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Vertex-Disjoint Cycles of Order Eight with Chords in a Bipartite Graph

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Abstract. Let $G = (V_1, V_2; E)$ be a bipartite graph with $\mid V_1 \mid = \mid V_2 \mid = 4k$, where k is a positive integer. In this paper, it is proved that if the minimum degree of G is at least $3k+1$, then G contains k vertex-disjoint cycles of order eight such that each of them has at least two chords.

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1. Introduction

Let G be a simple graph and $k \geq 1$ be an integer. The minimum degree of G is denoted by $\delta(G)$. Corrádi and Hajanal [2] proved that if G is of order at least $3k$ and the minimum degree of it is at least $2k$, then G contains k vertex-disjoint cycles. When the order of G is exactly $3k$, then G contains k vertex-disjoint triangles. In [5], Wang considered vertex-disjoint cycles in a bipartite graph and gave the following conjecture.

Conjecture 1.1. *Let $G = (V_1, V_2; E)$ be a bipartite graph with $\mid V_1 \mid = \mid V_2 \mid = n = sk$, where n, k and s are integers with $s \geq 2$ and $k \geq 1$. If the minimum degree of G is at least $(s-1)k+1$, then G contains k vertex-disjoint subgraphs isomorphic to $K_{s,s}$.*

Wang verified this conjecture for $k \leq 4$ in [4, 5]. For $s = 2$, Wang [3] proved that if $G = (V_1, V_2; E)$ is a bipartite graph with $\mid V_1 \mid = \mid V_2 \mid = 2k$ and the minimum degree of G is at least $k+1$, then G contains $k-1$ vertex-disjoint quadrilaterals and a path of order 4 such that the path is vertex disjoint of all the $k-1$ quadrilaterals. For $s = 3$, Wang [5] proved that if $G = (V_1, V_2; E)$ is a bipartite graph with $\mid V_1 \mid = \mid V_2 \mid = 3k$ and the minimum degree of G is at least $2k+1$, then G contains k vertex-disjoint cycles of order six such that each of them has at least two chords. When the order of G is large enough, Zhao [6] proved a result stronger than Conjecture 1.1.

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Theorem 1.1. *For each $s \geq 2$, there exists k_0 such that the following holds for all $k \geq k_0$. Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n = sk$ such that the minimum degree*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1, & \text{if } k \text{ is even}, \\ \frac{n+3s}{2} - 2, & \text{if } k \text{ is odd}. \end{cases}$$

Then G contains k vertex-disjoint subgraphs isomorphic to $K_{s,s}$.

In this paper, we consider the case $s = 4$ and show the following result.

Theorem 1.2. *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 4k$, where k is a positive integer. If the minimum degree of G is at least $3k+1$, then G contains k vertex-disjoint cycles of order eight such that each of them has at least two chords.*

We will use the following terminology and notation, where any undefined notation follows that of Bondy and Murty [1]. Let G be a simple graph. The order of G is $|V(G)|$ and its size is $e(G) = |E|$. A set of graphs is said to be independent if no two of them have any common vertex. If $A_1, A_2, ..., A_n$ are subsets of $V(G)$, we use $\langle A_1, A_2, ..., A_n \rangle$ to denote the subgraph of G induced by $A_1 \bigcup A_2 \bigcup ... \bigcup A_n$. For $x \in V(G)$, let $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$. If H is a subgraph of G, then $N_H(x) = N_G(x) \bigcap V(H)$, $d(x, H) = |N_H(x)|$. Let T be a simple graph and k be a positive integer, then $G \supseteq kT$ means that G contains k independent subgraphs isomorphic to T. Let X and Y be two independent subgraphs of G or two disjoint subsets of $V(G)$. We define $G[X]$ to be the subgraph of G induced by X, and $e(X, Y)$ to be the number of edges between X and Y. A k-cycle is a cycle of order k and a m-path is a path of order m, denoted by C^k and P^m, respectively. Particularly, a quadrilateral is a cycle of order 4, and a triangle is a cycle of order 3. For a k-cycle $C = x_1 x_2 ... x_k x_1$, $x_i x_{i+1}$ is an edge in C. For a cycle C of G, a chord of C is an edge of $G - E(C)$ which joins two vertices of C.

The structure of the paper is as follows. First we will show some useful lemmas in Section 2, then prove the main result in Section 3.

2. Lemmas

In this section, we will prove some useful lemmas. Let $G = (V_1, V_2; E)$ be a bipartite graph.

Lemma 2.1. *Let $P_1 = a_1 b_1 a_2$, $P_2 = y_1 x_2 y_2 x_3 y_3$, P_1 and P_2 are independent, where $\{a_1, x_2\} \subseteq V_1$. If $e(a_1 a_2, y_1 y_3) \geq 3$, then $G[P_1 \bigcup P_2]$ contains an 8-cycle with at least a chord.*

Proof. These are easily verified. ∎

Lemma 2.2. *Let $P = a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4$ be an 8-path, $C = x_1 y_1 x_2 y_2 x_3 y_3 x_4 y_4 x_1$ be an 8-cycle, P and C are independent, where $\{a_1, x_1\} \subseteq V_1$. If $e(P, C) \geq 25$, $d(a_1, C) > 0$, $d(b_4, C) > 0$, then $G[P \bigcup C]$ contains two independent 8-cycles.*

Proof. Suppose on the contrary that $G[P \bigcup C] \not\supseteq 2C^8$. Since $e(P, C) \geq 25$, there exists a vertex $x \in V(C)$, such that $d(x, P) = 4$. W.l.o.g., say $d(x_1, P) = 4$. Since $G[P - a_1 + x_1] \supseteq C^8$, $G[C - x_1 + a_1] \not\supseteq C^8$. Therefore, $d(a_1, C) \leq 3$ and $\{a_1 y_1, a_1 y_4\} \not\subseteq E$. We distinguish three cases: $d(a_1, C) = 1$, $d(a_1, C) = 2$ or $d(a_1, C) = 3$.

Case 1. $d(a_1, C) = 1$. By symmetry, we distinguish two cases: $a_1 y_1 \in E$ or $a_1 y_2 \in E$.

Case 1.1. $a_1 y_1 \in E$. Since $d(a_1, C) = 1$, we have $a_1 y_2 \notin E$, $a_1 y_3 \notin E$ and $a_1 y_4 \notin E$. Then $\{b_2 x_3, a_3 y_3\} \not\subseteq E$, for otherwise $G[P \bigcup C]$ contains two independent 8-cycles $a_1 y_1 x_2 y_2 x_3 b_2 a_2$

b_1a_1 and $a_3b_3a_4b_4x_1\ y_4x_4y_3a_3$. If $\{b_4x_2, a_4y_4\} \subseteq E$, then $G[P\bigcup C]$ contains two independent 8-cycles $a_4b_4x_2y_2x_3\ y_3x_4y_4a_4$ and $a_1b_1a_2b_2a_3b_3x_1y_1a_1$, a contradiction. Therefore, $\{b_4x_2, a_4y_4\} \not\subseteq E$. With the same proof, we can get $\{b_3x_2, a_4y_2\} \not\subseteq E$, $\{b_1x_4, a_2y_4\} \not\subseteq E$.

Suppose $b_4x_4 \in E$. Then $\langle b_2a_3b_3a_4b_4, x_4y_4x_1\rangle \supseteq C^8$ and therefore $\langle a_1b_1a_2, y_1x_2y_2x_3y_3\rangle \not\supseteq C^8$. This implies $a_2y_3 \notin E$. Therefore, we get $e(x_2y_2x_4y_4, P) \leq 11$ and $e(x_3y_3, P) \leq 5$. Thus, $e(P,C) \leq 24$, a contradiction.

Hence $b_4x_4 \notin E$. Thus $e(x_2y_2x_4y_4, P) \leq 10$ and $e(x_3y_3, P) \leq 6$. This implies $e(P,C) \leq 24$, a contradiction.

Case 1.2. $a_1y_2 \in E$. Since $d(a_1, C) = 1$, it follows that $a_1y_1 \notin E$, $a_1y_3 \notin E$, $a_1y_4 \notin E$. Since $\langle a_1b_1a_2b_2, x_1y_1x_2\ y_2\rangle \supseteq C^8$ and $G[P\bigcup C] \not\supseteq 2C^8$, we have $\{a_3y_4, x_3b_4\} \not\subseteq E$. Similarly, we can get $\{a_2y_1, b_4x_2\} \not\subseteq E$.

If $a_3y_1 \in E$, then $d(x_3, b_3b_4) = 0$ as $\langle a_1b_1a_2b_2a_3, y_1x_2y_2\rangle \supseteq C^8$. If $a_2y_4 \in E$, then $d(x_2, b_2b_4) = 0$ as $\langle a_1b_1a_2, y_2x_3y_3x_4y_4\rangle \supseteq C^8$. Therefore, if $\{a_3y_1, a_2y_4\} \subseteq E$, then $d(x_3, b_3b_4) = 0$ and $d(x_2, b_2b_4) = 0$. This implies $e(P,C) \leq 25$. Since $e(P,C) \geq 25$, $d(x_2, P) = 2$, $d(y_4, P) = 3$ and $d(x_4, P) = 4$. In particular, $x_2b_3 \in E$, $a_3y_4 \in E$, $a_4y_4 \in E$ and $b_2x_4 \in E$. Therefore, $G[P\bigcup C]$ contains two independent 8-cycles $y_1x_2b_3a_3y_4a_4b_4x_1y_1$ and $a_1b_1a_2b_2x_4\ y_3x_3y_2a_1$, a contradiction. So $\{a_3y_1, a_2y_4\} \not\subseteq E$ and $d(y_1, P) + d(y_4, P) \leq 5$.

Suppose $b_4x_4 \in E$. Then $\{b_3x_2, a_3y_3\} \not\subseteq E$, for otherwise $\langle a_1b_1a_2b_2a_3, y_2x_3y_3\rangle \supseteq C^8$ and $\langle b_3a_4b_4, x_2y_1x_1y_4x_4\rangle \supseteq C^8$. Since $e(P,C) \geq 25$, we have $e(y_1x_2x_3y_3y_4, P) = 13$. This implies either $a_3y_1 \in E$ or $a_2y_4 \in E$. If $a_3y_1 \in E$, then $d(x_3, b_3b_4) = 0$; if $a_2y_4 \in E$, then $d(x_2, b_2b_4) = 0$. In each case, $e(P,C) \leq 24$, a contradiction.

Therefore, $b_4x_4 \notin E$. Similarly, if $a_3y_1 \in E$, then $d(x_3, b_3b_4) = 0$; if $a_2y_4 \in E$, then $d(x_2, b_2b_4) = 0$. In each case, $e(P,C) \leq 24$, a contradiction. Therefore, $a_3y_1 \notin E$ and $a_2y_4 \notin E$. Thus $e(P,C) \leq 24$, a contradiction.

Case 2. $d(a_1, C) = 2$. By symmetry, we divide the proof into three cases: $a_1y_1 \in E$, $a_1y_2 \in E$; $a_1y_2 \in E$, $a_1y_3 \in E$ or $a_1y_1 \in E$, $a_1y_3 \in E$.

Case 2.1. $a_1y_1 \in E$, $a_1y_2 \in E$. Since $d(a_1, C) = 2$, it follows that $a_1y_3 \notin E$, $a_1y_4 \notin E$. Since $a_1y_1x_1y_4x_4y_3x_3y_2a_1$ is an 8-cycle in $G[P\bigcup C]$, we have $\{x_2b_1, x_2b_4\} \not\subseteq E$. Similarly, we get $\{b_2x_3, a_3y_3\} \not\subseteq E$ and $\{a_3y_4, x_3b_4\} \not\subseteq E$.

If $a_3y_1 \in E$, we have $d(x_3, b_3b_4) = 0$. Since $\langle a_1b_1a_2b_2a_3, y_1x_2y_2\rangle \supseteq C^8$. If $a_2y_4 \in E$, we have $d(x_2, b_2b_4) = 0$. Since $\langle a_1b_1a_2, y_2x_3y_3x_4y_4\rangle \supseteq C^8$. Therefore, if $\{a_3y_1, a_2y_4\} \subseteq E$, we have $d(x_2, P) = 2$, $d(x_4, P) = 4$ and $d(y_4, P) = 3$ as $e(P,C) \geq 25$. In particular, $a_3y_4 \in E$, $b_3x_2 \in E$ and $b_4x_4 \in E$. Then $G[P\bigcup C]$ contains two independent 8-cycles $a_1b_1a_2b_2a_3y_4x_1y_1a_1$ and $b_4a_4b_3x_2y_2x_3y_3x_4b_4$, a contradiction. Therefore, $\{a_3y_1, a_2y_4\} \not\subseteq E$.

Suppose $b_4x_4 \in E$. Since $\langle b_2a_3b_3a_4b_4, x_1y_4x_4\rangle \supseteq C^8$, we have $a_2y_3 \notin E$. As $e(P,C) \geq 25$, either $a_3y_1 \in E$ or $a_2y_4 \in E$. If $a_3y_1 \in E$, then $d(x_3, b_3b_4) = 0$; if $a_2y_4 \in E$, then $d(x_2, b_2b_4) = 0$. In each case, $e(P,C) \leq 24$, a contradiction.

Now we may assume $b_4x_4 \notin E$. Since $e(P,C) \geq 25$, it follows that either $a_3y_1 \in E$ or $a_2y_4 \in E$. Similarly, in each case, we can get $e(P,C) \leq 24$, a contradiction.

Case 2.2. $a_1y_2 \in E$, $a_1y_3 \in E$. Since $d(a_1, C) = 2$, we have $a_1y_1 \notin E$, $a_1y_4 \notin E$. If $\{a_2y_1, b_4x_2\} \subseteq E$, then $\langle a_2b_2a_3b_3a_4b_4,\ y_1x_2\rangle \supseteq C^8$ and $\langle a_1b_1, x_1y_4x_4y_3x_3y_2\rangle \supseteq C^8$, a contradiction. Therefore, $\{a_2y_1, b_4x_2\} \not\subseteq E$. Similarly, $\{a_2y_4, b_4x_4\} \not\subseteq E$ and $\{x_3b_1, x_3b_4\} \not\subseteq E$.

If $a_3y_1 \in E$, we have $d(x_3, b_3b_4) = 0$. Since $\langle a_1b_1a_2b_2a_3, y_1x_2y_2\rangle \supseteq C^8$. If $a_2y_4 \in E$, we have $d(x_2, b_2b_4) = 0$. Since $\langle a_1b_1a_2, y_2x_3y_3x_4y_4\rangle \supseteq C^8$. Therefore, if $\{a_3y_1, a_2y_4\} \subseteq E$, it follows that $d(x_2,P)=2$ and $d(x_4,P)+d(y_4,P)=6$. Since $e(P,C)\geq 25$. In particular, $a_3y_4\in E$, $b_3x_2\in E$, $x_4b_2\in E$ and $a_4y_4\in E$. Then $G[P\bigcup C]$ contains two independent 8-cycles $a_1b_1a_2b_2x_4y_3x_3y_2a_1$ and $y_1x_2b_3a_3y_4a_4b_4x_1y_1$, a contradiction. Therefore, $\{a_3y_1,a_2y_4\}\not\subseteq E$.

Similarly, if $a_3y_4\in E$, we have $d(x_3,b_3b_4)=0$. Since $\langle a_1b_1a_2b_2a_3,y_3x_4y_4\rangle\supseteq C^8$. If $a_2y_1\in E$, we have $d(x_4,b_2b_4)=0$. Since $\langle a_1b_1a_2,y_1x_2y_2x_3y_3\rangle\supseteq C^8$. Therefore, if $\{a_3y_4, a_2y_1\}\subseteq E$, then $d(x_3,b_3b_4)=0$ and $d(x_4,b_2b_4)=0$. Note that $\{a_3y_1,a_2y_4\}\not\subseteq E$ and $\{a_2y_1,b_4x_2\}\not\subseteq E$, we have $e(P,C)\leq 24$, a contradiction. Thus $\{a_3y_4,a_2y_1\}\not\subseteq E$.

If $a_2y_1\in E$, then $d(x_4,b_2b_4)=0$. Note that $\{a_2y_1,b_4x_2\}\not\subseteq E$ and $\{a_3y_4,a_2y_1\}\not\subseteq E$, then $b_4x_2\notin E$ and $a_3y_4\notin E$. Therefore, $e(P,C)\leq 24$, a contradiction. Thus, $a_2y_1\notin E$. With the same proof, we can get $a_2y_4\notin E$.

Suppose $a_3y_1\notin E$ and $a_3y_4\notin E$. Since $e(P,C)\geq 25$, we have $d(x_2,P)=d(y_2,P)=d(y_3,P)=d(x_4,P)=4$. By Lemma 2.1, $\langle x_2y_1x_1y_4x_4,b_3a_4b_4\rangle\supseteq C^8$ and $\langle y_2x_3y_3,a_1b_1a_2b_2a_3\rangle\supseteq C^8$, a contradiction.

Now we have either $a_3y_1\in E$ or $a_3y_4\in E$. By symmetry, say $a_3y_1\in E$. Then $d(x_3,b_3b_4)=0$ as we proved before. If $\{b_4x_4,b_2x_2\}\subseteq E$, then $\langle a_3b_3a_4b_4,x_4y_4x_1y_1\rangle\supseteq C^8$ and $\langle a_1b_1a_2b_2, x_2y_2x_3y_3\rangle\supseteq C^8$, a contradiction. Therefore, $\{b_4x_4,b_2x_2\}\not\subseteq E$. Since $e(P,C)\geq 25$, $d(x_2,P)+d(x_4,P)=7$ and $d(y_2,P)=d(y_3,P)=4$. This implies $e(x_2x_4,b_3b_4)\geq 3$ and $e(y_2y_3,a_1a_3)=4$. Then $\langle y_2x_3y_3,a_1b_1a_2b_2a_3\rangle\supseteq C^8$ and $\langle b_3a_4b_4,x_2y_1x_1y_4x_4\rangle\supseteq C^8$ by Lemma 2.1, a contradiction.

Case 2.3. $a_1y_1\in E$, $a_1y_3\in E$. Since $d(a_1,C)=2$, we have $a_1y_2\notin E$ and $a_1y_4\notin E$. If $\{a_3y_1,b_4x_3\}\subseteq E$, then $G[P\bigcup C]$ contains two independent 8-cycles $a_3b_3a_4b_4x_3y_2x_2y_1a_3$ and $a_1b_1a_2b_2x_1y_4x_4y_3a_1$, a contradiction. This implies that $\{a_3y_1,b_4x_3\}\not\subseteq E$. Similarly, $\{b_2x_3,a_3y_3\}\not\subseteq E$, $\{a_4y_4,b_4x_2\}\not\subseteq E$.

If $a_3y_4\in E$, we have $d(x_3,b_3b_4)=0$. Since $\langle a_1b_1a_2b_2a_3,y_3x_4y_4\rangle\supseteq C^8$. If $d(a_2,y_1y_3)>0$, we have $d(x_4,b_2b_4)=0$. Since $\langle a_1b_1a_2,y_1x_2y_2x_3y_3\rangle\supseteq C^8$. Therefore, if $a_3y_4\in E$ and $d(a_2,y_1y_3)>0$, then $d(x_3,b_3b_4)=0$ and $d(x_4,b_2b_4)=0$. Thus, $e(P,C)\leq 24$, a contradiction. This implies either $a_3y_4\notin E$ or $d(a_2,y_1y_3)=0$.

Suppose $d(a_2,y_1y_3)=0$. Since $e(P,C)\geq 25$, it follows that $d(x_2,P)+d(y_4,P)=6$ and $d(y_2,P)=3$. In particular, $x_2b_3\in E$ and $y_2a_4\in E$. Therefore, $\langle a_1b_1a_2b_2a_3b_3,y_1x_2\rangle\supseteq C^8$ and $\langle a_4b_4,x_1y_4x_4y_3x_3y_2\rangle\supseteq C^8$, a contradiction.

Now we may assume $a_3y_4\notin E$ and $d(a_2,y_1y_3)>0$. As we proved before, $d(x_4,b_2b_4)=0$. Therefore, $e(P,C)\leq 24$, a contradiction.

Case 3. $d(a_1,C)=3$. Since $\{a_1y_1,a_1y_4\}\not\subseteq E$, w.l.o.g., say $\{a_1y_1,a_1y_2,a_1y_3\}\subseteq E$, $a_1y_4\notin E$. Then $\{x_2b_1,x_2b_4\}\not\subseteq E$, for otherwise $G[P\bigcup C]$ contains two independent 8-cycles $a_1y_1x_1y_4x_4y_3x_3y_2a_1$ and $x_2b_1a_2b_2a_3b_3a_4b_4x_2$. If $\{x_2b_3,y_2a_4\}\subseteq E$, then $G[P\bigcup C]$ contains two independent 8-cycles $a_1b_1a_2b_2a_3b_3x_2y_1a_1$ and $a_4b_4x_1y_4x_4y_3x_3y_2a_4$, a contradiction. Thus, $\{x_2b_3,y_2a_4\}\not\subseteq E$. With the same proof, we can get $\{x_3b_1,x_3b_4\}\not\subseteq E$, $\{x_4b_1,y_4a_2\}\not\subseteq E$. If $d(a_2,y_1y_3)>0$, then $d(x_4,b_2b_4)=0$, for otherwise $\langle a_1b_1a_2,y_1x_2y_2x_3y_3\rangle\supseteq C^8$ and $\langle b_2a_3b_3a_4b_4,x_1y_4x_4\rangle\supseteq C^8$. Therefore, either $d(a_2,y_1y_3)=0$ or $d(x_4,b_2b_4)=0$. Since $e(P,C)\geq 25$, $d(y_1,P)+d(y_3,P)+d(x_4,P)+d(y_4,P)=12$ and $d(x_3,P)=3$. In particular, $a_3y_3\in E$ and $b_2x_3\in E$. Then $\langle a_3b_3a_4b_4,x_1y_4x_4y_3\rangle\supseteq C^8$ and $\langle a_1b_1a_2b_2,y_1x_2y_2x_3\rangle\supseteq C^8$, a contradiction. This completes the proof of Lemma 2.2. ∎

Lemma 2.3. *Let $P = a_1b_1a_2b_2a_3b_3a_4b_4a_5b_5$ be a 10-path, $C = x_1y_1x_2y_2x_3y_3x_4y_4x_1$ be an 8-cycle, P and C are independent, where $\{a_1,x_1\} \subseteq V_1$. If $e(P,C) \geq 31$, then $G[P \bigcup C - \{a_1,b_1\}] \supseteq 2C^8$, or $G[P \bigcup C - \{a_5,b_5\}] \supseteq 2C^8$, or $G[P \bigcup C - \{a_1,b_5\}] \supseteq 2C^8$.*

Proof. Suppose on the contrary that the lemma fails. Let $P_1 = P - a_1 - b_1$, $P_2 = P - a_5 - b_5$, $P_3 = P - a_1 - b_5$.

Suppose that $e(P_3,C) \geq 25$. Since $G[P_3 \bigcup C] \not\supseteq 2C^8$, we have either $d(b_1,C) = 0$ or $d(a_5,C) = 0$ by Lemma 2.2. W.l.o.g., say $d(b_1,C) = 0$. This implies $e(P_3,C) \leq 28$ and therefore $e(a_1b_5,C) \geq 3$. Since $e(P_3,C) \geq 25$, we have $d(x,C) \geq 1$ for all $x \in V(P_3 - b_1)$. Thus, $e(P_1,C) \geq 31 - d(a_1,C) \geq 27$. Note that $G[P_1 \bigcup C] \not\supseteq 2C^8$ and $d(a_2,C) \geq 1$, we get $d(b_5,C) = 0$ by Lemma 2.2. This implies $d(a_1,C) \geq 3$ and $e(P_2,C) \geq 27$. Note that $d(b_4,C) > 0$, then $G[P_2 \bigcup C] \supseteq 2C^8$ by Lemma 2.2, a contradiction. Hence $e(P_3,C) \leq 24$ and $e(a_1b_5,C) \geq 7$. The following proof is divided into two cases.

Case 1. Either $e(P_1,C) \geq 25$ or $e(P_2,C) \geq 25$. By symmetry, say $e(P_1,C) \geq 25$. Since $e(a_1b_5,C) \geq 7$, we have $d(b_5,C) \geq 3$ and $d(a_1,C) \geq 3$. Then $d(a_2,C) = 0$, for otherwise $G[P_1 \bigcup C] \supseteq 2C^8$ by Lemma 2.2. Since $e(P_1,C) \geq 25$, it follows that $d(u,C) > 0$ for every $u \in V(P_1 - a_2)$. As $d(a_1,C) > 0$, $d(b_4,C) > 0$ and $G[P_2 \bigcup C] \not\supseteq 2C^8$, we get $e(P_2,C) \leq 24$ by Lemma 2.2.

Since $d(a_1,C) \geq 3$ and $d(b_3,C) \geq 1$, w.l.o.g., we may say $\{a_1y_1, a_1y_2, b_3x_1\} \subseteq E$. If $\{a_4y_4, b_4x_2\} \subseteq E$, then $G[P_2 \bigcup C]$ contains two independent 8-cycles $a_1b_1a_2b_2a_3b_3x_1y_1a_1$ and $a_4b_4x_2y_2x_3y_3x_4y_4a_4$, a contradiction. Therefore, we have $\{a_4y_4, b_4x_2\} \not\subseteq E$. Similarly, $\{b_4x_3, a_3y_2\} \not\subseteq E$, for otherwise $G[P_2 \bigcup C]$ contains two independent 8-cycles $a_1b_1a_2b_2a_3y_2 x_2y_1a_1$ and $b_3a_4b_4x_3y_3x_4y_4x_1b_3$.

Suppose $b_3x_2 \in E$. Then $\{a_4y_2, b_4x_1\} \not\subseteq E$, for otherwise $G[P_2 \bigcup C]$ contains two independent 8-cycles $a_1b_1a_2b_2a_3b_3x_2y_1a_1$ and $a_4y_2x_3y_3x_4y_4x_1b_4a_4$. Similarly, $\{a_4y_1, b_4x_3\} \not\subseteq E$. Since $e(P_1,C) \geq 25$, we have $d(a_3,C) = d(b_2,C) = d(b_3,C) = 4$ and $d(a_4,C) + d(b_4,C) = 5$. In particular, $\{a_3y_3, a_3y_4, a_4y_3, b_2x_3, b_3x_4, b_4x_4\} \subseteq E$. Thus $G[P_2 \bigcup C]$ contains two independent 8-cycles $a_1y_1x_2y_2x_3b_2a_2b_1a_1$ and $b_4x_4b_3x_1y_4a_3y_3a_4b_4$, a contradiction.

So $b_3x_2 \notin E$. Since $e(P_1,C) \geq 25$, $d(b_2,C) = 4$, $d(b_3,C) = 3$ and $d(a_4,C) + d(b_4,C) + d(a_3,C) = 10$. In particular, $b_4x_1 \in E$, $a_3y_3 \in E$, $x_3b_2 \in E$. Then $G[P_2 \bigcup C]$ contains two independent 8-cycles $a_1b_1a_2b_2x_3y_2x_2y_1a_1$ and $b_4x_1y_4x_4y_3a_3b_3a_4b_4$, a contradiction.

Case 2. $e(P_1,C) \leq 24$ and $e(P_2,C) \leq 24$. Since $e(P,C) \geq 31$, we have $e(a_1b_1,C) \geq 7$, $e(a_1b_5, C) \geq 7$ and $e(a_5b_5,C) \geq 7$. W.l.o.g., say $d(a_1,C) = 4$, $d(b_5,C) \geq 3$. If $x_i \in N(b_1,C) \bigcap N(b_4, C)$ for some $i \in \{1,2,3,4\}$, then $\langle x_i, b_1a_2b_2a_3b_3a_4b_4\rangle \supseteq C^8$ and $\langle a_1, V(C) - x_i\rangle \supseteq C^8$, a contradiction. Therefore, we have $N(b_1,C) \bigcap N(b_4,C) = \emptyset$ and $d(b_1,C) + d(b_4,C) \leq 4$.

Suppose $d(b_5,C) = 4$. Just as the same proof before, if $N(a_2,C) \bigcap N(a_5,C) \neq \emptyset$, then $G[P_1 \bigcup C]$ contains two independent 8-cycles, a contradiction. Therefore, we have $N(a_2,C) \bigcap N(a_5,C) = \emptyset$ and $d(a_2,C) + d(a_5,C) \leq 4$. Since $e(P,C) \geq 31$, it follows that $d(b_2,C) + d(a_3,C) + d(b_3,C) + d(a_4,C) \geq 15$. This implies either $e(b_2a_3,C) = 8$ or $e(b_3a_4,C) = 8$, w.l.o.g., say $e(b_2a_3,C) = 8$. Therefore, $d(b_2,C) = d(a_3,C) = 4$, $e(b_3a_4,C) \geq 7$. Since $d(b_1,C) \geq 3$ and $d(a_5,C) \geq 3$, we have $e(b_1b_2, x_1x_3) \geq 3$ and $e(a_3a_5, y_3y_4) \geq 3$. Therefore, $\langle b_1a_2b_2, x_1y_1x_2y_2x_3\rangle \supseteq C^8$ and $\langle y_3x_4y_4, a_3b_3a_4b_4a_5\rangle \supseteq C^8$ by Lemma 2.1. This implies $G[P_3 \bigcup C]$ contains two independent 8-cycles, a contradiction.

Now we have $d(b_5,C) = 3$. Since $d(a_5,C) + d(b_5,C) \geq 7$, we have $d(a_5,C) = 4$. Since $d(b_1,C) \geq 3$, w.l.o.g., say $N(b_1,C) \supseteq \{x_1,x_2,x_3\}$. If $d(b_3,x_1x_2) > 0$, then $\langle b_1a_2b_2a_3b_3, x_1y_1x_2\rangle \supseteq C^8$ and therefore $\langle y_2x_3y_3x_4y_4, a_4b_4a_5\rangle \not\supseteq C^8$. This implies that $d(a_4,\ y_2y_4) =$

0. Similarly, if $d(b_3,x_3x_4)>0$, then $d(a_4,y_1y_3)=0$. Therefore, we have $d(b_3,C)+d(a_4,C)\le 4$. Since $e(P,C)\ge 31$, it follows that $d(a_3,C)=d(b_2,C)=d(a_5,C)=4$ and $d(b_1,C)+d(b_4,C)=4$. Therefore, $e(b_1b_2,x_1x_3)=4$ and $e(a_3a_5,y_3y_4)=4$. By Lemma 2.1, $\langle b_1a_2b_2,x_1y_1x_2y_2x_3\rangle\supseteq C^8$ and $\langle y_3x_4y_4,a_3b_3a_4b_4a_5\rangle\supseteq C^8$, a contradiction. ∎

Lemma 2.4. *Let $C_1=a_1b_1a_2b_2a_3b_3a_4b_4a_1$, $C_2=x_1y_1x_2y_2x_3y_3x_4y_4x_1$ be two independent 8-cycles, where $\{a_1,x_1\}\subseteq V_1$. If $e(C_1,C_2)\ge 25$, then $G[C_1\bigcup C_2]$ contains two independent 8-cycles C', C'' such that each of them has a chord.*

Proof. Suppose on the contrary that the lemma fails. Since $e(C_1,C_2)\ge 25$, we have $d(u,C_2)=4$ and $d(v,C_1)=4$ for some $u\in V(C_1)$, $v\in V(C_2)$. If $\{u,v\}\subseteq V_i$ for some $i\in\{1,2\}$, then C_1-u+v, C_2-v+u contain two independent 8-cycles with each having a chord, a contradiction. Therefore, we may assume

$$d(a_1,C_2)=4,\quad d(y_1,C_1)=4.$$

$$d(b_i,C_2)\le 3,\quad d(x_i,C_1)\le 3\quad \text{for every } i\in\{1,2,3,4\}$$

Suppose $d(a_3,C_2)\ge 3$. Then $e(a_1a_3,C_2)\ge 7$. Therefore, $e(a_1a_3,y_1y_2)\ge 3$ and $e(a_1a_3,y_1y_4)\ge 3$. By Lemma 2.1, both $\langle a_1b_1a_2b_2a_3,y_1x_2y_2\rangle$ and $\langle a_1b_1a_2b_2a_3,y_1x_1y_4\rangle$ contain an 8-cycle with each having a chord. Therefore, neither $\langle b_3a_4b_4,x_3y_3x_4y_4x_1\rangle$ nor $\langle b_3a_4b_4,x_2y_2x_3y_3x_4\rangle$ contain an 8-cycle with each having a chord. This implies $e(b_3b_4,x_1x_3)\le 2$ and $e(b_3b_4,x_2x_4)\le 2$ by Lemma 2.1. In particular, $e(b_3b_4,C_2)\le 4$. Similarly, we have both $\langle a_1b_4a_4b_3a_3,y_1x_2y_2\rangle$ and $\langle a_1b_4a_4b_3a_3,y_1x_1y_4\rangle$ contain an 8-cycle with each having a chord. This implies $e(b_1b_2,C_2)\le 4$. Then $\sum_{i=1}^{4}d(b_i,C_2)\le 8$ and therefore $e(C_1,C_2)\le 24$, a contradiction.

Now we have $d(a_3,C_2)\le 2$. With the same proof, if $e(a_2a_4,C_2)\ge 7$, then $e(C_1,C_2)\le 24$, a contradiction. Thus, $e(a_2a_4,C_2)\le 6$. This implies that $\sum_{i=1}^{4}d(a_i,C_2)\le 12$. Note that $d(b_i,C_2)\le 3$ for every $i\in\{1,2,3,4\}$, then $e(C_1,C_2)\le 24$, a contradiction. This completes the proof of Lemma 2.4. ∎

Lemma 2.5. *Let $C=a_1b_1a_2b_2a_3b_3a_1$ be a 6-cycle, $P=x_1y_1$ be a 2-path, C and P are independent, where $\{a_1,x_1\}\subseteq V_1$. If $d(x_1,C)\ge 2$, $d(y_1,C)>0$, then $G[C\bigcup P]$ contains an 8-cycle with at least two chords.*

Proof. If $d(x_1,C)=3$, obviously $G[C\bigcup P]$ contains an 8-cycle with at least two chords. Now we may assume $d(x_1,C)=2$, w.l.o.g., say $N(x_1,C)=\{b_1,b_2\}$. Since $d(y_1,C)>0$, we have either $a_1y_1\in E$, or $a_2y_1\in E$, or $a_3y_1\in E$. If $a_1y_1\in E$, then $G[C\bigcup P]$ contains an 8-cycle $a_1b_3a_3b_2a_2b_1x_1y_1a_1$ with two chords a_1b_1 and b_2x_1; if $a_2y_1\in E$, then $G[C\bigcup P]$ contains an 8-cycle $x_1b_1a_1b_3a_3b_2a_2y_1x_1$ with two chords a_2b_1 and x_1b_2; if $a_3y_1\in E$, then $G[C\bigcup P]$ contains an 8-cycle $y_1a_3b_3a_1b_1a_2b_2x_1y_1$ with two chords b_1x_1 and a_3b_2. In each case, $G[C\bigcup P]$ contains an 8-cycle with at least two chords. ∎

Lemma 2.6. *Let $P_1=a_1b_1a_2b_2a_3$ be a 5-path, $P_2=y_1x_2y_2$ be a 3-path, P_1 and P_2 are independent, where $\{a_1,x_2\}\subseteq V_1$. Then the following two statements hold.*

(1) *If $e(a_1a_3,y_1y_2)=4$, then $G[P_1\bigcup P_2]$ contains an 8-cycle with two chords.*
(2) *If $e(a_1a_3,y_1y_2)\ge 3$ and $a_1b_2\in E$, then $G[P_1\bigcup P_2]$ contains an 8-cycle with two chords.*

Proof. Easy to check. ∎

Lemma 2.7. *Let C_1 and C_2 be two independent 8-cycles in G such that each of them has a chord. If $e(C_1,C_2) \geq 25$, then $G[C_1 \bigcup C_2]$ contains two independent 8-cycles such that each of them has at least two chords.*

Proof. For simplicity, we will use K to denote an 8-cycle with two chords in the following. Suppose on the contrary that the lemma fails. Let $C_1 = a_1b_1a_2b_2a_3b_3a_4b_4a_1$ with chord a_1b_2 and $C_2 = x_1y_1x_2y_2x_3y_3x_4y_4x_1$ with chord x_1y_2, $\{a_1,x_1\} \subseteq V_1$, w.l.o.g. As $e(C_1,C_2) \geq 25$, we can assume $d(a_i,C_2) = 4$ for some i. If $d(x_j,C_1) = 4$ for some j, then $C_1 - a_i + x_j$, $C_2 - x_j + a_i$ contain two independent 8-cycles with each having at least two chords, a contradiction. Therefore, $d(x_j,C_1) \leq 3$ for every $j \in \{1,2,3,4\}$. Similarly, if $d(b_t,C_2) = 4$ for some t, then $d(y_j,C_1) \leq 3$ for every $j \in \{1,2,3,4\}$. This implies that $e(C_1,C_2) \leq 24$, a contradiction. Thus, we get $d(b_t,C_2) \leq 3$ for every $t \in \{1,2,3,4\}$.

As $G[C_1 \bigcup C_2]$ doesn't contain two independent 8-cycles such that each of them has at least two chords, then either $\langle a_2b_1, C_2 - x_2 - y_1\rangle \not\supseteq K$ or $\langle x_2y_1, C_1 - a_2 - b_1\rangle \not\supseteq K$, w.l.o.g., say $\langle x_2y_1, C_1 - a_2 - b_1\rangle \not\supseteq K$. The following proof is divided into three cases.

Case 1. $d(x_2, C_1 - a_2 - b_1) \leq 2$ and $d(y_1, C_1 - a_2 - b_1) \leq 2$. Since $\langle x_2y_1, C_1 - a_2 - b_1\rangle \not\supseteq K$, we have $e(x_2y_1, C_1 - a_2 - b_1) \leq 2$ by Lemma 2.5. This implies $d(x_2,C_1) + d(y_1,C_1) \leq 4$. Since $e(C_1,C_2) \geq 25$, $d(y_2,C_1) = d(y_4,C_1) = 4$ and $d(x_3,C_1) = d(x_4,C_1) = 3$. If $e(x_3x_4, b_2b_4) \geq 3$, then $\langle b_4a_1b_1a_2b_2, x_3y_3x_4\rangle \supseteq K$ and $\langle a_3b_3a_4, y_4x_1y_1x_2y_2\rangle \supseteq K$ by Lemma 2.6, a contradiction. Thus, $e(x_3x_4, b_2b_4) \leq 2$. Note that $d(x_3,C_1) = d(x_4,C_1) = 3$, then $e(x_3x_4, b_1b_3) = 4$. By Lemma 2.6, $\langle x_3y_3x_4, b_1a_2b_2a_3b_3\rangle \supseteq K$ and $\langle a_1b_4a_4, y_4x_1y_1x_2y_2\rangle \supseteq K$, a contradiction.

Case 2. $d(x_2, C_1 - a_2 - b_1) = 3$. Since $\langle x_2y_1, C_1 - a_2 - b_1\rangle \not\supseteq K, d(y_1, C_1 - a_2 - b_1) = 0$ by Lemma 2.5. This implies that $d(y_1,C_1) \leq 1$. Since $e(C_1,C_2) \geq 25$, it follows that $d(y_2,C_1) = d(y_4,C_1) = 4$ and $d(x_3,C_1) = d(x_4,C_1) = 3$. The rest of the proof is just same as that in Case 1.

Case 3. $d(y_1, C_1 - a_2 - b_1) = 3$. Obviously, $a_1y_1 \in E$, $a_3y_1 \in E$ and $a_4y_1 \in E$. Since $\langle x_2y_1, C_1 - a_2 - b_1\rangle \not\supseteq K$, we have $d(x_2, C_1 - a_2 - b_1) = 0$ by Lemma 2.5. This implies $x_2b_2 \notin E$, $x_2b_3 \notin E$ and $x_2b_4 \notin E$.

Suppose $x_2b_1 \notin E$. Then $d(x_2,C_1) = 0$. Since $e(C_1,C_2) \geq 25$, $d(y_2,C_1) = d(y_4,C_1) = 4$ and $d(x_3,C_1) = d(x_4,C_1) = 3$. The rest of the proof is just same as that in Case 1.

Now we get $x_2b_1 \in E$ and $d(x_2,C_1) = 1$. Since $e(C_1,C_2) \geq 25$, we have $e(y_1y_2y_3y_4, C_1) \geq 15$. Since $x_2b_1a_1b_4a_4b_3a_3y_1x_2$ is an 8-cycle in G with two chords a_1y_1 and a_4y_1, $\langle a_2b_2, x_1y_2x_3y_3x_4y_4\rangle \not\supseteq K$. Note that $e(y_1y_2y_3y_4, C_1) \geq 15$, then $d(a_2, y_2y_3y_4) \geq 2$. By Lemma 2.5, $d(b_2, x_1x_3x_4) = 0$. This implies $x_1b_2 \notin E$, $x_3b_2 \notin E$ and $x_4b_2 \notin E$. Since $e(y_1y_2y_3y_4, C_1) \geq 15$, we have $e(y_3y_4, a_1a_3) \geq 3$. By Lemma 2.6, $\langle a_1b_1a_2b_2a_3, y_3x_4y_4\rangle \supseteq K$ and therefore $\langle b_3a_4b_4, x_1y_1x_2y_2x_3\rangle \not\supseteq K$. This implies $e(b_3b_4, x_1x_3) \leq 2$. Note that $x_1b_2 \notin E$ and $x_3b_2 \notin E$, then $e(x_1x_3, C_1) \leq 4$. Since $d(x_2,C_1) = 1$ and $d(x_4,C_1) \leq 3$, we have $e(C_1,C_2) \leq 24$, a contradiction. ∎

3. Proof of Theorem 1.2

In this section, we will prove the Theorem 1.2. Let $G = (V_1,V_2;E)$ be a bipartite graph with $|V_1| = |V_2| = 4k$ and the minimum degree $\delta(G) \geq 3k+1$, where k is a positive integer.

We first claim that $G \supseteq kC^8$. Suppose on the contrary that this is not true. We can assume that G is a maximal counterexample, i.e., $G + xy \supseteq kC^8$ for every edge $xy \notin E(G)$.

Then G contains $k-1$ independent 8-cycles $C_1, C_2, \cdots, C_{k-1}$ and an 8-path P such that P is independent of $H = \bigcup_{i=1}^{k-1} C_i$. Denote $P = a_1b_1a_2b_2a_3b_3a_4b_4$ with $a_1 \in V_1$. As $G \not\supseteq kC^8$, we have $G[V(P)] \not\supseteq C^8$. Then $e(G[P]) \leq 16 - 3 = 13$ and $e(a_1b_4, P) \leq 4$. Hence, $e(a_1b_4, H) \geq 6(k-1) + 4$. Then there exists an 8-cycle $C_i \subseteq H$ such that $d(a_1b_4, C_i) \geq 7$. W.l.o.g., say $d(a_1b_4, C_1) \geq 7$, $C_1 = x_1y_1x_2y_2x_3y_3x_4y_4x_1$ with $x_1 \in V_1$. We may assume $d(a_1, C_1) = 4$ and $d(b_4, C_1) \geq 3$. In particular, w.l.o.g., let $\{x_1, x_2, x_3\} \subseteq N(b_4, C_1)$. Denote $P' = y_1Px_1$. Since $G[P \bigcup C_1] \not\supseteq 2C^8$, we have $e(P, C_1) \leq 24$ by Lemma 2.2. Thus, $\sum_{x \in V(P)} d(x, P \bigcup C_1) \leq 50$. Since $e(x_1y_1, P \bigcup C_1) \leq 16$, it follows that $\sum_{x \in V(P')} d(x, P \bigcup C_1) \leq 66$. Then we have $e(P', H - C_1) \geq 10(3k+1) - 66 = 30(k-2) + 4$. This means that there exists an 8-cycle $C_j \in H - C_1$ such that $e(P', C_j) \geq 31$. W.l.o.g., say $j = 2$. Since $G[P \bigcup C_2] \not\supseteq 2C^8$, we have either $G[P' \bigcup C_2 - x_1 - b_4] \supseteq 2C^8$ or $G[P' \bigcup C_2 - a_1 - y_1] \supseteq 2C^8$ by Lemma 2.3. In the former case, $x_1b_4x_2y_2x_3y_3x_4y_4x_1$ is an 8-cycle in G. In the latter case, $a_1y_1x_2y_2x_3y_3x_4y_4a_1$ is an 8-cycle in G. Therefore, $G[P \bigcup C_1 \bigcup C_2] \supseteq 3C^8$, a contradiction. This proves our claim.

Now we choose k independent 8-cycles in G with as many having a chord as possible. Let $C_1, C_2, \cdots, C_k$ be such a choice. We claim that C_i has a chord for every $i \in \{1, 2, \cdots, k\}$. If not, assume C_i has no chord for some i, say $i = 1$. This implies $e(C_1) = 8$. Thus, $\sum_{x \in V(C_1)} d(x, \bigcup_{i=2}^{k} C_i) \geq 8(3k+1) - 16 = 24(k-1) + 16$. Therefore, there exists an 8-cycle $C_j (j \neq 1)$, such that $e(C_1, C_j) \geq 25$. By Lemma 2.4, $G[C_1 \bigcup C_j]$ contains two independent 8-cycles with each having a chord. This contradicts to our choice of $C_1, C_2, \cdots, C_k$.

Finally, we choose k independent 8-cycles in G such that each 8-cycle has a chord. Subject to this, we choose k independent 8-cycles in G with as many having at least two chords as possible. Let $C_1, C_2, \cdots, C_k$ be such a choice. If C_i does not have two chords for some $i \in \{1, 2, \cdots, k\}$, say $i = 1$. This implies $e(C_1) = 9$. Thus, $\sum_{x \in V(C_1)} d(x, \bigcup_{i=2}^{k} C_i) \geq 8(3k+1) - 18 = 24(k-1) + 14$. Therefore, there exists an 8-cycle $C_j (j \neq 1)$, such that $e(C_1, C_j) \geq 25$. By Lemma 2.7, $G[C_1 \bigcup C_j]$ contains two independent 8-cycles with each having at least two chords. This contradicts to our choice of $C_1, C_2, \cdots, C_k$. This completes the whole proof.

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