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On the Total $\{k\}$-Domination and Total $\{k\}$-Domatic Number of Graphs

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Abstract. For a positive integer k, a *total* $\{k\}$*-dominating function* of a graph G without isolated vertices is a function f from the vertex set $V(G)$ to the set $\{0,1,2,\ldots,k\}$ such that for any vertex $v \in V(G)$, the condition $\sum_{u\in N(v)} f(u) \geq k$ is fulfilled, where $N(v)$ is the open neighborhood of v. The *weight* of a total $\{k\}$-dominating function f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *total* $\{k\}$*-domination number*, denoted by $\gamma_t^{\{k\}}(G)$, is the minimum weight of a total $\{k\}$-dominating function on G. A set $\{f_1, f_2, \ldots, f_d\}$ of total $\{k\}$-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *total* $\{k\}$*-dominating family* (of functions) on G. The maximum number of functions in a total $\{k\}$-dominating family on G is the *total* $\{k\}$*-domatic number* of G, denoted by $d_t^{\{k\}}(G)$. Note that $d_t^{\{1\}}(G)$ is the classic total domatic number $d_t(G)$. In this paper, we present bounds for the total $\{k\}$-domination number and total $\{k\}$-domatic number. In addition, we determine the total $\{k\}$-domatic number of cylinders and we give a Nordhaus-Gaddum type result.

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1. Introduction

In this paper, G is a simple graph with no isolated vertices and with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. If $S \subseteq V(G)$, then $G[S]$ is the subgraph of G induced by S. The complement of a graph G is denoted by $\overline{G}$. Consult [3, 6] for the notation and terminology which are not defined here.

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A subset S of vertices of G is a *total dominating set* if $N(S) = V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. A total domatic partition is a partition of V into total dominating sets, and the total domatic number $d_t(G)$ is the largest number of sets in a total domatic partition. The total domatic number was introduced by Cockayne *et al.* in [2].

For a positive integer k, a *total $\{k\}$-dominating function* (T$\{k\}$DF) of a graph G without isolated vertices is a function f from the vertex set $V(G)$ to the set $\{0,1,2,\ldots,k\}$ such that for any vertex $v \in V(G)$, the condition $\sum_{u\in N(v)} f(u) \geq k$ is fulfilled. The *weight* of a T$\{k\}$DF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *total $\{k\}$-domination number* of a graph G, denoted by $\gamma_t^{\{k\}}(G)$, is the minimum weight of a T$\{k\}$DF of G. A $\gamma_t^{\{k\}}(G)$*-function* is a total $\{k\}$-dominating function of G with weight $\gamma_t^{\{k\}}(G)$. Note that $\gamma_t^{\{1\}}(G)$ is the classical total domination number $\gamma_t(G)$. The total $\{k\}$-domination number was introduced by Li and Hou [4].

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct total $\{k\}$-dominating functions of G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *total $\{k\}$-dominating family* (of functions) on G. The maximum number of functions in a total $\{k\}$-dominating family (T$\{k\}$D family) on G is the *total $\{k\}$-domatic number* of G, denoted by $d_t^{\{k\}}(G)$. The total $\{k\}$-domatic number is well-defined and

$$d_t^{\{k\}}(G) \geq 1 \tag{1.1}$$

for all graphs G without isolated vertices, since the set consisting of the function $f : V(G) \to \{0,1,2,\ldots,k\}$ defined by $f(v) = k$ for each $v \in V(G)$, forms a T$\{k\}$D family on G. The total $\{k\}$-domatic number was introduced by Sheikholeslami and Volkmann [5] and has also been studied in [1].

In this paper, we continue the study of the total $\{k\}$-domination number and total $\{k\}$-domatic number in graphs. We first study bounds for the total $\{k\}$-domination number and total $\{k\}$-domatic number. Then we determine the total $\{k\}$-domatic number of some cylinders and we present a Nordhaus-Gaddum type result.

The following known results are useful for our investigations.

Theorem 1.1 (Chen, Hou, Li [1]). *Let G be a graph without isolated vertices and $\delta = \delta(G)$. If $\delta \mid k$, then $d_t^{\{k\}}(G) \geq \delta - 1$, and if $\delta \nmid k$, then $d_t^{\{k\}}(G) \geq \lfloor k / \lceil \frac{k}{\delta} \rceil \rfloor$.*

Theorem 1.2 (Sheikholeslami, Volkmann [5]). *If G is a graph of order n without isolated vertices, then*

$$\gamma_t^{\{k\}}(G) \cdot d_t^{\{k\}}(G) \leq kn.$$

Moreover, if $\gamma_t^{\{k\}}(G) \cdot d_t^{\{k\}}(G) = kn$, then for each T$\{k\}$D family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d_t^{\{k\}}(G)$, each function f_i is a $\gamma_t^{\{k\}}(G)$-function and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V$.

Theorem 1.3 (Sheikholeslami, Volkmann [5]). *For every graph G without isolated vertices,*

$$d_t^{\{k\}}(G) \leq \delta(G).$$

Moreover, if $d_t^{\{k\}}(G) = \delta(G)$, then for each function of any T$\{k\}$D family $\{f_1, f_2, \cdots, f_d\}$ and for all vertices v of degree $\delta(G)$, $\sum_{u\in N(v)} f_i(u) = k$ and $\sum_{i=1}^{d} f_i(u) = k$ for every $u \in N(v)$.

Theorem 1.4 (Sheikholeslami, Volkmann [5]). *If G is a graph of order n without isolated vertices and k a positive integer, then*

$$\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \leq nk + 1.$$

Theorem 1.5 (Sheikholeslami, Volkmann [5]). *Let G be a graph of order n without isolated vertices and k a positive integer. If $d_t^{\{k\}}(G) \geq 2$, then*

$$\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \leq \frac{kn}{2} + 2.$$

If each component of a graph G has at least three vertices, then we can improve Theorem 1.4 a little bit.

Proposition 1.1. *Let $k \geq 2$ be an integer, and let G be a graph of order n. If each component of G has at least three vertices, then*

$$\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \leq \frac{2kn}{3} + 1 \leq kn - 1.$$

Proof. In view of [2], the inequality $\gamma_t(G) \leq 2n/3$ is valid. This implies that

$$\gamma_t^{\{k\}}(G) \leq k\gamma_t(G) \leq \frac{2kn}{3}.$$

If $d_t^{\{k\}}(G) = 1$, then it follows that

$$\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \leq \frac{2kn}{3} + 1 \leq kn - 1.$$

If $d_t^{\{k\}}(G) \geq 2$, then we deduce from Theorem 1.5 that

$$\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \leq \frac{kn}{2} + 2 \leq \frac{2kn}{3} + 1 \leq kn - 1. \qquad \blacksquare$$

Observation 1.1. If $G = P_r \times P_t$ is a grid of order $n = rt$ such that $2 \leq r \leq t$, then $d_t^{\{k\}}(G) = 2$.

Proof. According to Theorem 1.3, $d_t^{\{k\}}(G) \leq 2$. Now let $V(G) = \{x_{i,j} | 1 \leq i \leq r \text{ and } 1 \leq j \leq t\}$ be the vertex set of G. Define $f, g : V(G) \to \{0, 1, 2, \ldots, k\}$ by $f(x_{i,j}) = k$ if i is odd and $f(x_{i,j}) = 0$ if i is even and $g(x_{i,j}) = k$ if i is even and $g(x_{i,j}) = 0$ if i is odd. Now $\{f, g\}$ is a T$\{k\}$D family on G. Therefore $d_t^{\{k\}}(G) \geq 2$ and thus $d_t^{\{k\}}(G) = 2$. ∎

2. Total $\{k\}$-domination and domatic numbers of p-partite graphs

Theorem 2.1. *Let G be a p-partite graph without isolated vertices and $p \geq 2$. If $k \geq 1$ is an integer, then*

$$\gamma_t^{\{k\}}(G) \geq \left\lceil \frac{pk}{p-1} \right\rceil. \tag{2.1}$$

Proof. Let f be a $\gamma_t^{\{k\}}(G)$-function, and let $V_1, V_2, \ldots, V_p$ be the partite sets of G. If $w_i \in V_i$ for $1 \leq i \leq p$, then the definition implies that $\sum_{x \in N(w_i)} f(x) \geq k$ for $1 \leq i \leq p$. It follows that

$$(p-1)\omega(f) = (p-1) \sum_{x \in V(G)} f(x) = \sum_{i=1}^{p} \sum_{x \in (V(G) - V_i)} f(x) \geq \sum_{i=1}^{p} \sum_{x \in N(w_i)} f(x) \geq pk$$

and thus $\gamma_t^{\{k\}}(G) \geq \lceil pk/(p-1) \rceil$. ∎

Since each graph without isolated vertices is p-partite for some $p \geq 2$, the next corollary follows immediately from Theorem 2.1.

Corollary 2.1 (Sheikholeslami, Volkmann [5]). *For each positive integer k and any graph G without isolated vertices, $\gamma_t^{\{k\}}(G) \geq k+1$.*

The next examples will demonstrate that inequality (2.1) is sharp.

Let $k \geq 1$ be an integer, and let H be a complete p-partite ($p \geq 2$) graph with the partite sets $V_1, V_2, \ldots, V_p$ such that $v_i \in V_i$ for $i = 1, 2, \ldots, p$.

Assume first that $k = s(p-1)$ with an integer $s \geq 1$. Define $f : V(H) \to \{0, 1, 2, \ldots, k\}$ by $f(v_i) = s$ for $i = 1, 2, \ldots, p$ and $f(x) = 0$ for $x \in V(H) - \{v_1, v_2, \ldots, v_p\}$. We observe that $\sum_{v \in N(u)} f(x) \geq (p-1)s = k$ for each vertex $u \in V(H)$, and therefore f is a T$\{k\}$DF. It follows that $\gamma_t^{\{k\}}(H) \leq ps = \lceil pk/(p-1) \rceil$ and thus Theorem 2.1 implies that $\gamma_t^{\{k\}}(H) = \lceil pk/(p-1) \rceil$.

Assume second that $k = s(p-1) + r$ with integers $s \geq 0$ and $1 \leq r \leq p-2$. Define $f : V(H) \to \{0, 1, 2, \ldots, k\}$ by $f(v_1) = f(v_2) = \ldots = f(v_{r+1}) = s+1$, $f(v_{r+2}) = f(v_{r+3}) = \ldots = f(v_p) = s$ and $f(x) = 0$ for $x \in V(H) - \{v_1, v_2, \ldots, v_p\}$. We see that $\sum_{v \in N(u)} f(x) \geq (p-1)s + r = k$ for each vertex $u \in V(H)$, and therefore f is a T$\{k\}$DF. It follows that

$$\begin{aligned}\gamma_t^{\{k\}}(H) &\leq ps + r + 1 = ps + r + \left\lceil \frac{r}{p-1} \right\rceil = ps + \left\lceil \frac{(p-1)r + r}{p-1} \right\rceil \\ &= ps + \left\lceil \frac{pr}{p-1} \right\rceil = \left\lceil \frac{ps(p-1) + pr}{p-1} \right\rceil = \left\lceil \frac{pk}{p-1} \right\rceil\end{aligned}$$

and thus Theorem 2.1 implies that $\gamma_t^{\{k\}}(H) = \lceil pk/(p-1) \rceil$.

Proposition 2.1. *Let G be a bipartite graph without isolated vertices. If $k \geq 1$ is an integer and X and Y are the partite sets of G, then $\gamma_t^{\{k\}}(G) \geq 2k$ with equality if and only if there exist two vertices $u \in X$ and $v \in Y$ such that $N(u) = Y$ and $N(v) = X$.*

Proof. It follows from Theorem 2.1 that $\gamma_t^{\{k\}}(G) \geq 2k$. If there exist two vertices $u \in X$ and $v \in Y$ such that $N(u) = Y$ and $N(v) = X$, then define $f : V(G) \to \{0, 1, 2, \ldots, k\}$ by $f(u) = f(v) = k$ and $f(x) = 0$ for $x \in V(G) - \{u, v\}$. Obviously, f is a total $\{k\}$-dominating function of G. This implies that $\gamma_t^{\{k\}}(G) \leq 2k$ and so $\gamma_t^{\{k\}}(G) = 2k$.

Conversely, assume that $\gamma_t^{\{k\}}(G) = 2k$, and let f be a $\gamma_t^{\{k\}}(G)$-function. It follows that

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(y) = k.$$

Now let $X^+ \subseteq X$ be such that $\sum_{x \in X^+} f(x) = k$ and $f(x) \geq 1$ for $x \in X^+$ and $Y^+ \subseteq Y$ be such that $\sum_{y \in Y^+} f(x) = k$ and $f(x) \geq 1$ for $y \in X^+$. Then $Y^+ \subseteq N(x)$ for each vertex $x \in X$ and $X^+ \subseteq N(y)$ for each vertex $y \in Y$. This leads to $N(x) = Y$ for each vertex $x \in X^+$ and $N(y) = X$ for each vertex $y \in Y^+$, and the proof is complete. ∎

Corollary 2.2. *If k is a positive integer, and G is a bipartite graph of order n without isolated vertices, then*

$$d_t^{\{k\}}(G) \leq \frac{n}{2},$$

with equality only if n is even and $\gamma_t^{\{k\}}(G) = 2k$.

Proof. According to Theorem 2.1, we have $\gamma_t^{\{k\}}(G) \geq 2k$. Therefore it follows from Theorem 1.2 that

$$d_t^{\{k\}}(G) \leq \frac{kn}{\gamma_t^{\{k\}}(G)} \leq \frac{kn}{2k} = \frac{n}{2},$$

and this is the desired inequality.

Assume that $d_t^{\{k\}}(G) = n/2$. The inequality chain above shows that $\gamma_t^{\{k\}}(G) = 2k$ and that n is even. ∎

Let G be isomorphic to the complete bipartite graph $K_{p,p}$ with the partite sets $\{u_1, u_2, \ldots, u_p\}$ and $\{v_1, v_2, \ldots, v_p\}$. Define $f_i : V(G) \to \{0, 1, 2, \ldots, k\}$ by $f_i(u_i) = f_i(v_i) = k$ and $f_i(x) = 0$ when $x \in V(G) - \{u_i, v_i\}$ for $1 \leq i \leq p$. Now $\{f_1, f_2, \ldots, f_p\}$ is a T$\{k\}$D family on G and thus $d_t^{\{k\}}(G) \geq p$. By Corollary 2.2, $d_t^{\{k\}}(G) \leq p$ and thus $d_t^{\{k\}}(G) = p$. This example shows that Corollary 2.2 is sharp.

3. Cylinder and torus

The *Cartesian product* $G = G_1 \times G_2$ of two disjoint graphs G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. The Cartesian product of a cycle $C_r = (x_1x_2 \ldots x_r)$ and a path $P_t = y_1y_2 \ldots y_t$ is called a *cylinder* and the Cartesian product of two cycles $C_r = (x_1x_2 \ldots x_r)$ and $C_t = (y_1y_2 \ldots y_t)$ is called a *torus*. If G is a cylinder (or torus), then let $V(G) = \{x_{i,j} | 1 \leq i \leq r \text{ and } 1 \leq j \leq t\}$ be the vertex set of G.

In this section we determine the total $\{k\}$-domination and domatic number of some cylinders and torus. First we determine the exact value of $d_t^{\{k\}}(C_n \times P_2)$. We start with the following proposition.

Proposition 3.1. *If $G = C_{3r} \times P_t$ is a cylinder of order $n = 3rt$ such that $2 \leq t$, then $d_t^{\{k\}}(G) = 3$.*

Proof. According to Theorem 1.3, $d_t^{\{k\}}(G) \leq 3$. Define $f, g, h : V(G) \to \{0, 1, 2, \ldots, k\}$ by $f(x_{i,j}) = k$ if $i \equiv 1 \pmod 3$ and $f(x_{i,j}) = 0$ otherwise, $g(x_{i,j}) = k$ if $i \equiv 2 \pmod 3$ and $g(x_{i,j}) = 0$ otherwise and $h(x_{i,j}) = k$ if $i \equiv 0 \pmod 3$ and $h(x_{i,j}) = 0$ otherwise. Now $\{f, g, h\}$ is a T$\{k\}$D family on G. Therefore $d_t^{\{k\}}(G) \geq 3$ and thus $d_t^{\{k\}}(G) = 3$. ∎

Proposition 3.2. *For $n \geq 3$,*

$$d_t^{\{k\}}(C_n \times P_2) = \begin{cases} 3 & \text{if} \quad n \equiv 0 \pmod 3 \\ 2 & \qquad \text{otherwise.} \end{cases}$$

Proof. If $n \equiv 0 \pmod 3$, then the result follows from Proposition 3.1.

Let now $n \not\equiv 0 \pmod 3$. Suppose that $\{f, g, h\}$ is a T$\{k\}$D family of $C_n \times P_2$. By Theorem 1.3, $\sum_{u \in N(v)} f(u) = k$ for each $v \in V(C_n \times P_2)$. Assume that $f(x_{1,1}) = a, f(x_{1,2}) = a', f(x_{2,1}) = b$ and $f(x_{2,2}) = b'$. Since $\sum_{u \in N(x_{2,1})} f(u) = k$ and $\sum_{u \in N(x_{2,2})} f(u) = k$, we have $f(x_{3,1}) = k - a - b'$ and $f(x_{3,2}) = k - a' - b$. Since also $\sum_{u \in N(x_{3,1})} f(u) = k$ and $\sum_{u \in N(x_{3,2})} f(u) = k$, we have $f(x_{4,1}) = a'$ and $f(x_{4,2}) = a$. By repeating this process, we distinguish four cases.

Case 1. Assume that $n \equiv 4 \pmod 6$.
Then $f(x_{n-2,1}) = b, f(x_{n-2,2}) = b', f(x_{n-1,1}) = k - a - b', f(x_{n-1,2}) = k - a' - b, f(x_{n,1}) = a'$

and $f(x_{n,2}) = a$. By Theorem 1.3,

$$k = \sum_{u \in N(x_{n,1})} f(u) = a + k - b', \tag{3.1}$$

$$k = \sum_{u \in N(x_{n,2})} f(u) = a' + k - b, \tag{3.2}$$

$$k = \sum_{u \in N(x_{1,1})} f(u) = 2a' + b, \tag{3.3}$$

$$k = \sum_{u \in N(x_{1,2})} f(u) = 2a + b'. \tag{3.4}$$

It follows from (3.1) and (3.4) that $a = b' = \frac{k}{3}$ and from (3.2) and (3.3) that $a' = b = k/3$. This implies that $f(x_{i,j}) = k/3$ for each i and j. An argument similar to that described above shows that $g(x_{i,j}) = k/3$ for each i and j which leads to the contradiction $f = g$.

Case 2. Assume that $n \equiv 5 \pmod 6$.
Then $f(x_{n-2,1}) = k - a - b', f(x_{n-2,2}) = k - a' - b, f(x_{n-1,1}) = a', f(x_{n-1,2}) = a, f(x_{n,1}) = b'$ and $f(x_{n,2}) = b$.

Case 3. Assume that $n \equiv 1 \pmod 6$.
Then $f(x_{n-2,1}) = b', f(x_{n-2,2}) = b, f(x_{n-1,1}) = k - a' - b, f(x_{n-1,2}) = k - a - b', f(x_{n,1}) = a$ and $f(x_{n,2}) = a'$.

Case 4. Assume that $n \equiv 2 \pmod 6$.
Then $f(x_{n-2,1}) = k - a' - b, f(x_{n-2,2}) = k - a - b', f(x_{n-1,1}) = a, f(x_{n-1,2}) = a', f(x_{n,1}) = b$ and $f(x_{n,2}) = b'$.

Using the same arguments as in Case 1, the Cases 2, 3 and 4 lead to a contradiction too. It follows that $d_t^{\{k\}}(C_n \times P_2) \le 2$. In addition, if we define $f, g : V(C_n \times P_2) \to \{0, 1, 2, \dots, k\}$ by $f(x_{i,1}) = k$ and $f(x_{i,2}) = 0$ and $g(x_{i,1}) = 0$ and $g(x_{i,2}) = k$ for $1 \le i \le n$, then $\{f, g\}$ is a T$\{k\}$D family on $C_n \times P_2$. Therefore $d_t^{\{k\}}(C_n \times P_2) \ge 2$ and thus $d_t^{\{k\}}(C_n \times P_2) = 2$ in these four cases, and the proof is complete. ∎

Proposition 3.3. *If $G = C_{3r+1} \times P_t$ is a cylinder of order $n = (3r+1)t$, $t \ge 3$ and k is even, then $d_t^{\{k\}}(G) = 3$.*

Proof. According to Theorem 1.3, $d_t^{\{k\}}(G) \le 3$. Define $f, g, h : V(G) \to \{0, 1, 2, \dots, k\}$ as follows: $f(x_{1,1}) = f(x_{1,t}) = k/2$, $f(x_{3m+2,j}) = f(x_{3m+4,j}) = k/2$ if $0 \le m \le r-1$, $1 \le j \le t$ and $f(x_{i,j}) = 0$ otherwise; $g(x_{2,1}) = g(x_{2,t}) = k/2$, $g(x_{1,j}) = g(x_{3,j}) = g(x_{3m+2,j}) = g(x_{3m+3,j}) = k/2$ for $1 \le j \le t$, $1 \le m \le r-1$ when $r \ge 2$ and $g(x_{i,j}) = 0$ otherwise and $h(x_{3m+3,j}) = h(x_{3m+4,j}) = k/2$ if $0 \le m \le r-1$, $1 \le j \le t$ and $h(x_{1,3s+2}) = h(x_{2,3s+2}) = k/2$ for $0 \le s \le (t-3)/3$ if $t \equiv 0 \pmod 3$, $h(x_{1,2}) = h(x_{2,2}) = h(x_{1,3s+3}) = h(x_{2,3s+3}) = k/2$ for $0 \le s \le (t-4)/3$ if $t \equiv 1 \pmod 3$, $h(x_{1,2}) = h(x_{2,2}) = h(x_{1,3s+4}) = h(x_{2,3s+4}) = k/2$ for $0 \le s \le (t-5)/3$ if $t \equiv 2 \pmod 3$, $h(x_{i,j}) = 0$ otherwise. Now it is easy to verify that $\{f, g, h\}$ is a T$\{k\}$D family on G. Therefore $d_t^{\{k\}}(G) \ge 3$ and thus $d_t^{\{k\}}(G) = 3$. ∎

Proposition 3.4. *If $G = C_{3r+2} \times P_t$ is a cylinder of order $n = (3r+2)t$, $t \ge 3$ and k is even, then $d_t^{\{k\}}(G) = 3$.*

Proof. According to Theorem 1.3, $d_t^{\{k\}}(G) \le 3$. Consider two cases.

Case 1. Assume that $t \equiv 1 \pmod 2$. Then $t = 3+2m$ for some $m \geq 0$. Define $f,g,h : V(G) \to \{0,1,2,\dots,k\}$ as follows: $f(x_{2,2s+1}) = f(x_{3,2s+1}) = k/2$ if $0 \leq s \leq m+1$, $f(x_{1,j}) = f(x_{4,j}) = k/2$ if $1 \leq j \leq t$, $f(x_{3l+6,j}) = f(x_{3l+7,j}) = k/2$ if $0 \leq l \leq r-2$ $(r \geq 2)$, $1 \leq j \leq t$, and $f(x_{i,j}) = 0$ otherwise, $g(x_{3,2s+1}) = g(x_{4,2s+1}) = k/2$ if $0 \leq s \leq m+1$, $g(x_{2,j}) = g(x_{5,j}) = k/2$ if $1 \leq j \leq t$, $g(x_{3l+7,j}) = g(x_{3l+8,j}) = k/2$ if $0 \leq l \leq r-2$ $(r \geq 2)$, $1 \leq j \leq t$, and $g(x_{i,j}) = 0$ otherwise, and $h(x_{2,2s}) = h(x_{4,2s}) = k/2$, $h(x_{3,2s}) = k$ if $1 \leq s \leq m+1$, $h(x_{1,j}) = h(x_{5,j}) = k/2$ if $1 \leq j \leq t$, $h(x_{3l+6,j}) = h(x_{3l+8,j}) = k/2$ if $0 \leq l \leq r-2$ $(r \geq 2)$, $1 \leq j \leq t$, and $h(x_{i,j}) = 0$ otherwise. It is easy to verify that $\{f,g,h\}$ is a T$\{k\}$D family on G. Therefore $d_t^{\{k\}}(G) \geq 3$ and so $d_t^{\{k\}}(G) = 3$.

Case 2. Assume that $t \equiv 0 \pmod 2$. Then $t = 4+2m$ for some $m \geq 0$. Define $f,g,h : V(G) \to \{0,1,2,\dots,k\}$ as follows: $f(x_{2,1}) = f(x_{3,1}) = k/2$, $f(x_{2,2s+4}) = f(x_{3,2s+4}) = k/2$ if $0 \leq s \leq m$, $f(x_{3l+6,j}) = f(x_{3l+7,j}) = k/2$ if $0 \leq l \leq r-2$ $(r \geq 2)$, $1 \leq j \leq t$, $f(x_{1,j}) = f(x_{4,j}) = k/2$ if $1 \leq j \leq t$ and $f(x_{i,j}) = 0$ otherwise, $g(x_{3,1}) = g(x_{4,1}) = k/2$, $g(x_{3,2s+4}) = g(x_{4,2s+4}) = k/2$ if $0 \leq s \leq m$, $g(x_{3l+7,j}) = g(x_{3l+8,j}) = k/2$ if $0 \leq l \leq r-2$ $(r \geq 2)$, $1 \leq j \leq t$, $g(x_{2,j}) = g(x_{5,j}) = k/2$ if $1 \leq j \leq t$ and $g(x_{i,j}) = 0$ otherwise, and $h(x_{2,2}) = h(x_{2,2s+1}) = k/2$, $h(x_{3,2}) = h(x_{3,2s+3}) = k$ if $0 \leq s \leq m$, $h(x_{3l+5,j}) = h(x_{3l+6,j}) = k/2$ if $0 \leq l \leq r-2$ $(r \geq 2)$, $1 \leq j \leq t$, $h(x_{1,j}) = h(x_{3r+2,j}) = k/2$ if $1 \leq j \leq t$ and $h(x_{i,j}) = 0$ otherwise. It is easy to verify that $\{f,g,h\}$ is a T$\{k\}$D family on G. Therefore $d_t^{\{k\}}(G) \geq 3$ and so $d_t^{\{k\}}(G) = 3$. ∎

Proposition 3.5. *If $G = C_{4r} \times P_{3t}$ is a cylinder of order $n = 12rt$ such that $t,r \geq 1$, then $d_t^{\{k\}}(G) = 3$.*

Proof. According to Theorem 1.3, $d_t^{\{k\}}(G) \leq 3$. Define $f,g,h : V(G) \to \{0,1,2,\dots,k\}$ by $f(x_{i,j}) = k$ if $i \equiv 1 \pmod 4$ and $j \equiv 1,2 \pmod 3$ and $f(x_{i,j}) = k$ if $i \equiv 3 \pmod 4$ and $j \equiv 0,2 \pmod 3$ and $f(x_{i,j}) = 0$ otherwise, $g(x_{i,j}) = k$ if $i \equiv 2 \pmod 4$ and $j \equiv 1,2 \pmod 3$ and $g(x_{i,j}) = k$ if $i \equiv 0 \pmod 4$ and $j \equiv 0,2 \pmod 3$ and $g(x_{i,j}) = 0$ otherwise, $h(x_{i,j}) = k$ if $i \equiv 0,3 \pmod 4$ and $j \equiv 1 \pmod 3$ and $h(x_{i,j}) = k$ if $i \equiv 1,2 \pmod 4$ and $j \equiv 0 \pmod 3$ and $h(x_{i,j}) = 0$ otherwise. Now $\{f,g,h\}$ is a T$\{k\}$D family on G. Therefore $d_t^{\{k\}}(G) \geq 3$ and thus $d_t^{\{k\}}(G) = 3$. ∎

Proposition 3.6. *If $G = C_{4r} \times P_{2t+1}$ is a cylinder of order $n = 4r(2t+1)$ such that $t,r \geq 1$, then $d_t^{\{k\}}(G) = 3$.*

Proof. According to Theorem 1.3, $d_t^{\{k\}}(G) \leq 3$. Define $f,g,h : V(G) \to \{0,1,2,\dots,k\}$ by $f(x_{4m+1,4l+1}) = f(x_{4m+2,4l+1}) = (k+1)/2$, $0 \leq m \leq r-1$, $0 \leq l \leq \lfloor t/2 \rfloor$, $f(x_{4m+1,4l+3}) = f(x_{4m+2,4l+3}) = (k-1)/2$, $0 \leq m \leq r-1$, $0 \leq l \leq \lceil t/2 \rceil - 1$, $f(x_{4m+3,4l+1}) = f(x_{4m+4,4l+1}) = (k-1)/2$, $0 \leq m \leq r-1$, $0 \leq l \leq \lfloor t/2 \rfloor$, $f(x_{4m+3,4l+3}) = f(x_{4m+4,4l+3}) = (k+1)/2$, $0 \leq m \leq r-1$, $0 \leq l \leq \lceil t/2 \rceil - 1$, $f(x_{i,j}) = 0$ otherwise, $g(x_{i,j}) = k - f(x_{i,j})$ when $f(x_{i,j}) \neq 0$ and $g(x_{i,j}) = 0$ otherwise, and $h(x_{i,2s}) = k$ if $1 \leq i \leq 4r$, $1 \leq s \leq t$ and $h(x_{i,j}) = 0$ otherwise. Clearly $\{f,g,h\}$ is a T$\{k\}$D family on G. Therefore $d_t^{\{k\}}(G) \geq 3$ and thus $d_t^{\{k\}}(G) = 3$. ∎

Theorem 3.1. *Let $p,r \geq 2$ be two integers, and let G be a p-regular graph of order pr. If $V(G)$ has a partition in p sets $\{u_1^i, u_2^i, \dots, u_r^i\}$ such that the subgraph $G[\{u_1^i, u_2^i, \dots, u_r^i\}]$ has no isolated vertices and $N(u_1^i) \cup N(u_2^i) \cup \dots \cup N(u_r^i) = V(G)$ for $i = 1,2,\dots,p$, then $d_t^{\{k\}}(G) = p$.*

Proof. According to Theorem 1.3, $d_t^{\{k\}}(G) \le p$. Define $f_i : V(G) \to \{0,1,2,\dots,k\}$ by $f_i(u_1^i) = f_i(u_2^i) = \dots = f_i(u_r^i) = k$ and $f_i(x) = 0$ for $x \in V(G) - \{u_1^i, u_2^i, \dots, u_r^i\}$ for $i = 1,2,\dots,p$. The hypothesis shows that $\{f_1, f_2, \dots, f_p\}$ is a T$\{k\}$D family on G. Therefore $d_t^{\{k\}}(G) \ge p$ and thus $d_t^{\{k\}}(G) = p$. ∎

The complete bipartite graph $K_{p,p}$ and the torus $C_4 \times C_4$ are examples which fulfil the conditions of Theorem 3.1. Furthermore, one can show that $C_{4s} \times C_{4t}$ fulfils the condition of Theorem 3.1 for $s,t \ge 1$ and therefore $d_t^{\{k\}}(C_{4s} \times C_{4t}) = 4$.

Proposition 3.7. *If $G = C_{2n} \times C_{2m}$ is a torus of order $4nm$ such that $n,m \ge 2$ and k is even, then $d_t^{\{k\}}(G) = 4$.*

Proof. According to Theorem 1.3, $d_t^{\{k\}}(G) \le 4$. Define $f_s : V(G) \to \{0,1,2,\dots,k\}$ by $f_1(x_{2i-1,j}) = k/2$, $f_2(x_{2i,j}) = k/2$ if $1 \le i \le n$, $1 \le j \le 2m$, and $f_3(x_{i,2j-1}) = k/2$, $f_4(x_{i,2j}) = k/2$ if $1 \le i \le 2n$, $1 \le j \le m$ and $f_s(x_i, x_j) = 0$ otherwise for $s = 1,2,3,4$. Clearly, $\{f_1, f_2, f_3, f_4\}$ is a T$\{k\}$D family on G. Therefore $d_t^{\{k\}}(G) \ge 4$ and thus $d_t^{\{k\}}(G) = 4$. ∎

Proposition 3.8. *If $G = C_n \times C_m$ is a torus of order nm such that $4 \nmid nmk$, then $d_t^{\{k\}}(G) \le 3$.*

Proof. Let $4 \nmid nmk$ and let f belong to a T$\{k\}$D family on G. Since $C_n \times C_m$ is 4-regular, according to Theorem 1.3, $d_t^{\{k\}}(G) \le 4$. Suppose to the contrary that $d_t^{\{k\}}(G) = 4$. By Theorem 1.3,

$$nmk = \sum_{v \in V(G)} \sum_{u \in N(v)} f(u) = 4 \sum_{u \in V(G)} f(u) = 4w(f).$$

It follows that $4 \mid nmk$ which is a contradiction. Hence $d_t^{\{k\}}(G) \le 3$. ∎

We conclude this section with two problems.

Problem 3.1. Prove or disprove: If $G = C_n \times P_m$ is a cylinder of order nm such that $m,n \ge 3$, then $d_t^{\{k\}}(G) = 3$.

Problem 3.2. Prove or disprove: Let $G = C_n \times C_m$ be a torus of order nm. If $4 \nmid nmk$, then $d_t^{\{k\}}(G) = 3$ and $d_t^{\{k\}}(G) = 4$ otherwise.

4. A Nordhaus-Gaddum bound

In this section we present a lower bound on the sum $d_t^{\{k\}}(G) + d_t^{\{k\}}(\overline{G})$.

Theorem 4.1. *For every δ-regular graph of order $n \ge 5$ in which neither G nor $\overline{G}$ have isolated vertices,*

$$d_t^{\{k\}}(G) + d_t^{\{k\}}(\overline{G}) \ge \min\left\{k+1, \left\lceil \frac{n-2}{2} \right\rceil\right\}.$$

Proof. Let $\delta = \delta(G)$ and $\overline{\delta} = \delta(\overline{G})$. Since G is δ-regular, we observe that $\delta + \overline{\delta} = n-1$. If we assume, without loss of generality, that $\delta \ge \overline{\delta}$, then $\delta \ge (n-1)/2$.
Case 1. Assume that $k \le (n-1)/2$. Thus $\delta \ge k$. If $\delta = k$, then Theorem 1.1 implies that $d_t^{\{k\}}(G) \ge \delta - 1$ and therefore

$$d_t^{\{k\}}(G) + d_t^{\{k\}}(\overline{G}) \ge \delta - 1 + 1 = \delta \ge \frac{n-1}{2} \ge \min\left\{k+1, \left\lceil \frac{n-2}{2} \right\rceil\right\}.$$

If $\delta > k$, then it follows from Theorem 1.1 that

$$d_t^{\{k\}}(G) \geq \left\lfloor \frac{k}{\lceil \frac{k}{\delta} \rceil} \right\rfloor = k$$

and hence

$$d_t^{\{k\}}(G) + d_t^{\{k\}}(\overline{G}) \geq k+1 \geq \min\left\{k+1, \left\lceil \frac{n-2}{2} \right\rceil \right\}.$$

Case 2. Assume that $k > (n-1)/2$. If $\delta > k$, then we obtain as above the desired bound. Finally, assume that $k \geq \delta$. If $\delta \,|\, k$, then Theorem 1.1 leads to $d_t^{\{k\}}(G) \geq \delta - 1 \geq (\delta-1)/2$, and if $\delta \not| k$, then Theorem 1.1 implies that

$$d_t^{\{k\}}(G) \geq \left\lfloor \frac{k}{\lceil \frac{k}{\delta} \rceil} \right\rfloor > \frac{k}{\lceil \frac{k}{\delta} \rceil} - 1 \geq \frac{k}{\frac{k}{\delta}+1} - 1 = \frac{k\delta}{k+\delta} - 1 \geq \frac{\delta}{2} - 1.$$

Consequently, $d_t^{\{k\}}(G) \geq (\delta-1)/2$ in every case. As $k \geq \overline{\delta}$, we obtain analogously the inequality

$$d_t^{\{k\}}(\overline{G}) \geq \frac{\overline{\delta}-1}{2}$$

and therefore

$$d_t^{\{k\}}(G) + d_t^{\{k\}}(\overline{G}) \geq \frac{\delta-1}{2} + \frac{\overline{\delta}-1}{2} = \frac{n-3}{2} \geq \min\left\{k+1, \frac{n-3}{2}\right\}.$$

If n is even, then this inequality chain leads to the desired bound. If n is odd, then it follows that δ and $\overline{\delta}$ are even and thus $d_t^{\{k\}}(G) \geq \delta/2$ and $d_t^{\{k\}}(\overline{G}) \geq \overline{\delta}/2$ and so

$$d_t^{\{k\}}(G) + d_t^{\{k\}}(\overline{G}) \geq \frac{\delta}{2} + \frac{\overline{\delta}}{2} = \frac{n-1}{2} \geq \min\left\{k+1, \left\lceil \frac{n-2}{2} \right\rceil \right\}. \qquad \blacksquare$$

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