order.

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1. Introduction

In this study we are concerned with estimating the radius of the convergence ball of a modified Newton's method which computes derivatives every other step in Banach space and is used to solve the nonlinear operator equation

$$F(x) = 0, \tag{1.1}$$

where F is defined on an open convex subset D of a Banach space X with values in a Banach space Y.

There are kinds of method to find a solution of (1.1). Iterative methods are often used to solve this problem. If we use the famous Newton's method (see [16]), we can do as

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad (n \geq 0)\ (x_0 \in D). \tag{1.2}$$

To improve the convergence order, many modified methods have been presented (see [6, 7, 11, 12, 15, 16, 23, 26]). Among these, a modified Newton's method which compute derivatives every other step is one of the most popular methods and is defined as follows

$$y_n = x_n - F'(x_n)^{-1}F(x_n), \quad x_{n+1} = y_n - F'(x_n)^{-1}F(y_n), \quad (n \geq 0)\ (x_0 \in D). \tag{1.3}$$

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Two benefits of this method bring us: it computes less derivatives and less inverses of derivatives. Therefore, the method is valuable especially as the computing cost of derivatives or inverses of derivatives is big.

The convergence of (1.3) to a solution of (1.1) has been studied by other authors [3–5, 16, 21]. Reference [21] discussed the point estimates on the modified Newton's method. References [3–5] established some semilocal convergence theorems using recurrence relations for method (1.3), and gave an abundant of application for solving boundary value problems. Now we consider the problem from the different way. We suppose that the nonlinear operator equation (1.1) has a solution $x_\star$. An interesting problem is to estimate the radius of the convergence ball of this modified Newton's method. An open ball $B(x_\star, r) \subset X$ with center $x_\star$ and radius r is called a convergence ball of an iterative method, if the sequence generated by this iterative method starting from any initial values in it converges. The convergence ball of an iterative method is very important, because it shows the extent of difficulty to choose initial points for the iterative methods.

References [20] and [22] gave an exact estimate of radius of the convergence ball $r = 2/(3K)$ respectively for Newton's method (1.2) under the following Lipschitz continuous condition

$$\|F'(x_\star)^{-1}(F'(x) - F'(y))\| \le K\|x - y\|, \quad \forall\, x, y \in D,\ for\ some\ K > 0. \tag{1.4}$$

Reference [14] generalized this result for Newton's method to a type of Hölder continuous condition. Reference [2] obtained the estimate of radius of Newton's method using Lipschitz type assumptions on the second Fréchet derivative. References [13, 17, 18, 24, 25] gave the estimates of radius for the Secant method, a modified secant method, a deformed Newton's method and Müller's Method.

In this study, under the following Hölder continuous condition

$$\|F'(x_\star)^{-1}(F'(x) - F'(y))\| \le K\|x - y\|^p, \quad \forall\, x, y \in D,\ for\ some\ K > 0, \tag{1.5}$$

where $0 < p \le 1$, the radius of the convergence ball of the modified Newton's method (1.3) is proved to be $\sqrt[p]{s_p/K}$ at least, where s_p is the minimum positive root of a cubic equation. The error analysis is also given which shows the convergence order of the modified Newton's method (1.3) is at least $1 + 2p$.

Throughout the paper we denote $B(x, r) = \{y \in X; \|y - x\| < r\}$ and $\overline{B(x, r)} = \{y \in X; \|y - x\| \le r\}$.

2. Convergence ball and error analysis

In this section, we give the radius of convergence and error analysis of the modified Newton's method.

Theorem 2.1. *Suppose F has Fréchet derivatives on an open convex subset D of a Banach space X with values in a Banach space Y, $x_\star \in D$, $F(x_\star) = 0$, $F'(x_\star)^{-1}$ exists, the Hölder continuous condition (1.5) holds, $0 < p \le 1$, and $B(x_\star, \sqrt[p]{(1+p)/K}) \subseteq D$. Suppose s_p is the minimum positive root of the following cubic equation:*

$$h_p(s) = 2(1+p)^2(1-s)^3 + (p+2)s^3 - 2(1+p)s^2 = 0. \tag{2.1}$$

Denote $r_p = \sqrt[p]{s_p/K}$, then the sequence $\{x_n\}$ generated by the modified Newton's method (1.3) *starting from any initial point x_0 in $B(x_\star, r_p)$ converges to the unique solution $x_\star$ in*

$B(x_\star, \sqrt[p]{(1+p)/K})$ that is bigger than $B(x_\star, r_p)$. Moreover, the following error estimate is satisfied

$$\|x_n - x_\star\| \le r_p \left(\frac{\|x_0 - x_\star\|}{r_p}\right)^{(1+2p)^n}, \quad (n \ge 0). \tag{2.2}$$

We need following lemmas in order to prove the above theorem.

Lemma 2.1. *Let $0 < p \le 1$, then equation (2.1) always has at least one positive root, and the minimum positive root s_p of the equation satisfies: $0 < s_p < 1$.*

Proof. As $0 < p \le 1$, we have

$$h_p(0) = 2(1+p)^2 > 0, \tag{2.3}$$

$$h_p(1) = p + 2 - 2(1+p) = -p < 0. \tag{2.4}$$

Hence, $h_p(s)$ has at least a zero in $(0,1)$. By the definition of s_p, we have $0 < s_p < 1$, which completes the proof. ∎

Lemma 2.2. *Let us suppose $x_\star \in D$, $F'(x_\star)^{-1}$ exists, the Hölder continuous condition (1.5) holds, $0 < p \le 1$, and $B(x_\star, \sqrt[p]{\frac{1}{K}}) \subseteq D$. Then for any $x \in B(x_\star, \sqrt[p]{1/K})$, $F'(x)$ is invertible, and the following estimate holds*

$$\|(F'(x_\star)^{-1}F'(x))^{-1}\| \le \frac{1}{1 - K\|x - x_\star\|^p}. \tag{2.5}$$

Proof. We consider to estimate $\|I - F'(x_\star)^{-1}F'(x)\|$. From (1.5) we have

$$\|I - F'(x_\star)^{-1}F'(x)\| = \|F'(x_\star)^{-1}(F'(x) - F'(x_\star))\| \le K\|x - x_\star\|^p. \tag{2.6}$$

Hence, by the Banach lemma, this lemma holds. ∎

Lemma 2.3. *For the modified Newton's method (1.3), if x_n is well defined, and n is a non-negative integer, $F'(x_n)^{-1}$ exists, $x_\star \in D$, $F(x_\star) = 0$ and $F'(x_\star)^{-1}$ exists. Then, we have the following formula:*

$$\begin{aligned} x_{n+1} - x_\star = {}& \left(F'(x_\star)^{-1}F'(x_n)\right)^{-1} F'(x_\star)^{-1} \left(F'(x_n) - \int_0^1 F'(ty_n + (1-t)x_\star)dt\right) \\ & \left(F'(x_\star)^{-1}F'(x_n)\right)^{-1} F'(x_\star)^{-1} \left(F'(x_n) - \int_0^1 F'(tx_n + (1-t)x_\star)dt\right)(x_n - x_\star). \end{aligned} \tag{2.7}$$

Proof. By (1.3) and the fundamental theorem of calculus, we get

$$\begin{aligned} x_{n+1} - x_\star &= y_n - x_\star - F'(x_n)^{-1}(F(y_n) - F(x_\star)) \\ &= F'(x_n)^{-1}\left(F'(x_n) - \int_0^1 F'(ty_n + (1-t)x_\star)dt\right)(y_n - x_\star) \\ &= F'(x_n)^{-1}\left(F'(x_n) - \int_0^1 F'(ty_n + (1-t)x_\star)dt\right)\left(x_n - x_\star - F'(x_n)^{-1}F(x_n)\right) \\ &= F'(x_n)^{-1}\left(F'(x_n) - \int_0^1 F'(ty_n + (1-t)x_\star)dt\right)F'(x_n)^{-1}(F'(x_n) \\ &\quad - \int_0^1 F'(tx_n + (1-t)x_\star)dt)(x_n - x_\star). \end{aligned} \tag{2.8}$$

By (2.8), it is obvious that (2.7) holds. ∎

Lemma 2.4. *For the modified Newton's method* (1.3)*, if x_n is well defined, and n is a non-negative integer,* $0 < p \le 1$*,* $x_\star \in D$*,* $x_n \in B(x_\star, \sqrt[p]{1/K}) \subseteq D$*,* $F(x_\star) = 0$*,* $F'(x_\star)^{-1}$ *exists and the Hölder continuous condition* (1.5) *holds, then the following estimate holds:*

$$\left\| F'(x_\star)^{-1}\left(F'(x_n) - \int_0^1 F'(tx_n + (1-t)x_\star)dt\right)\right\| \le \frac{K\|x_n - x_\star\|^p}{1+p}, \tag{2.9}$$

$$\begin{aligned}&\left\| F'(x_\star)^{-1}\left(F'(x_n) - \int_0^1 F'(ty_n + (1-t)x_\star)dt\right)\right\| \\ &\le K\|x_n - x_\star\|^p \left(1 + \frac{pK\|x_n - x_\star\|^p}{2(1+p)(1-K\|x_n - x_\star\|^p)}\right).\end{aligned} \tag{2.10}$$

Proof. By the Hölder continuous condition (1.5), we have

$$\begin{aligned}&\left\| F'(x_\star)^{-1}\left(F'(x_n) - \int_0^1 F'(tx_n + (1-t)x_\star)dt\right)\right\| \\ &= \left\| \int_0^1 F'(x_\star)^{-1}(F'(x_n) - F'(tx_n + (1-t)x_\star))dt \right\| \\ &\le K\int_0^1 \|x_n - tx_n - (1-t)x_\star\|^p dt = K\|x_n - x_\star\|^p \int_0^1 (1-t)^p dt = \frac{K\|x_n - x_\star\|^p}{1+p}.\end{aligned} \tag{2.11}$$

So, (2.9) holds. Analogously, we have

$$\begin{aligned}&\left\| F'(x_\star)^{-1}\left(F'(x_n) - \int_0^1 F'(ty_n + (1-t)x_\star)dt\right)\right\| \\ &= \left\| \int_0^1 F'(x_\star)^{-1}(F'(x_n) - F'(ty_n + (1-t)x_\star))dt \right\| \\ &\le K\int_0^1 \|x_n - ty_n - (1-t)x_\star\|^p dt.\end{aligned} \tag{2.12}$$

Based on the definition of the modified Newton's method, we get

$$\begin{aligned}&x_n - ty_n - (1-t)x_\star = x_n - x_\star - t(y_n - x_\star) \\ &= x_n - x_\star - t(x_n - x_\star - F'(x_n)^{-1}F(x_n)) \\ &= x_n - x_\star - tF'(x_n)^{-1}\left(F'(x_n)(x_n - x_\star) - \int_0^1 F'(tx_n + (1-t)x_\star)dt(x_n - x_\star)\right) \\ &= x_n - x_\star - tF'(x_n)^{-1}\left(F'(x_n) - \int_0^1 F'(tx_n + (1-t)x_\star)dt\right)(x_n - x_\star).\end{aligned} \tag{2.13}$$

Since $x_n \in B(x_\star, \sqrt[p]{1/K})$, from Lemma 2.2, $F'(x_n)^{-1}$ exists and

$$\|(F'(x_\star)^{-1}F'(x_n))^{-1}\| \le \frac{1}{1 - K\|x_n - x_\star\|^p}. \tag{2.14}$$

By (2.13), (2.14) and (2.9), we get

$$\begin{aligned}\|x_n - ty_n - (1-t)x_\star\| &\le \|x_n - x_\star\| + t\frac{\|x_n - x_\star\|}{1 - K\|x_n - x_\star\|^p}\frac{K\|x_n - x_\star\|^p}{1+p} \\ &= \|x_n - x_\star\|\left(1 + t\frac{K\|x_n - x_\star\|^p}{(1+p)(1-K\|x_n - x_\star\|^p)}\right).\end{aligned} \tag{2.15}$$

Now by (2.12) and (2.15), we have

$$\begin{aligned}
&\left\| F'(x_\star)^{-1}\left(F'(x_n)-\int_0^1 F'(ty_n+(1-t)x_\star)dt\right)\right\| \\
&\le K\|x_n-x_\star\|^p\int_0^1\left(1+t\frac{K\|x_n-x_\star\|^p}{(1+p)(1-K\|x_n-x_\star\|^p)}\right)^p dt \qquad (2.16)\\
&=(1-K\|x_n-x_\star\|^p)\left(\left(1+\frac{K\|x_n-x_\star\|^p}{(1+p)(1-K\|x_n-x_\star\|^p)}\right)^{p+1}-1\right).
\end{aligned}$$

Define a function $g(v)=v^{p+1}, v\in[a,b]$, where $a=1, b=1+(K\|x_n-x_\star\|^p)/((1+p)(1-K\|x_n-x_\star\|^p))$. Then,

$$g'(v)=(p+1)v^p,\quad g''(v)=(p+1)pv^{p-1}. \tag{2.17}$$

By Taylor formula, $\exists\ \xi\in[a,b]$, such that the following formula holds

$$\begin{aligned}
g(b)-g(a)&=g'(a)(b-a)+\frac{g''(\xi)}{2}(b-a)^2\\
&=(p+1)\frac{K\|x_n-x_\star\|^p}{(1+p)(1-K\|x_n-x_\star\|^p)}+\frac{p(1+p)}{2\xi^{1-p}}\frac{K^2\|x_n-x_\star\|^{2p}}{(1+p)^2(1-K\|x_n-x_\star\|^p)^2}\\
&=\frac{K\|x_n-x_\star\|^p}{1-K\|x_n-x_\star\|^p}+\frac{p}{2\xi^{1-p}}\frac{K^2\|x_n-x_\star\|^{2p}}{(1+p)(1-K\|x_n-x_\star\|^p)^2}. \qquad (2.18)
\end{aligned}$$

Since $\xi\in[a,b]$, and $a=1, 0<p\le 1$, it is obvious $1/(\xi^{1-p})\le 1$. Then by (2.18), we have

$$g(b)-g(a)\le\frac{K\|x_n-x_\star\|^p}{1-K\|x_n-x_\star\|^p}+\frac{p}{2(1+p)}\frac{K^2\|x_n-x_\star\|^{2p}}{(1-K\|x_n-x_\star\|^p)^2}. \tag{2.19}$$

By (2.16) and (2.19), we obtain

$$\begin{aligned}
&\left\|F'(x_\star)^{-1}(F'(x_n)-\int_0^1 F'(ty_n+(1-t)x_\star)dt)\right\|\\
&\le K\|x_n-x_\star\|^p\left(1+\frac{pK\|x_n-x_\star\|^p}{2(1+p)(1-K\|x_n-x_\star\|^p)}\right).
\end{aligned} \tag{2.20}$$

Hence, (2.10) holds. The proof is completed. ∎

Proof of Theorem 2.1.

We will prove this theorem by induction. Firstly, by Lemma 2.1, $0<s_p<1$. Hence, $r_p=\sqrt[p]{s_p/K}<\sqrt[p]{1/K}$. By $x_0\in B(x_\star,r_p)$, we have $x_0\in B(x_\star,\sqrt[p]{1/K})$. By Lemma 2.2, $F'(x_0)^{-1}$ exists. So, x_1 is well defined, and

$$\|(F'(x_\star)^{-1}F'(x_0))^{-1}\|\le\frac{1}{1-K\|x_0-x_\star\|^p}. \tag{2.21}$$

By Lemma 2.3, we have

$$\begin{aligned}
x_1-x_\star&=(F'(x_\star)^{-1}F'(x_0))^{-1}F'(x_\star)^{-1}(F'(x_0)-\int_0^1 F'(ty_0+(1-t)x_\star)dt)\\
&(F'(x_\star)^{-1}F'(x_0))^{-1}F'(x_\star)^{-1}(F'(x_0)-\int_0^1 F'(ty_0+(1-t)x_\star)dt)(x_0-x_\star). \qquad (2.22)
\end{aligned}$$

Now by Lemma 2.4 and (2.21), we obtain

$$\|x_1 - x_\star\| \le \frac{\|x_0 - x_\star\|}{(1-K\|x_0-x_\star\|^p)^2} K\|x_0-x_\star\|^p \left(1+\frac{p}{2(1+p)}\frac{K\|x_0-x_\star\|^p}{(1-K\|x_0-x_\star\|^p)}\right)\frac{K\|x_0-x_\star\|^p}{1+p}$$

(2.23)

$$= \frac{K^2\|x_0-x_\star\|^{1+2p}}{(1-K\|x_0-x_\star\|^p)^2(1+p)}\left(1+\frac{p}{2(1+p)}\frac{K\|x_0-x_\star\|^p}{1-K\|x_0-x_\star\|^p}\right).$$

Since $x_0 \in B(x_\star, r_p)$, by (2.23), we know

$$\begin{aligned}\|x_1 - x_\star\| &< \frac{(Kr_p^p)^2}{(1-Kr_p^p)^2(1+p)}\left(1+\frac{p}{2(1+p)}\frac{Kr_p^p}{1-Kr_p^p}\right)r_p\\ &= \frac{(Kr_p^p)^2}{2(1-Kr_p^p)^3(1+p)^2}(pKr_p^p+2(1+p)(1-Kr_p^p))r_p \qquad (2.24)\\ &= \frac{1}{2(1-Kr_p^p)^3(1+p)^2}\left(2(1+p)(Kr_p^p)^2-(p+2)(Kr_p^p)^3\right)r_p.\end{aligned}$$

Since $r_p = \sqrt[p]{s_p/K}$, that is $s_p = Kr_p^p$, and s_p is the minimum positive root of the cubic equation (2.1), i.e.,

$$\frac{2(1+p)(Kr_p^p)^2-(p+2)(Kr_p^p)^3}{2(1-Kr_p^p)^3(1+p)^2} = 1. \tag{2.25}$$

Hence, by (2.24) and (2.25), we obtain

$$\|x_1 - x_\star\| < r_p. \tag{2.26}$$

This means $x_1 \in B(x_\star, r_p)$. Generally, we suppose x_l $(l \le n)$ has been generated by the modified Newton's method, and $x_l \in B(x_\star, r_p)$ $(l \le n)$, where n is a natural number. Then proceeding similar to the existence of $F'(x_0)^{-1}$, we can see $F'(x_n)^{-1}$ also exists, and the following estimate formula holds

$$\|(F'(x_\star)^{-1}F'(x_n))^{-1}\| \le \frac{1}{1-K\|x_n-x_\star\|^p}. \tag{2.27}$$

Analogously, such as the estimate of $\|x_1 - x_\star\|$, we get

$$\|x_{n+1} - x_\star\| \le \frac{K^2\|x_n-x_\star\|^{1+2p}}{(1-K\|x_n-x_\star\|^p)^2(1+p)}\left(1+\frac{p}{2(1+p)}\frac{K\|x_n-x_\star\|^p}{1-K\|x_n-x_\star\|^p}\right). \tag{2.28}$$

Since $x_n \in B(x_\star, r_p)$, proceeding similarly such as (2.24) and (2.25), we obtain

$$\|x_{n+1} - x_\star\| < \frac{2(1+p)((Kr_p)^2-(p+2)(Kr_p^p)^3)}{2(1-Kr_p^p)^3(1+p)^2}r_p = r_p. \tag{2.29}$$

This means $x_{n+1} \in B(x_\star, r_p)$. By induction, $\{x_n\}$ is well defined, and $x_n \in B(x_\star, r_p) (n \ge 0)$. Moreover, (2.28) holds for $n \ge 0$. Hence, we have

$$\|x_{n+1} - x_\star\| \le \frac{(Kr_p^p)^2\left(1+\frac{p}{2(1+p)}\frac{Kr_p^p}{1-Kr_p^p}\right)}{(1-Kr_p^p)^2(1+p)}\cdot\frac{\|x_n-x_\star\|^{1+2p}}{(r_p^p)^2}, \quad (n \ge 0). \tag{2.30}$$

By the process to deduce (2.24), we can see

$$\frac{(Kr_p^p)^2}{(1-Kr_p^p)^2(1+p)}\left(1+\frac{p}{2(1+p)}\frac{Kr_p^p}{1-Kr_p^p}\right) = 1. \tag{2.31}$$

Hence, by (2.30), we obtain

$$\|x_{n+1} - x_\star\| \le \frac{\|x_n - x_\star\|^{1+2p}}{r_p^{2p}}, \quad (n \ge 0). \tag{2.32}$$

Then, we have

$$\frac{\|x_{n+1} - x_\star\|}{r_p} \le \left(\frac{\|x_n - x_\star\|}{r_p}\right)^{1+2p}, \quad (n \ge 0). \tag{2.33}$$

Moreover, we easily get

$$\frac{\|x_n - x_\star\|}{r_p} \le \left(\frac{\|x_0 - x_\star\|}{r_p}\right)^{(1+2p)^n}, \quad (n \ge 0). \tag{2.34}$$

So we have

$$\|x_n - x_\star\| \le r_p \left(\frac{\|x_0 - x_\star\|}{r_p}\right)^{(1+2p)^n}, \quad (n \ge 0). \tag{2.35}$$

This estimate formula indicates that (2.2) holds.

Now we prove $x_\star$ is a unique solution in $B(x_\star, \sqrt[p]{(1+p)/K})$. Suppose there exists another solution $y_\star \in B(x_\star, \sqrt[p]{(1+p)/K})$,and $F(y_\star) = 0$. Denote an operator $A = \int_0^1 F'(ty_\star + (1-t)x_\star)dt$, then $F(y_\star) - F(x_\star) = A(y_\star - x_\star) = 0$. Hence, if we prove A is invertible, then $y_\star = x_\star$. By the Hölder continuous condition (1.5), we have

$$\begin{aligned} \|I - F'(x_\star)^{-1}A\| &= \|F'(x_\star)^{-1} \int_0^1 (F'(x_\star) - F'(ty_\star + (1-t)x_\star))dt\| \\ &\le K \int_0^1 t^p \|x_\star - y_\star\|^p dt = \frac{K\|x_\star - y_\star\|^p}{p+1} < 1. \end{aligned} \tag{2.36}$$

By the Banach lemma, A is invertible. Since $0 < s_p < 1$, $B(x_\star, \sqrt[p]{(1+p)/K})$ is bigger than $B(x_\star, r_p)$. The proof of Theorem 2.1 is completed. ∎

Taking $p = 1$ in the above theorem, noting also that $s_1 = (5 - \sqrt{5})/5$ is the minimum positive root of the following cubic equation

$$5s^3 - 20s^2 + 24s - 8 = 0, \tag{2.37}$$

we can get the following corollary at once:

Corollary 2.1. *Suppose $x_\star \in D, F(x_\star) = 0, F'(x_\star)^{-1}$ exists, the Lipschitz continuous condition* (1.4) *holds. Denote $r_1 = (5 - \sqrt{5})/5K$, then the sequence $\{x_n\}$ generated by the modified Newton's method* (1.3) *starting from any initial point $x_0 \in B(x_\star, r_1)$ converges to the unique solution $x_\star \in B(x_\star, 2/K)$ that is bigger than $B(x_\star, r_1)$. Moreover, the following error estimate is satisfied*

$$\|x_n - x_\star\| \le r_1 \left(\frac{\|x_0 - x_\star\|}{r_1}\right)^{3^n}, \quad n \ge 0. \tag{2.38}$$

Remark 2.1. By Theorem 2.1, under the hypotheses in Theorem 2.1, the convergence order of the modified Newton's method reaches to at least $1 + 2p$. When $p = 1$, the convergence order of the method reaches to at least 3. This is accordant with the convergence order obtained by other authors (see [16]).

Table 1. Values of s_p

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
s_p	0.50622	0.51230	0.51817	0.52380	0.52919	0.53434	0.53926	0.54397	0.54847	0.55279

Remark 2.2. It needs further study to decide whether the estimate radius r_p of the convergence ball of the modified Newton's method established in Theorem 2.1 is optimal.

Remark 2.3. In Table 1, we list values of s_p for $p = 0.1, 0.2, ..., 1.0$. From the table, we see that s_p increases as p increases, i.e., r_p increases as p increases.

3. Numerical examples

In this section, we apply the convergence ball result and show some numerical examples.

Example 3.1. Let us consider

$$F(x) = e^x - 1, \quad x \in D = [-1,1]. \tag{3.1}$$

Then, $F'(x) = e^x$, $F(x)$ has a zero $x_\star = 0$ in D, and $F'(x_\star) = 1$. We easily obtain

$$|F'(x_\star)^{-1}(F'(x) - F'(y))| = |e^x - e^y| \le e|x-y|, \quad \forall\, x,y \in D. \tag{3.2}$$

Hence, the Lipschitz condition (1.4) holds with $K = e$ and $p = 1$. On the other hand, $B(x_\star, \sqrt[p]{(1+p)/K}) = (-2/e, 2/e) \subseteq D = [-1,1]$. From Table 1, we obtain $s_p = 0.55279$. By Theorem 2.1, the radius of the convergence ball of the modified Newton's method is at least $r_p = \sqrt[p]{s_p/K} \approx 0.20336$.

Example 3.2. Let us consider

$$F(x) = \frac{2}{3}x^{\frac{3}{2}} - x, \quad x \in D = [1,3]. \tag{3.3}$$

Then, $F'(x) = x^{\frac{1}{2}} - 1$, $F(x)$ has a zero $x_\star = 9/4$ in D, and $F'(x_\star) = 1/2$. We easily obtain

$$|F'(x_\star)^{-1}(F'(x) - F'(y))| = 2|x^{\frac{1}{2}} - y^{\frac{1}{2}}| \le 2|x-y|^{\frac{1}{2}}, \quad \forall\, x,y \in D. \tag{3.4}$$

Hence, the Hölder continuous condition (1.5) holds with $K = 2$ and $p = \frac{1}{2}$. On the other hand, $B(x_\star, \sqrt[p]{(1+p)/K}) = (27/16, 45/16) \subseteq D = [1,3]$. By Table 1, we obtain $s_p = 0.52919$. By Theorem 2.1, the radius of the convergence ball of the modified Newton's method is at least $r_p = \sqrt[p]{s_p/K} \approx 0.07001$.

Example 3.3. Let us consider to solve the following nonlinear system:

$$\begin{cases} 2x_1 - \frac{1}{9}{x_1}^{1+p} - x_2 = 0, \\ -x_1 + 2x_2 - \frac{1}{9}{x_2}^{1+p} = 0, \end{cases} \tag{3.5}$$

where $0 < p \le 1$. This nonlinear system comes from the following nonlinear boundary value problem of second-order:

$$\begin{cases} x'' + x^{1+p} = 0, \qquad 0 < p \le 1, \\ x(0) = x(1) = 0, \end{cases} \tag{3.6}$$

which has been used as the example by many authors [1, 8–10, 13].

Now we define an operator $F : R^2 \to R^2$ such that $F = (F_1, F_2)$. For $x = (x_1, x_2) \in D = R^2$, we take $F_1(x_1, x_2) = 2x_1 - 1/9x_1{}^{1+p} - x_2$, $F_2(x_1, x_2) = -x_1 + 2x_2 - 1/9x_2{}^{1+p}$. Then, noticing $0 < p \le 1$, it is easy to see F is Fréchet differentiable in R^2, and we have

$$F'(x) = \begin{pmatrix} 2 - \frac{1}{9}(1+p)x_1^p & -1 \\ -1 & 2 - \frac{1}{9}(1+p)x_2^p \end{pmatrix}. \tag{3.7}$$

Let $x = (x_1, x_2) \in R^2$ and $\|x\| = \|x\|_\infty = \max_{1\le i\le 2} |x_i|$. The corresponding norm on $A \in R^2 \times R^2$ is

$$\|A\| = \max_{1\le i\le 2} \sum_{j=1}^{2} |a_{ij}|. \tag{3.8}$$

It can be verified easily that $x_\star = (9^{1/p}, 9^{1/p})$ is a solution of (3.5). From (3.7), we get

$$F'(x_\star) = \begin{pmatrix} 1-p & -1 \\ -1 & 1-p \end{pmatrix}. \tag{3.9}$$

As $0 < p \le 1$, it is easy to see $F'(x_\star)$ is invertible. Similarly to [8], we can deduce that the Hölder continuous condition (1.5) is true for $K = (1+p)/(9p(2-p))$. Setting $p = 0.3$, we obtain $s_p = 0.51817$ from Table 1. Hence, all conditions in Theorem 2.1 are satisfied. By Theorem 2.1, the radius of the convergence ball of the modified Newton's method is at least $r_p = \sqrt[p]{s_p/K} \approx 7.48983$.

Example 3.4. Let us consider the two point boundary value problem:

$$\begin{cases} x'' + x^{1+p} + x^2 = 0, & 0 < p \le 1, \\ x(0) = x(1) = 0. \end{cases} \tag{3.10}$$

We shall solve this problem by using finite differences. Let $h = 1/n$, where n is natural integer, and set $t_i = ih$, $i = 1, 2, \dots, n-1$. It follows from the boundary conditions that $x_0 = x_n = 0$. We approximate the second derivative $x''(t)$ by the usual estimate [16]:

$$\begin{aligned} &x''(t) \approx [x(t+h) - 2x(t) + x(t-h)]/h^2, \\ &x''(t_i) = (x_{i+1} - 2x_i + x_{i-1})/h^2, \quad i = 1, 2, \dots, n-1. \end{aligned} \tag{3.11}$$

A substitution into (3.10) of (3.11) leads to the following system of equations:

$$\begin{cases} 2x_1 - h^2 x_1^{1+p} - h^2 x_1^2 - x_2 = 0, \\ -x_{i-1} + 2x_i - h^2 x_i^{1+p} - h^2 x_i^2 - x_{i+1} = 0, & i = 2, 3, \dots, n-2, \\ -x_{n-2} + 2x_{n-1} - h^2 x_{n-1}^{1+p} - h^2 x_{n-1}^2 = 0. \end{cases} \tag{3.12}$$

Let us define operator $F : R^{n-1} \to R^{n-1}$ by

$$F(x) = A(x) - h^2 f(x), \tag{3.13}$$

where,

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}, \tag{3.14}$$

and

$$f(x) = [x_1^{1+p} + x_1^2, x_2^{1+p} + x_2^2, \ldots, x_{n-1}^{1+p} + x_{n-1}^2]^t. \tag{3.15}$$

Using (3.13), we obtain

$$F'(x) = A - h^2(1+p)\begin{pmatrix} x_1^p & 0 & \ldots & 0 \\ 0 & x_2^p & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_{n-1}^p \end{pmatrix} - 2h^2\begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_{n-1} \end{pmatrix}. \tag{3.16}$$

We introduce norms as follows: for $x \in R^{n-1}$, let $\|x\| = \max_{1 \le i \le n-1} |x_i|$. The corresponding norm on $M \in R^{n-1} \times R^{n-1}$ is

$$\|M\| = \max_{1 \le i \le n-1} \sum_{j=1}^{n-1} |m_{ij}|. \tag{3.17}$$

Let $x, y \in R^{n-1}$ with $|x_i| > 0$, $|y_i| > 0$, $i = 1, 2, \ldots, n-1$. Then using the above norms, we obtain in turn:

$$\begin{aligned} \|F'(x) - F'(y)\| &= \|\operatorname{diag}\{(1+p)(y_i^p - x_i^p) + 2(y_i - x_i)\}h^2\| \\ &= \max_{1 \le i \le n-1} |(1+p)(y_i^p - x_i^p) + 2(y_i - x_i)|h^2 \\ &\le [(1+p)\max_{1 \le i \le n-1} |y_i^p - x_i^p| + 2\max_{1 \le i \le n-1} |y_i - x_i|]h^2 \\ &\le [(1+p)(\max_{1 \le i \le n-1} |y_i - x_i|)^p + 2\|y - x\|]h^2 \\ &= [(1+p) + 2\|y - x\|^{1-p}]\|y - x\|^p h^2. \end{aligned} \tag{3.18}$$

In view of (1.4) and (3.18), we obtain for any $x, y \in D = B(x_\star, \alpha)$, $\alpha > 0$, and $b \ge \|F'(x_\star)^{-1}\|$:

$$\begin{aligned} bh^2(1 + p + 2\|y - x\|^{1-p}) &\le bh^2(1 + p + 2(\|y - x_\star\| + \|x_\star - x\|)^{1-p}) \\ &\le bh^2(1 + p + 2(2\alpha)^{1-p}) = K. \end{aligned} \tag{3.19}$$

We shall use the modified Newton's method (1.3) to approximate the solution of equation $F(x) = 0$, where operator F is given in (3.13). If $n = 10$, and $p = 1/2$, then using (3.12) we obtain nine equations. Using Newton's method (1.2), Reference [19] obtained the solution $x_\star$ of system (3.12):

$$x_\star = \begin{pmatrix} 2.394640795 \\ 4.694882371 \\ 6.672977547 \\ 8.033409359 \\ 8.520791424 \\ 8.033409359 \\ 6.672977547 \\ 4.694882371 \\ 2.394640795 \end{pmatrix}. \tag{3.20}$$

Using the definition of the constant b, we can set $b = \|F'(x_\star)^{-1}\| = 15.753443793$. In view of the definition of Hölder constant K, we have for $\alpha = 0.8$:

$$K = 0.634835764. \tag{3.21}$$

From Table 1, we obtain $s_p = 0.52919$. Hence, all conditions in Theorem 2.1 are satisfied. By Theorem 2.1, the radius of the convergence ball of the modified Newton's method is at least $r_p = \sqrt[p]{s_p/K} \approx 0.69487$.

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