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Improved Converse Theorems and Fractional Moduli of Smoothness in Orlicz Spaces

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Abstract. In the present work converse theorems of trigonometric approximation of functions and its fractional derivatives in certain Orlicz spaces are improved with respect to the fractional moduli of smoothness. An improved Marchaud inequality is also given.

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1. Introduction

There are many results in approximation theory concerning the relationship of the best approximation of periodic function and its smoothness characteristics. It is well known that for functions f from Lebesgue spaces $L^p(\mathbb{T})$, $1 \leq p < \infty$, or $C(\mathbb{T})$ for $p = \infty$, the classical Jackson theorem

$$E_n(f)_p := \inf_{T \in \mathscr{T}_n} \|f - T\|_p \leq C(r)\, \omega_r\left(f, \frac{1}{n}\right)_p, \quad n \in \mathbb{Z}^+ \tag{1.1}$$

and its weak converse

$$\omega_r\left(f, \frac{1}{n}\right)_p \leq \frac{C(r)}{n^r} \sum_{\nu=1}^{n} \nu^{r-1} E_{\nu-1}(f)_p, \quad n \in \mathbb{Z}^+ \tag{1.2}$$

hold, where $r \in \mathbb{Z}^+$, $\omega_r(f,\delta)_p := \sup_{0<h\leq\delta} \|(T_h - I)^r f\|_p$ is the *rth moduli of smoothness* of the function $f \in L^p(\mathbb{T})$, $T_h f(\circ) := f(\circ + h)$ is translation operator, I is identity operator and $\mathscr{T}_n$ is the class of trigonometric polynomials of degree not greater than n.

These two inequalities have been generalized to many directions. We shall not mention all results but we can refer to books [8, 9, 11, 18, 21, 27, 31].

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In 1958, Timan proved [32] that inequality (1.2) be refined as follows

$$\omega_r\left(f,\frac{1}{n}\right)_p \leq \frac{C(r,p)}{n^r}\left\{\sum_{\nu=1}^{n}\nu^{rq-1}E_{\nu-1}^q(f)_p\right\}^{1/q}, \tag{1.3}$$

$$1<p<\infty,\ n\in\mathbb{Z}^+,\ q=\min\{2,p\}$$

and the value $\min\{2,p\}$ in (1.3) is optimal [34].

Equivalently the classical *Marchaud inequality* [20]

$$\omega_r(f,t)_p \leq C(r,p)t^r\left\{\int_t^\infty \frac{\omega_k(f,u)_p}{u^{r+1}}du\right\};\quad 1\leq p\leq\infty;\ r<k;\ r,k,n\in\mathbb{Z}^+;\ t>0 \tag{1.4}$$

improved to

$$\omega_r(f,t)_p \leq C(r,p)t^r\left\{\int_t^\infty \frac{\omega_k^q(f,u)_p}{u^{qr+1}}du\right\}^{1/q},\quad q=\min\{2,p\};\ 1\leq p<\infty;\ r<k;\ r,k\in\mathbb{Z}^+.$$

Similar inequalities for the Orlicz spaces were considered in [1, 10, 15, 16, 23, 33] and for Smirnov-Orlicz spaces in [1, 3, 4, 12–14].

In 1977, the concept of *fractional moduli of smoothness* $\omega_r(f,\cdot)_p$, $r>0$, $p\in[1,\infty]$, defined in the papers [5] and [29]. Inequalities (1.1) and (1.2) with $r>0$ were proved in [5], [29] and inequality (1.4) with $r>0$ was proved in [5].

In [2] some direct and converse inequalities for $r>0$ was proved in $L_M(\mathbb{T})$ to obtain analogues of the these inequalities on complex domains.

In this paper we improve the converse inequality [2, Theorem 2]

$$\omega_r\left(f,\frac{1}{n+1}\right)_M \leq \frac{C(M,r)}{(n+1)^r}\sum_{\nu=0}^{n}(\nu+1)^{r-1}E_\nu(f)_M,\ r>0,\ n\in\mathbb{N}_0$$

to

$$\omega_r\left(f,\frac{\pi}{n+1}\right)_M \leq \frac{C(M,r)}{(n+1)^r}\left\{\sum_{\nu=0}^{n}(\nu+1)^{rq-1}E_\nu^q(f)_M\right\}^{1/q} \tag{1.5}$$

in case $M\left(u^{1/q}\right)$ is convex for some $1<q\leq 2$.

Inequality (1.5) with $r\in\mathbb{Z}^+$ was proved in [15] for convex $M(\sqrt{u})$ and in [16] under the condition that $M\left(u^{1/\alpha}\right)$ is a convex function for some $1<\alpha\leq 2$. These two results use the Littlewood-Paley theory. In [7] some results about the sharp Jackson and Marchaud inequalities are proved for classical Lebesgue spaces. Embedding results of some function classes of periodic functions are considered in [28].

Throughout this work by $C(x,y,\ldots)$ we denote constants which are different in different places and depends only on the parameters $x,y,\ldots$ given in the brackets.

In this paper we will use the following notations: $\mathbb{T}:=[0,2\pi]$, $\mathbb{R}:=(-\infty,\infty)$, $\mathbb{R}^+:=(0,\infty)$, $\mathbb{Z}^+:=\{1,2,3,\ldots\}$, $\mathbb{N}_0:=\mathbb{Z}^+\cup\{0\}$, $\mathbb{Z}_n^*:=\{\pm1,\pm2,\pm3,\ldots,\pm n\}$, $\mathbb{Z}^*:=\{\pm1,\pm2,\pm3,\ldots\}$, $A(x)\asymp B(x)\iff \exists c_1,c_2>0: c_1B(x)\leq A(x)\leq c_2B(x)$.

2. Preliminaries and results

Let M be an *N-function*, that is, the mapping $M:\mathbb{R}\to\mathbb{R}^+$ is even, convex and satisfies

$$M(x)=0\iff x=0;\ \lim_{x\to\infty}\frac{M(x)}{x}=\infty \text{ and } \lim_{x\to 0}\frac{M(x)}{x}=0.$$

We denote by $L_M(\mathbb{T})$ the class of Lebesgue measurable functions $f:\mathbb{T}\to\mathbb{R}$ satisfying the condition

$$\int_{\mathbb{T}} M(C|f(x)|)\,dx<\infty$$

for some constant $C>0$, depending on f.

The linear space $L_M(\mathbb{T})$ becomes a Banch space with the *Luxemburg* norm

$$\|f\|_{(M)}:=\inf\left\{\tau>0:\int_{\mathbb{T}} M\left(\frac{|f|}{\tau}\right)dx\leq 1\right\} \tag{2.1}$$

or *Orlicz* norm

$$\|f\|_{L_M(\mathbb{T})}:=\|f\|_M:=\sup\left\{\int_{\mathbb{T}}|f(x)g(x)|\,dx:\int_{\mathbb{T}}\tilde{M}(|g|)\,dx\leq 1\right\} \tag{2.2}$$

where $\tilde{M}(y):=\sup_{x\geq 0}(x|y|-M(x))$, $y\in\mathbb{R}$, is the *complementary N-function* of M.

Banach space $L_M(\mathbb{T})$ is called *Orlicz space* and it satisfies embedding

$$L_M(\mathbb{T})\subset L^1(\mathbb{T}). \tag{2.3}$$

Norms (2.1) and (2.2) are equivalent since

$$\|f\|_{(M)}\leq\|f\|_M\leq 2\|f\|_{(M)}. \tag{2.4}$$

Also (see [24, p.22]),

$$\|f\|_M=\inf_{k>0}\left(\frac{1}{k}+\frac{1}{k}\int_{\mathbb{T}} M(k|f|)\,dx\right). \tag{2.5}$$

We note that $L_M(\mathbb{T})$ is *reflexive* if and only if both of the complementary pair of N-functions M and N satisfy Δ_2 *condition*. Recall that $M\in\Delta_2$ if there are constants $x_0>0$ and $C>2$ such that $M(2x)\leq CM(x)$ for $x\geq x_0$.

Let

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{2.6}$$

be *Fourier series* of a function $f\in L^1(\mathbb{T})$.

For a given $f\in L^1(\mathbb{T})$, assuming

$$\int_{\mathbb{T}} f(x)\,dx=0, \tag{2.7}$$

we define α*-th fractional* $(\alpha>0)$ *integral* of f as [35, v.2, p.134]

$$I_\alpha(x,f):=\sum_{k\in\mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx}=\frac{1}{2\pi}\int_{\mathbb{T}} f(t)\Psi_\alpha(x-t)\,dt, \tag{2.8}$$

where $\Psi_\alpha(u):=\sum_{k\in\mathbb{Z}^*} e^{iku}(ik)^{-\alpha}$ and

$$(ik)^{-\alpha}:=|k|^{-\alpha}e^{(-1/2)\pi i\alpha\,\mathrm{sign}\,k}$$

as principal value.

Let $\alpha>0$ be given. We define *fractional* (Weyl's) *derivative* of a function $f\in L^1(\mathbb{T})$, satisfying (2.7), as

$$f^{(\alpha)}(x):=\frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}}I_{1+\alpha-[\alpha]}(x,f)$$

provided the right hand side exists, where $[\alpha]$ is the integer part of real number $\alpha > 0$.

Suppose that $x, h \in \mathbb{R}$ and $r > 0$. The Binomial series

$$\Delta_h^r f(x) := \sum_{k=0}^{\infty} (-1)^k \binom{r}{k} f(x - kh), \quad f \in L^1(\mathbb{T}) \tag{2.9}$$

converges in $L^1(\mathbb{T})$ and $\Delta_h^r f(\cdot)$ is a measurable function. Since Binomial coefficients $\binom{r}{k}$ satisfy [26, p. 14]

$$\left|\binom{r}{k}\right| = \left|\frac{r(r-1)\dots(r-k+1)}{k!}\right| \le \frac{c_8}{k^{r+1}}, \quad k \in \mathbb{Z}^+,$$

we have

$$C(r) := \sum_{k=0}^{\infty} \left|\binom{r}{k}\right| < \infty. \tag{2.10}$$

Therefore

$$\|\Delta_h^r f\|_B \le C(r) \|f\|_B \tag{2.11}$$

for any Banach space of Lebesgue integrable functions on $\mathbb{T}$ (or $\mathbb{R}$) with norm $\|\cdot\|_B$ satisfying $\|f(\cdot + h)\|_B = \|f(\cdot)\|_B$ for $\forall h \in \mathbb{R}$ and $\|f(\cdot + h) - f(\cdot)\|_B = o(1)$ for $h \to 0$. Throughout the paper such spaces will be denoted by B.

Taking into account (2.3), (2.9), (2.10) and (2.11) we can define *r-th fractional* $(r > 0)$ *modulus of smoothness* of a function $f \in B$ as

$$\omega_r(f, \delta)_B := \sup_{0 < h \le \delta} \|\Delta_h^r f\|_B. \tag{2.12}$$

The modulus of smoothness $\omega_r(f, \delta)_B$ has the following usual properties:
(i) $\omega_r(f, \delta)_B$ is non-negative and non-decreasing function of $\delta \ge 0$,
(ii) $\lim_{\delta \to 0^+} \omega_r(f, \delta)_B = 0$ and (iii) $\omega_r(f_1 + f_2, \cdot)_B \le \omega_r(f_1, \cdot)_B + \omega_r(f_2, \cdot)_B$.

Let

$$E_n(f)_B := \inf_{T \in \mathscr{T}_n} \|f - T\|_B, \quad f \in B, \tag{2.13}$$

where $\mathscr{T}_n$ is the class of *trigonometric polynomials* of degree not greater than n.

Let B^β, $\beta > 0$, be the class of functions $f \in B$ such that $f^{(\beta)} \in B$. B^β, $\beta > 0$, becomes a Banach space with the norm $\|f\|_{B^\beta} := \|f\|_B + \left\|f^{(\beta)}\right\|_B$. In case of $B = L_M(\mathbb{T})$ we will use the notations $\omega_r(f, \delta)_B := \omega_r(f, \delta)_M$, $E_n(f)_B := E_n(f)_M$ and $B^\beta := W_{\beta,M}(\mathbb{T})$.

For $r, t \in \mathbb{R}^+$ and $f \in B$ we define the *fractional K-functional* as

$$K_r(f, t, B, B^r) := \inf_{g \in B^r} \|f - g\|_B + t^r \left\|g^{(r)}\right\|_B.$$

Our main results are the following.

Theorem 2.1. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space such that $M\left(u^{1/\alpha}\right)$ is a convex function for some $1 < \alpha \le 2$. Then for $n \in \mathbb{N}_0$, $r > 0$ and $f \in L_M(\mathbb{T})$ we have*

$$\omega_r\left(f, \frac{\pi}{n+1}\right)_M \le \frac{C(M,r)}{(n+1)^r} \left\{\sum_{\upsilon=0}^{n} (\upsilon+1)^{\alpha r - 1} E_\upsilon^\alpha(f)_M\right\}^{1/\alpha}$$

and equivalently, for $r < s$ $(r,s > 0)$ and $t > 0$

$$\omega_r(f,t)_M \le C(M,s,r)t^r \left\{ \int_t^\infty \frac{\omega_s(f,u)_M^\alpha}{u^{r\alpha+1}} du \right\}^{1/\alpha}.$$

Theorem 2.2. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space such that $M\left(u^{1/\alpha}\right)$ is a convex function for some $1 < \alpha \le 2$. If $f \in L_M(\mathbb{T})$ and*

$$\sum_{\nu=1}^{\infty} \nu^{\alpha\beta-1} E_\nu^\alpha(f)_M < \infty \tag{2.14}$$

for some $\beta \in \mathbb{R}^+$, then $f \in W_{\beta,M}(\mathbb{T})$. Furthermore, for $n \in \mathbb{N}$ we have

$$E_n(f^{(\beta)})_M \le C(M,\beta)\Big(n^\beta E_n(f)_M + \Big\{ \sum_{\upsilon=n+1}^{\infty} \upsilon^{\alpha\beta-1} E_\upsilon^\alpha(f)_M \Big\}^{1/\alpha}\Big).$$

In the particular case for the classical Lebesgue spaces the last inequality was proved in [28, (90)].

From Theorems 2.1 and 2.2 we have

Corollary 2.1. *Under the conditions of Theorems 2.1 and 2.2 we have for $n \in \mathbb{N}$ and $r > 0$*

$$\omega_r\left(f^{(\beta)}, \frac{\pi}{n}\right)_M \le C(M,\beta,r)\left(\Big(\sum_{\nu=n+1}^{\infty} \nu^{\alpha\beta-1} E_\nu^\alpha(f)_M\Big)^{\frac{1}{\alpha}} + \frac{1}{n^k}\Big(\sum_{\upsilon=1}^{n} \upsilon^{\alpha(r+\beta)-1} E_\upsilon^\alpha(f)_M\Big)^{\frac{1}{\alpha}}\right).$$

In the particular case for the classical Lebesgue spaces the last inequality was proved in [28, (91)].

Theorem 2.3. *If $r > 0$ and $f \in L_M(\mathbb{T})$ then*

$$\omega_r(f,1/n)_M \asymp K_r(f,1/n,L_M(\mathbb{T}),W_{r,M}(\mathbb{T})).$$

The following result is a corollary of Theorems 2.1, 2.2 and 2.3

Corollary 2.2. *If $s > r > 0$ $(s,r,\lambda,t \in \mathbb{R}^+)$ and $f \in L_M(\mathbb{T})$, then*

$$K_r(f,t,L_M(\mathbb{T}),W_{r,M}(\mathbb{T})) \le C(M,r,s)t^r \left\{ \int_t^\infty \frac{K_s(f,u,L_M(\mathbb{T}),W_{r,M}(\mathbb{T}))^\alpha}{u^{s\alpha+1}} du \right\}^{1/\alpha}$$

and

$$\omega_r(f,\lambda t)_M \le C(M,r)\lambda^r \omega_r(f,t)_M, \quad \lambda \ge 1. \tag{2.15}$$

3. Auxiliary results

We shall start with

Lemma 3.1. *Suppose that $r > 0$, $T_n \in \mathscr{T}_n$ and $0 < h < 2\pi/n$. Then*

$$\left\| T_n^{(r)} \right\|_B \le \left(\frac{n}{2\sin(nh/2)} \right)^r \|\Delta_h^r T_n\|_B.$$

In particular, if $h = \pi/n$, then

$$\left\| T_n^{(r)} \right\|_B \le 2^{-r} n^r \left\| \Delta_{\pi/n}^r T_n \right\|_B. \tag{3.1}$$

Proof. Let $n \in \mathbb{N}$ and we set

$$g(t) := \left(\frac{t}{2\sin\frac{h}{2}t} \right)^r \quad \text{for } -n \le t \le n \text{ and } g(0) := h^{-r}.$$

Then for $x \in \mathbb{R}$, $h \in (0, 2\pi/n)$ we find [29]

$$T_n^{(r)}(x) = \sum_{k=-\infty}^{\infty} d_k \Delta_h^r T_n \left(x + \frac{k\pi}{n} + \frac{r}{2}h \right)$$

where

$$g(t) = \sum_{k=-\infty}^{\infty} d_k e^{ik\pi t/n}$$

converges uniformly in $[-n, n]$ and $(-1)^k d_k \ge 0$. Hence

$$\left\| T_n^{(r)} \right\|_B \le \|\Delta_h^r T_n\|_B \sum_{k=-\infty}^{\infty} \left| d_k e^{ik\pi} \right| = \left(\frac{n}{2\sin(nh/2)} \right)^r \|\Delta_h^r T_n\|_B$$

and Lemma 3.1 is proved. ∎

Lemma 3.2. *If $r, \delta \in \mathbb{R}^+$ and $f \in W_{r,M}(\mathbb{T})$ then*

$$\omega_r(f, \delta)_M \le C(r)\, \delta^r \left\| f^{(r)} \right\|_M . \tag{3.2}$$

Proof. For the function $\chi_r(\cdot, h) \in L^1(\mathbb{T})$ of [26, (20.15), p.376] we define

$$(A_h^r f)(x) := (f * \chi_r(\cdot, h))(x) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x-u)\, \chi_r(u,h)\, du, \quad x \in \mathbb{T},\, h \in \mathbb{R}^+.$$

Then using Fubini's theorem we get

$$\|A_h^r f\|_M = \sup \left\{ \int_{\mathbb{T}} \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(x-u)\, \chi_r(u,h)\, du\, g(x) \right| dx : \int_{\mathbb{T}} \tilde{M}(|g|)\, dx \le 1 \right\} \tag{3.3}$$

$$\le \|\chi_r(\cdot, h)\|_{L^1(\mathbb{T})} \|f\|_M \le C(r) \|f\|_M .$$

Since [5, (2.5) and (2.12)]

$$(\Delta_h^r f)(x) = h^r (A_h^r f)^{(r)}(x) = h^r A_h^r \left(f^{(r)} \right)(x)$$

we have from (3.3) that

$$\sup_{0<h\le\delta} \|\Delta_h^r f\|_M = \sup_{0<h\le\delta} h^r \left\| A_h^r \left(f^{(r)} \right) \right\|_M \le C(r)\, \delta^r \left\| f^{(r)} \right\|_M$$

and we obtain (3.2). ∎

Lemma 3.3. *Let $\{f_n\}$ be a sequence such that every f_n is absolutely continuous. If the sequence $\{f_n\}$ converges to the function f in reflexive Orlicz space $L_M(\mathbb{T})$ and the sequence of first derivatives $\{f_n'\}$ converges to some function g in $L_M(\mathbb{T})$, then f is absolutely continuous and $f'(x) = g(x)$ a.e.*

Proof. Since $\|f_n - f\|_M \to 0$ as $n \to \infty$ there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ satisfying $f_{n_k}(x) \to f(x)$ a.e. Let x_0 be a point of convergence. Using generalized Hölder's inequality [24, p.15]

$$\left| \int_{x_0}^{x} f'_{n_k}(t)\,dt - \int_{x_0}^{x} g(t)\,dt \right| \le C(\tilde{M}) \left\| f'_{n_k} - g \right\|_M$$

and since $\|f'_n - g\|_M \to 0$ as $n \to \infty$, we have

$$\left| \int_{x_0}^{x} f'_{n_k}(t)\,dt - \int_{x_0}^{x} g(t)\,dt \right| \to 0.$$

Then

$$\int_{x_0}^{x} g(t)\,dt = \lim_{k\to\infty} \int_{x_0}^{x} f'_{n_k}(t)\,dt = \lim_{k\to\infty} \left(f_{n_k}(x) - f_{n_k}(x_0)\right) = f(x) - f(x_0)$$

and Lemma 3.3 is proved. ∎

Lemma 3.4. *Let $f_n, f \in L_M(\mathbb{T})$, $n \in \mathbb{N}_0$. If for some $r > 0$ the function $I_{r-[r]}(\cdot, f_n)$ is absolutely continuous on $\mathbb{T}$, $\left(I_{r-[r]}(\cdot, f_n)\right)' = f_n^{(r)} \in L_M(\mathbb{T})$ for $n \in \mathbb{Z}^+$ and*

$$\|f_n - f\|_M \to 0 \ ; \ \left\| \left(I_{r-[r]}(\cdot, f_n)\right)' - \varphi \right\|_M \to 0, \quad \text{as } n \to \infty$$

for some $\varphi \in L_M(\mathbb{T})$. Then, $I_{r-[r]}(\cdot, f)$ is absolutely continuous on $\mathbb{T}$ and $f^{(r)} \in L_M(\mathbb{T})$.

Proof. We prove that if $f \in L_M(\mathbb{T})$ with Fourier series (2.6) satisfying (2.7), then $I_{r-[r]}(\cdot, f) \in L_M(\mathbb{T})$ and

$$\left\| I_{r-[r]}(\cdot, f) \right\|_M \le C(r, M) \|f\|_M . \tag{3.4}$$

By (2.8), Fubini's theorem and (8.15) of [35, p. 136] we have

$$\begin{aligned} \left\| I_{r-[r]}(\cdot, f) \right\|_M &= \sup \left\{ \int_{\mathbb{T}} \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \Psi_{r-[r]}(x-t)\,dt\, g(x) \right| dx : \int_{\mathbb{T}} \tilde{M}(|g|)\,dx \le 1 \right\} \\ &\le \frac{1}{2\pi} \|f\|_M \left\| \Psi_{r-[r]}(\cdot) \right\|_{L^1(\mathbb{T})} \\ &\le \frac{C(M)}{2\pi} \|f\|_M \int_0^{\pi} u^{r-[r]-1}\,du = C(r, M) \|f\|_M . \end{aligned}$$

On the other hand, since

$$\|f_n - f\|_M \to 0, \quad \text{as } n \to \infty,$$

from linearity of α-th integral and (3.4), we get

$$\left\| I_{r-[r]}(\cdot, f_n) - I_{r-[r]}(\cdot, f) \right\|_M \to 0, \quad \text{as } n \to \infty.$$

Hence, we conclude by Lemma 3.3 that $I_{r-[r]}(\cdot, f)$ is absolutely continuous and $f^{(r)} \in L_M(\mathbb{T})$. ∎

Lemma 3.5. [17, Theorem 2] *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space such that $M\left(x^{1/\alpha}\right)$ is a convex function for some $\alpha > 1$. If $f \in L_M(\mathbb{T})$ with corresponding Fourier series (2.6), then the modular inequality*

$$C \int_{\mathbb{T}} M\Big(\Big(\sum_{\mu=1}^{\infty} \left|\bar{\Delta}_\mu\right|^2\Big)^{1/2}\Big)dx \le \int_{\mathbb{T}} M(|f(x)|)dx \le C \int_{\mathbb{T}} M\Big(\Big(\sum_{\mu=1}^{\infty} \left|\bar{\Delta}_\mu\right|^2\Big)^{1/2}\Big)dx$$

hold with a constant C independent of f, where

$$\bar{\Delta}_\mu := \bar{\Delta}_\mu(x,f) := \sum_{|\nu|=2^{\mu-1}}^{2^\mu-1} c_\nu e^{i\nu x}.$$

4. Proof of theorems

Proof of Theorem 2.1. Let $f \in L_M(\mathbb{T})$ be such that $\int_0^{2\pi} f(x)\,dx = 0$ with corresponding Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{ikx} \sim f(x)$. We choose a $\sigma \in \mathbb{Z}^+$ so that $2^{\sigma-1} \le 2n+1 < 2^\sigma$ hold. Let us denote $S_n(x) := S_n(x,f) := \sum_{k=-n}^{n} c_k e^{ikx}$ and $\tilde{S}_n(x,f) := \sum_{k=-n}^{n} (-i\,\mathrm{sign}\,k)\, c_k e^{ikx}$. It is well known from [25] that

$$\|f(x) - S_m(x)\|_M \le C(M) E_m(f)_M.$$

Therefore

$$\|\Delta_h^r f\|_M \le C(M,r) E_{2n+1}(f)_M + \|\Delta_h^r S_{2^\sigma-1}\|_M$$

and if $0 < h < 2\pi/(2^\sigma - 1)$ then

$$\Delta_h^r S_{2^\sigma-1}\left(x + \frac{rh}{2}\right) = \sum_{\nu=-2^\sigma+1}^{2^\sigma-1} \left(2i \sin\frac{\nu h}{2}\right)^r c_\nu e^{i\nu x}.$$

Since modular inequality is stronger than norm inequality we have from Lemma 3.5 that

$$C\left\|\left(\sum_{\mu=1}^{\infty} |\bar{\Delta}_\mu|^2\right)^{1/2}\right\|_M \le \|f\|_M \le C\left\|\left(\sum_{\mu=1}^{\infty} |\bar{\Delta}_\mu|^2\right)^{1/2}\right\|_M \tag{4.1}$$

hold. Now using (4.1) we have

$$\|\Delta_h^r S_{2^\sigma-1}\|_M \le C(M,r)\left\|\left\{\sum_{\mu=1}^{\sigma}\left|\sum_{|\nu|=2^{\mu-1}}^{2^\mu-1}\left(2i\sin\frac{\nu h}{2}\right)^r c_\nu e^{i\nu x}\right|^2\right\}^{1/2}\right\|_M. \tag{4.2}$$

Putting $\Psi(x) := M\left(x^{1/\alpha}\right)$ and

$$\delta_\mu := \delta_\mu(x,h,r) := \sum_{|\nu|=2^{\mu-1}}^{2^\mu-1}\left(2i\sin\frac{\nu h}{2}\right)^r c_\nu e^{i\nu x}$$

we get from (4.2), (2.4) and (2.1)

$$\|\Delta_h^r S_{2^\sigma-1}\|_M \le 2C(M,r)\inf\left\{\tau > 0 : \int_0^{2\pi} M\left(\frac{\left|\sum_{\mu=1}^{\sigma}\delta_\mu^2\right|^{1/2}}{\tau}\right)dx \le 1\right\}$$

$$= C(M,r)\inf\left\{\tau > 0 : \int_0^{2\pi} \Psi\left(\frac{\left|\sum_{\mu=1}^{\sigma}\delta_\mu^2\right|^{\alpha/2}}{\tau^\alpha}\right)dx \le 1\right\} =: I_1.$$

Using inequality

$$\left(\sum u\right)^{\alpha/2} \le \sum u^{\alpha/2}, \quad 1 < \alpha \le 2$$

and (2.4) we find

$$I_1 \leq C(M,r)\inf\left\{\tau>0:\int_0^{2\pi}\Psi\left(\frac{1}{\tau^\alpha}\sum_{\mu=1}^{\sigma}\left|\delta_\mu\right|^\alpha\right)dx\leq 1\right\}$$

$$(4.3)\qquad = C(M,r)\left\|\sum_{\mu=1}^{\sigma}\left|\delta_\mu\right|^\alpha\right\|_{(\Psi)}^{1/\alpha} \leq C(M,r)\left\{\sum_{\mu=1}^{\sigma}\left\|\left|\delta_\mu\right|^\alpha\right\|_{(\Psi)}\right\}^{\frac{1}{\alpha}}$$

$$= C(M,r)\left\{\sum_{\mu=1}^{\sigma}\left\|\delta_\mu\right\|_{(M)}^\alpha\right\}^{\frac{1}{\alpha}} \leq C(M,r)\left\{\sum_{\mu=1}^{\sigma}\left\|\delta_\mu\right\|_{M}^\alpha\right\}^{\frac{1}{\alpha}}.$$

Since

$$\left|\delta_\mu\right| = \sum_{\nu=2^{\mu-1}}^{2^\mu-1}\left(2\sin\frac{\nu h}{2}\right)^r 2\,\mathrm{Re}\left(c_\nu e^{i(\nu x+r\pi/2)}\right)$$

we have

$$\left\|\delta_\mu\right\|_M = 2^r\left\|\sum_{\nu=2^{\mu-1}}^{2^\mu-1}\left(\sin\frac{\nu h}{2}\right)^r U_\nu(x)\right\|_M$$

with $U_\nu(x) := 2\,\mathrm{Re}\left(c_\nu e^{i(\nu x+r\pi/2)}\right)$.

By Abel's transformation we get

$$(4.4)\qquad \begin{aligned} 2^{-r}\left\|\delta_\mu\right\|_M \leq & \sum_{\nu=2^{\mu-1}}^{2^\mu-2}\left|\sin^r\frac{\nu h}{2}-\sin^r\frac{(\nu+1)h}{2}\right|\left\|\sum_{l=2^{\mu-1}}^{\nu}U_l(x)\right\|_M + \\ & + \left|\sin^r\frac{(2^\mu-1)h}{2}\right|\left\|\sum_{l=2^{\mu-1}}^{2^\mu-1}U_l(x)\right\|_M \end{aligned}$$

and for $2^{\mu-1}\leq\nu\leq 2^\mu-1$, $(\mu\in\mathbb{Z}^+)$ we have

$$\begin{aligned} \left\|\sum_{l=2^{\mu-1}}^{\nu}U_l(x)\right\|_M &= 2\left\|\mathrm{Re}\left(\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx}\cos\frac{r\pi}{2}\right)-\mathrm{Im}\left(\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx}\sin\frac{r\pi}{2}\right)\right\|_M \\ &\leq \left\|2\,\mathrm{Re}\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx}\right\|_M + \left\|2\,\mathrm{Im}\sum_{l=2^{\mu-1}}^{\nu}c_l e^{ilx}\right\|_M \\ &= \left\|S_\nu(x,f)-S_{2^{\mu-1}-1}(x,f)\right\|_M + \left\|\tilde{S}_\nu(x,f)-\tilde{S}_{2^{\mu-1}-1}(x,f)\right\|_M. \end{aligned}$$

From [25] we have

$$\left\|\tilde{S}_n(\cdot,f)\right\|_M \leq C\left\|S_n(\cdot,f)\right\|_M,\ n\in\mathbb{Z}^+$$

and hence

$$(4.5)\qquad \left\|\sum_{l=2^{\mu-1}}^{\nu}U_l(x)\right\|_M \leq (1+C(M,r))\left\|S_\nu(x,f)-S_{2^{\mu-1}-1}(x,f)\right\|_M$$

$$(4.6)\qquad \leq C(M,r)E_{2^{\mu-1}-1}(f)_M.$$

Using [30] for $q\geq 0$, $k>0$ and $h\in(0,\pi/2^\mu]$, $\mu\in\mathbb{Z}^+$ we have

$$\sum_{\nu=2^{\mu-1}}^{2^\mu-1}\left|\nu^q\sin^k\nu h-(\nu+1)^q\sin^k(\nu+1)h\right|\leq c2^{\mu(q+k)}h^k$$

and hence from (4.4) and (4.5) we get for $\mu \in \mathbb{Z}^+$

$$\left\|\delta_\mu\right\|_M \le C(M,r)2^{\mu r}h^r E_{2^{\mu-1}-1}(f)_M .$$

Therefore using (4.3) we obtain

$$\begin{aligned}
\|\Delta_h^r S_{2^\sigma-1}\|_M &\le C(M,r)h^r\left\{\sum_{\mu=1}^{\sigma} 2^{\mu r\alpha}E^{\alpha}_{2^{\mu-1}-1}(f)_M\right\}^{1/\alpha} \\
&\le C(M,r)h^r\left\{2^{\alpha r}E_0^{\alpha}(f)_M\right\}^{1/\alpha} \\
&\quad + C(M,r)h^r\left\{\sum_{\mu=2}^{\sigma}\sum_{\nu=2^{\mu-2}}^{2^{\mu-1}-1}\nu^{\alpha r-1}E_\nu^{\alpha}(f)_M\right\}^{1/\alpha} \qquad (4.7)\\
&\le C(M,r)h^r\left\{\sum_{\nu=0}^{2^{\sigma-1}-1}(\nu+1)^{\alpha r-1}E_\nu^{\alpha}(f)_M\right\}^{1/\alpha}, \quad 0<h\le 2\pi/2^{\sigma}.
\end{aligned}$$

On the other hand, using (2.15) and (4.7) we find

$$\begin{aligned}
&\omega_r\left(f,\frac{\pi}{n+1}\right)_M \le C(M,r)\omega_r\left(f,\frac{2\pi}{2^\sigma}\right)_M \le C(M,r)\sup_{0<h\le 2\pi/2^\sigma}\|\Delta_h^r f\|_M \\
&\le C(M,r)E_{2n+1}(f)_M + C(M,r)\left(\frac{2\pi}{2^\sigma}\right)^r\left\{\sum_{\nu=0}^{2^{\sigma-1}-1}(\nu+1)^{\alpha r-1}E_\nu^{\alpha}(f)_M\right\}^{1/\alpha} \\
&\le C(M,r)E_{2n+1}(f)_M + C(M,r)\left(\frac{\pi}{n+1}\right)^r\left\{\sum_{\nu=0}^{2n}(\nu+1)^{\alpha r-1}E_\nu^{\alpha}(f)_M\right\}^{1/\alpha} \\
&\le \frac{C(M,r)}{(n+1)^r}\left\{\sum_{\nu=0}^{n}(\nu+1)^{\alpha r-1}E_\nu^{\alpha}(f)_M\right\}^{1/\alpha}
\end{aligned}$$

and Theorem 2.1 is proved. ∎

Proof of Theorem 2.2. Let $T_n \in \mathscr{T}_n$ such that $E_n(f)_M = \|f-T_n\|_M$ and we take

$$\mathscr{U}_0(x) := T_1(x) - T_0(x); \, \mathscr{U}_\nu(x) := T_{2^\nu}(x) - T_{2^{\nu-1}}(x), \quad \nu \in \mathbb{Z}^+.$$

In this case we have

$$T_{2^N}(x) = T_0(x) + \sum_{\nu=0}^{N}\mathscr{U}_\nu(x), \quad N \in \mathbb{N}_0.$$

For a given $\varepsilon > 0$, by (2.14) there exists $\eta \in \mathbb{Z}^+$ such that

$$\sum_{\nu=2^\eta}^{\infty}\nu^{\alpha\beta-1}E_\nu^{\alpha}(f)_M < \varepsilon. \qquad (4.8)$$

From Lemma 3.1 we have

$$\left\|\mathscr{U}_\nu^{(\beta)}(x)\right\|_M \le C(M,\beta)2^{\nu\beta}\|\mathscr{U}_\nu(x)\|_M \le C(M,\beta)2^{\nu\beta}E_{2^{\nu-1}}(f)_M, \quad \nu\in\mathbb{Z}^+.$$

On the other hand it is easily seen that

$$2^{\nu\beta}E_{2^{\nu-1}}(f)_M \leq C(M)2^{2\beta}\left\{\sum_{\mu=2^{\nu-2}+1}^{2^{\nu-1}}\mu^{\alpha\beta-1}E_{\mu}^{\alpha}(f)_M\right\}^{1/\alpha}, \quad \nu=2,3,\ldots.$$

For the positive integers satisfying $K<N$

$$T_{2^N}^{(\beta)}(x)-T_{2^K}^{(\beta)}(x)=\sum_{\nu=K+1}^{N}U_{\nu}^{(\beta)}(x), \quad x\in\mathbb{T}$$

and hence if K,N are large enough we obtain from (4.8)

$$\left\|T_{2^N}^{(\beta)}(x)-T_{2^K}^{(\beta)}(x)\right\|_M \leq \sum_{\nu=K+1}^{N}\left\|\mathscr{U}_{\nu}^{(\beta)}(x)\right\|_M \leq C(M,\beta)\sum_{\nu=K+1}^{N}2^{\nu\beta}E_{2^{\nu-1}}(f)_M$$

$$\leq C(M,\beta)\sum_{\nu=K+1}^{N}\left\{\sum_{\mu=2^{\nu-2}+1}^{2^{\nu-1}}\mu^{\alpha\beta-1}E_{\mu}^{\alpha}(f)_M\right\}^{1/\alpha}$$

$$=C(M,\beta)\left\{\sum_{\mu=2^{K-1}+1}^{2^{N-1}}\mu^{\alpha\beta-1}E_{\mu}^{\alpha}(f)_M\right\}^{1/\alpha}<C(M,\beta)\varepsilon^{1/\alpha}.$$

Therefore $\left\{T_{2^N}^{(\beta)}\right\}$ is a Cauchy sequence in $L_M(\mathbb{T})$. Then there exists a $\varphi\in L_M(\mathbb{T})$ satisfying

$$\left\|T_{2^N}^{(\beta)}-\varphi\right\|_M \to 0, \quad \text{as } N\to\infty.$$

On the other hand we have

$$\|T_{2^N}-f\|_M=E_{2^N}(f)_M\to 0, \quad \text{as } N\to\infty.$$

Then from Lemma 3.4 we obtain that $I_{\beta-[\beta]}$ is absolutely continuous on $\mathbb{T}$ and $f^{(\beta)}\in L_M(\mathbb{T})$. Therefore $f\in W_{\beta,M}(\mathbb{T})$.

We note that

$$\begin{aligned} E_n\left(f^{(\beta)}\right)_M &\leq \left\|f^{(\beta)}-S_nf^{(\beta)}\right\|_M \\ &\leq \left\|S_{2^{m+2}}f^{(\beta)}-S_nf^{(\beta)}\right\|_M+\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\beta)}-S_{2^k}f^{(\beta)}\right]\right\|_M. \end{aligned} \tag{4.9}$$

By Lemma 3.1 we get for $2^m<n<2^{m+1}$

$$\left\|S_{2^{m+2}}f^{(\beta)}-S_nf^{(\beta)}\right\|_M \leq C(M,\beta)2^{(m+2)\beta}E_n(f)_M \leq C(M,\beta)n^{\beta}E_n(f)_M. \tag{4.10}$$

By Lemma 3.5 we find

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\beta)}-S_{2^k}f^{(\beta)}\right]\right\|_M \leq C(M,\beta)\left\|\left\{\sum_{k=m+2}^{\infty}\left|\sum_{|\nu|=2^k+1}^{2^{k+1}}(i\nu)^{\beta}c_{\nu}e^{i\nu x}\right|^2\right\}^{1/2}\right\|_M$$

and therefore

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\beta)}-S_{2^k}f^{(\beta)}\right]\right\|_M\leq C(M,\beta)\left(\sum_{k=m+2}^{\infty}\left\|\sum_{|\nu|=2^k+1}^{2^{k+1}}(i\nu)^{\beta}c_\nu e^{i\nu x}\right\|_M^{\alpha}\right)^{1/\alpha}.$$

Putting

$$\left|\bar{\delta}_\nu\right|:=\left|\bar{\delta}_\nu(x,\beta)\right|:=\sum_{|\nu|=2^k+1}^{2^{k+1}}(i\nu)^{\beta}c_\nu e^{i\nu x}=\sum_{\nu=2^k+1}^{2^{k+1}}\nu^{\beta}2\operatorname{Re}\left(c_\nu e^{i(\nu x+\beta\pi/2)}\right)$$

we have

$$\left\|\bar{\delta}_\nu\right\|_M=\left\|\sum_{\nu=2^k+1}^{2^{k+1}}\nu^{\beta}U_\nu(x)\right\|_M$$

($U_\nu(x)=2\operatorname{Re}\left(c_\nu e^{i(\nu x+\beta\pi/2)}\right)$) and using Abel's transformation we get

$$\left\|\bar{\delta}_\nu\right\|_M\leq\sum_{\nu=2^k+1}^{2^{k+1}-1}\left|\nu^{\beta}-(\nu+1)^{\beta}\right|\left\|\sum_{l=2^k+1}^{\nu}U_l(x)\right\|_M+\left|\left(2^{k+1}\right)^{\beta}\right|\left\|\sum_{l=2^k+1}^{2^{k+1}-1}U_l(x)\right\|_M.$$

For $2^k+1\leq\nu\leq 2^{k+1}$, $(k\in\mathbb{Z}^+)$ we have

$$\left\|\sum_{l=2^k+1}^{\nu}U_l(x)\right\|_M\leq C(M,\beta)E_{2^k}(f)_M$$

and since

$$(\nu+1)^{\beta}-\nu^{\beta}\leq\begin{cases}\beta(\nu+1)^{\beta-1} & ,\beta\geq 1,\\ \beta\nu^{\beta-1} & ,0\leq\beta<1,\end{cases}$$

we obtain

$$\left\|\bar{\delta}_\nu\right\|_M\leq C(M,\beta)2^{k\beta}E_{2^k-1}(f)_M.$$

Therefore

$$(4.11)\qquad \begin{aligned}\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\beta)}-S_{2^k}f^{(\beta)}\right]\right\|_M&\leq C(M,\beta)\left\{\sum_{k=m+2}^{\infty}2^{k\beta\alpha}E_{2^k-1}^{\alpha}(f)_M\right\}^{1/\alpha}\\&\leq C(M,\beta)\left\{\sum_{\nu=n+1}^{\infty}\nu^{\alpha\beta-1}E_k^{\alpha}(f)_M\right\}^{1/\alpha}\end{aligned}$$

and using (4.9), (4.10) and (4.11), the proof is complete. ∎

Proof of Theorem 2.3. Starting with upper inequality we take a $t\in(0,2\pi)$. Then there exists $n\in\mathbb{Z}^+$ such that $\pi/n<t\leq 2\pi/n$. Let t_n^* be the best approximating trigonometric polynomial to $f\in L_M(\mathbb{T})$. Using Theorem 1 of [2] we get

$$(4.12)\qquad \|f-t_n^*\|_M=E_n(f)_M\leq C(r,M)\,\omega_r\left(f,\frac{\pi}{n}\right)_M.$$

From (2.11) and (3.1) we have

$$\begin{aligned}\left\|t_n^{*(r)}\right\|_M&\leq 2^{-r}n^r\left\|\Delta_{\pi/n}^r t_n^*\right\|_M\leq(\pi/t)^r\left\{C(M,r)\|f-t_n^*\|_M+\left\|\Delta_{\pi/n}^r f\right\|_M\right\}\\&\leq C(M,r)\,t^{-r}\omega_r\left(f,\frac{\pi}{n}\right)_M\end{aligned}$$

and therefore

$$K_r\left(f,t,L_M\left(\mathbb{T}\right),W_{r,M}\left(\mathbb{T}\right)\right) \leq \|f-t_n^*\|_M + t^r \left\|t_n^{*(r)}\right\|_M \leq C\left(M,r\right)\omega_r\left(f,t\right)_M.$$

The lower inequality is straighforward from (iii), (2.11), (4.12) and Lemma 3.2. ∎

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