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Submersion of Semi-Invariant Submanifolds of Trans-Sasakian Manifold

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Abstract. In this paper, we discuss submersion of semi-invariant submanifolds of trans-Sasakian manifold and derive some results on their differential geometry. We also discuss cohomology of semi-invariant submanifold of trans-Sasakian manifold under the submersion.

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1. Introduction

The study of Riemannian submersions was initiated by O'Neill [15]. Semi-Riemannian submersions were introduced by O'Neill in [16]. It is well known that semi-Riemannian submersions are of interest in physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theory [9, 11, 19, 20]. In [12], S. Kobayashi studied submersion of *CR*-submanifolds and obtained interesting results. In this paper we study submersion of semi-invariant submanifold of trans-Sasakian manifold.

Let $\bar{M}$ be an n-dimensional almost contact metric manifold with almost contact metric structure (ϕ, ξ, η, g). Then they satisfy

$$\phi^2 = -1 + \eta \otimes \xi, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

for any vector fields X, Y on $\bar{M}$.

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In 1985, Oubina introduced a new class of almost contact Riemannian manifold known as trans-Sasakian manifold [17]. An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called *trans-Sasakian* if it satisfies

$$(1.3) \qquad (\bar{\nabla}_X \phi)Y = \alpha\,[g(X,Y)\xi - \eta(Y)X] + \beta\,[g(\phi X,Y)\xi - \eta(Y)\phi X],$$

where α and β are non-zero constants on $\bar{M}$, $\bar{\nabla}$ is a Riemannian connection of g and we say that the trans-Sasakian structure is of type (α, β). A trans-Sasakian manifold is a generalization of both α-Sasakian and β-Kenmotsu manifold.

Let M be an n-dimensional isometrically immersed submanifold of $\bar{M}$ and tangent to ξ. Let g be the metric tensor field on $\bar{M}$ as well as the induced metric on M.

Definition 1.1. *An m-dimensional Riemannian submanifold M of a trans-Sasakian manifold $\bar{M}$ is called a semi-invariant submanifold if ξ is tangent to M and it is endowed with a pair of orthogonal differentiable distributions (D,$D^\perp$) which satisfies*

(i) $TM = D \oplus D^\perp \oplus \{\xi\}$, *where $\oplus$ denotes the orthogonal direct sum,*
(ii) *the distribution $D_x\ : x \longrightarrow D \subset T_xM$ is invariant under ϕ i.e. $\phi D_x \subset D_x$ for each $x \in M$,*
(iii) *the orthogonal complementary distribution $D^\perp : x \longrightarrow D^\perp \subset T_xM$ of the distribution D on M is totally real i.e., $\phi D^\perp \subset T_x^\perp M$ where $T_xM, T_x^\perp M$ are the tangent space and the normal space of M at x respectively.*

Let the dimension of D (resp. $D^\perp$) be $2p$(resp. q) where $2p+q = m-1$. If $p = 0$ (resp. $q = 0$) the submanifold M becomes *anti-invariant* (resp. *invariant*) submanifold. A *generic* submanifold M satisfies $\dim D^\perp = \dim T_x^\perp M$. A submanifold is called *proper* if it is neither invariant nor anti-invariant. It is easy to see that any hypersurface to which the vector field ξ is tangent is a typical example of semi-invariant submanifold.

Definition 1.2. *Let M be a semi-invariant submanifold of a trans-Sasakian manifold $\bar{M}$ and M' be an almost contact metric manifold with the almost contact metric structure (ϕ', ξ', η', g'). Assume that there is a submersion $\pi : M \longrightarrow M'$ such that*

(i) $D^\perp = \ker \pi_*$, *where $\pi_* : TM \longrightarrow TM'$ is the tangent mapping to π,*
(ii) $\pi_* : D_p \oplus \{\xi\} \longrightarrow T_{\pi(p)}M'$ *is an isometry for each $p \in M$ which satisfies $\pi_* \circ \phi = \phi' \circ \pi_*$; $\eta = \eta' \circ \pi_*$; $\pi_*(\xi_p) = \xi'_{\pi(p)}$, where $T_{\pi(p)}M'$ denotes the tangent space of M' at $\pi(p)$.*

Papaghuic studied submersion of semi-invariant submanifolds of a Sasakian manifold [18]. For trans-Sasakian manifold we prove

Theorem 1.1. *Let $\pi : M \longrightarrow M'$ be a submersion of semi-invariant submanifold of a trans-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold M'. Then M' is also a trans-Sasakian manifold.*

In particular, we obtain results on Sasakian manifold, Kenmotsu manifold, cosymplectic manifold, α-Sasakian manifold and β-Kenmotsu manifold through submersion of semi-invariant submanifolds. We also derive expressions relating curvatures of $\bar{M}$ and M' via submersions.

2. Preliminaries and some results

Let M be an n-dimensional isometrically immersed submanifold of trans-Sasakian manifold $\bar{M}$ and tangent to ξ and suppose $\bar{\nabla}$ (resp. ∇) be the covariant differentiation with respect to the Levi-Civita connection on $\bar{M}$ (resp. M). The Gauss and Weingarten formulae for M are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.1}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.2}$$

for $X,Y \in TM$, $N \in T^{\perp}M$, where h (resp. A) is the second fundamental form (resp. tensor) of M in $\bar{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover we have

$$g(h(X,Y),N) = g(A_N X,Y). \tag{2.3}$$

The projection of TM to D and $D^{\perp}$ are denoted by h and v respectively i.e., for any $X \in TM$ we have

$$X = hX + vX + \eta(X)\xi. \tag{2.4}$$

The normal bundle to M has the decomposition

$$T^{\perp}M = \phi D^{\perp} \oplus n_1, \tag{2.5}$$

where $g(\phi D^{\perp}, n_1) = \{0\}$. For any $U \in T^{\perp}M$, we put

$$U = nU + mU, \tag{2.6}$$

where $nU \in \phi D^{\perp}$, $mU \in n_1$. Making use of the above equation, we may write

$$\phi U = \phi nU + \phi mU, \quad U \in T^{\perp}M, \quad \phi nU \in D^{\perp}, \quad \phi mU \in n_1. \tag{2.7}$$

A vector field X on M is said to be *basic* if $X \in D_p \oplus \{\xi\}$ and X is π-related to a vector field on M' i.e., there exists a vector field $X_* \in TM'$ such that $\pi_*(X_p) = X_{*\pi(p)}$ for each $p \in M$. Note that, by condition (ii) of the above definition 1.2, we have that the structural vector field ξ is a basic vector field.

Lemma 2.1. [18] *Let X,Y be basic vector fields on M. Then*

(i) $g(X,Y) = g'(X_*,Y_*) \circ \pi$,
(ii) *the component $h([X,Y]) + \eta([X,Y])\xi$ of $[X,Y]$ is a basic vector field and corresponds to $[X_*,Y_*]$, i.e., $\pi_*(h([X,Y]) + \eta([X,Y])\xi) = [X_*,Y_*]$,*
(iii) *$[U,X] \in D^{\perp}$ for any $U \in D^{\perp}$,*
(iv) *$h(\nabla_X Y) + \eta(\nabla_X Y)\xi$ is a basic vector field corresponding to $\nabla^*_{X_*} Y_*$, where ∇^* denotes the Levi-Civita connection on M'.*

For basic vector fields on M, we define the operator $\tilde{\nabla}^*$ corresponding to ∇^* by setting $\tilde{\nabla}^*_X Y = h(\nabla_X Y) + \eta(\nabla_X Y)\xi$ for $X,Y \in (D \oplus \{\xi\})$. By (iv) of lemma 2.1, $\tilde{\nabla}^*_X Y$ is a basic vector field and we have

$$\pi_*(\tilde{\nabla}^*_X Y) = \nabla^*_{X_*} Y_*. \tag{2.8}$$

Define the tensor field C by

$$\nabla_X Y = \tilde{\nabla}^*_X Y + C(X,Y), \quad X,Y \in (D \oplus \{\xi\}), \tag{2.9}$$

where $C(X,Y)$ is the vertical part of $\nabla_X Y$. It is known that C is skew-symmetric and satisfies

$$C(X,Y) = \frac{1}{2}v[X,Y], \quad X,Y \in (D \oplus \{\xi\}). \tag{2.10}$$

The curvature tensors R, R^* of the connection ∇, ∇^* on M and M' respectively are related by [18]

$$\begin{aligned} R(X,Y,Z,W) &= R^*(X_*,Y_*,Z_*,W_*) - g(C(Y,Z),C(X,W)) + g(C(X,Z),C(Y,W)) \\ &\quad + 2g(C(X,Y),C(Z,W)) \quad X,Y,Z,W \in (D \oplus \{\xi\}), \end{aligned} \tag{2.11}$$

where $\pi_* X = X_*$, $\pi_* Y = Y_*$, $\pi_* Z = Z_*$ and $\pi_* W = W_* \in \chi(M')$.

First we prove the following.

Proposition 2.1. *Let $\pi : M \longrightarrow M'$ be a submersion of semi-invariant submanifold of a trans-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold M'. Then we have*

$$(\tilde{\nabla}^*_X \phi)Y = \alpha\,[g(X,Y)\xi - \eta(Y)X] + \beta\,[g(\phi X,Y)\xi - \eta(Y)\phi X], \tag{2.12}$$

$$C(X,\phi Y) = \phi nh(X,Y), \tag{2.13}$$

$$\phi C(X,Y) = nh(X,\phi Y), \tag{2.14}$$

$$\phi mh(X,Y) = mh(X,\phi Y) \tag{2.15}$$

for any $X,Y \in (D \oplus \{\xi\})$.

Proof. For any $X,Y \in (D \oplus \{\xi\})$ and by using Gauss formula (2.1), decomposition equation (2.6) and (2.9) we obtain

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X,Y) = \nabla_X Y + nh(X,Y) + mh(X,Y) \\ &= \tilde{\nabla}^*_X Y + C(X,Y) + nh(X,Y) + mh(X,Y). \end{aligned} \tag{2.16}$$

Hence

$$\phi \bar{\nabla}_X Y = \phi \tilde{\nabla}^*_X Y + \phi C(X,Y) + \phi nh(X,Y) + \phi mh(X,Y). \tag{2.17}$$

Putting $Y = \phi Y$ in (2.16), it follows

$$\bar{\nabla}_X \phi Y = \tilde{\nabla}^*_X \phi Y + C(X,\phi Y) + nh(X,\phi Y) + mh(X,\phi Y). \tag{2.18}$$

On the other hand, using the definition of trans-Sasakian manifold we find

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y = \alpha\,[g(X,Y)\xi - \eta(Y)X] + \beta\,[g(\phi X,Y)\xi - \eta(Y)\phi X]. \tag{2.19}$$

Substituting (2.17) and (2.18) in (2.19) we get

$$\begin{aligned} &\tilde{\nabla}^*_X \phi Y + C(X,\phi Y) + nh(X,\phi Y) + mh(X,\phi Y) - \phi \tilde{\nabla}^*_X Y - \phi C(X,Y) \\ &\qquad - \phi nh(X,Y) - \phi mh(X,Y) = \alpha\,[g(X,Y)\xi - \eta(Y)X] + \beta\,[g(\phi X,Y)\xi - \eta(Y)\phi X]. \end{aligned}$$

Comparing components of $(D \oplus \{\xi\})$, $D^\perp$, $\phi D^\perp$ and n_1 respectively on both sides in the above equation, we get the required results. ∎

Corollary 2.1. *Let $\pi : M \longrightarrow M'$ be a submersion of semi-invariant submanifold of (a) β-Kenmotsu (b) α-Sasakian (c) Kenmotsu (d) Sasakian (e) cosymplectic manifold $\bar{M}$ respectively onto an almost contact metric manifold M'. Then we have*

(i) (a$'$) $(\tilde{\nabla}^*_X \phi)Y = \beta\,[g(\phi X,Y)\xi - \eta(Y)\phi X]$,

(b$'$) $(\tilde{\nabla}^*_X\phi)Y = \alpha\,[g(X,Y)\xi - \eta(Y)X]$,
(c$'$) $(\tilde{\nabla}^*_X\phi)Y = [g(\phi X,Y)\xi - \eta(Y)\phi X]$,
(d$'$) $(\tilde{\nabla}^*_X\phi)Y = [g(X,Y)\xi - \eta(Y)X]$,
(e$'$) $(\tilde{\nabla}^*_X\phi)Y = 0$,

(ii) $C(X,\phi Y) = \phi nh(X,Y)$,
(iii) $\phi C(X,Y) = nh(X,\phi Y)$,
(iv) $\phi mh(X,Y) = mh(X,\phi Y)$

for any $X,Y \in (D \oplus \{\xi\})$.

Now we prove

Theorem 2.1. *Let* $\pi : M \longrightarrow M'$ *be a submersion of semi-invariant submanifold of a trans-Sasakian manifold* $\bar{M}$ *onto an almost contact metric manifold* M'. *Then* M' *is also a trans-Sasakian manifold.*

Proof. Using (2.12) of the Proposition 2.1, we write

$$(\tilde{\nabla}^*_X\phi)Y = \alpha\,[g(X,Y)\xi - \eta(Y)X] + \beta\,[g(\phi X,Y)\xi - \eta(Y)\phi X].$$

Applying π_* to the above equation and using Lemma 2.1, (2.8) and definition of submersion, we derive

$$(\tilde{\nabla}^*_{X_*}\phi')Y_* = \alpha\left[g'(X_*,Y_*)\xi' - \eta'(Y_*)X_*\right] + \beta\left[g'(\phi' X_*,Y_*)\xi' - \eta'(Y_*)\phi' X_*\right].$$

The above equation shows that M' is a trans-Sasakian manifold. ∎

Corollary 2.2. *Let* $\pi : M \longrightarrow M'$ *be a submersion of semi-invariant submanifold of (a)* β*-Kenmotsu (b)* α*-Sasakian (c) Kenmotsu (d) Sasakian (e) cosymplectic manifold* $\bar{M}$ *respectively onto an almost contact metric manifold* M'. *Then* M' *is also (a$'$)* β*-Kenmotsu (b$'$)* α*-Sasakian (c$'$) Kenmotsu (d$'$) Sasakian (e$'$) cosymplectic manifold.*

Proposition 2.2. *Let* $\pi : M \longrightarrow M'$ *be a submersion of semi-invariant submanifold of a trans-Sasakian manifold* $\bar{M}$ *onto an almost contact metric manifold* M'. *Then*

(i) $nh(\phi X,\phi Y) + nh(\phi X,Y) = 0$,
(ii) $nh(\phi X,\phi Y) = nh(X,Y)$,
(iii) $mh(\phi X,\phi Y) = -mh(X,Y)$,
(iv) $C(\phi X,\phi Y) = C(X,Y)$

for any $X,Y \in (D \oplus \{\xi\})$.

Proof.

(i) Interchanging X and Y in (2.14) gives

$$\phi C(Y,X) = nh(Y,\phi X) = nh(\phi X,Y).$$

Then

$$nh(X,\phi Y) + nh(\phi X,Y) = \phi C(X,Y) + \phi C(Y,X) = \phi C(X,Y) - \phi C(X,Y) = 0.$$

(ii) Putting $X = \phi X$ in (2.14), we get

$$nh(\phi X,\phi Y) = \phi C(\phi X,Y) = -\phi C(Y,\phi X).$$

Using (2.13) in the above equation, we deduce

$$nh(\phi X,\phi Y) = -\phi C(Y,\phi X) = -\phi(\phi nh(Y,X)) = -\phi^2 nh(Y,X)$$

$$= nh(Y,X) - \eta(h(X,Y))\xi = nh(Y,X).$$

(iii) Putting $X = \phi X$ in (2.15) and using again the same equation, we find

$$mh(\phi X,\phi Y) = \phi mh(\phi X,Y) = \phi mh(Y,\phi X) = \phi^2 mh(Y,X) = -mh(X,Y).$$

(iv) Putting $X = \phi X$ in (2.13) and then using (2.14) yields

$$\begin{aligned} C(\phi X,\phi Y) &= \phi nh(\phi X,Y) = \phi nh(Y,\phi X) = \phi^2 C(Y,X) \\ &= -C(Y,X) + \eta(C(Y,X))\xi = C(X,Y). \end{aligned}$$

∎

3. Curvature relations

Proposition 3.1. *Let $\pi : M \longrightarrow M'$ be a submersion of semi-invariant submanifold of a trans-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold M'. Then the ϕ-bisectional curvature of $\bar{M}$ and M' are related by*

$$\begin{aligned} \bar{B}(X,Y) = B'(X_*,Y_*) &- 2\|nh(X,Y)\|^2 - 2\|nh(X,\phi Y)\|^2 \\ &- 2g(nh(X,X),nh(Y,Y)) + 2\|mh(X,Y)\|^2, \end{aligned}$$

where $X,Y \in (D \oplus \{\xi\})$.

Proof. We know

$$\bar{B}(X,Y) = \bar{R}(X,\phi X,\phi Y,Y).$$

Put $Y = \phi X$, $Z = \phi Y$, $W = Y$ in Gauss equation

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)),$$

we get

$$\bar{R}(X,\phi X,\phi Y,Y) = R(X,\phi X,\phi Y,Y) - g(h(X,Y),h(\phi X,\phi Y)) + g(h(X,\phi Y),h(\phi X,Y)).$$

Substituting $h = nh + mh$, in the above equation, we arrive at

$$\begin{aligned} \bar{R}(X,\phi X,\phi Y,Y) &= R(X,\phi X,\phi Y,Y) - g(nh(X,Y) + mh(X,Y),nh(\phi X,\phi Y) + mh(\phi X,\phi Y)) \\ &\quad + g(nh(X,\phi Y) + mh(X,\phi Y),nh(\phi X,Y) + mh(\phi X,Y)) \\ &= R(X,\phi X,\phi Y,Y) - g(nh(X,Y),nh(\phi X,\phi Y)) - g(nh(X,Y),mh(\phi X,\phi Y)) \\ &\quad - g(mh(X,Y),nh(\phi X,\phi Y)) - g(mh(X,Y),mh(\phi X,\phi Y)) \\ &\quad + g(nh(X,\phi Y),nh(\phi X,Y)) + g(nh(X,\phi Y),mh(\phi X,Y)) \\ &\quad + g(mh(X,\phi Y),nh(\phi X,Y)) + g(mh(X,\phi Y),mh(\phi X,Y)) \\ &= R(X,\phi X,\phi Y,Y) - g(nh(X,Y),nh(\phi X,\phi Y)) - g(mh(X,Y),mh(\phi X,\phi Y)) \\ &\quad + g(nh(X,\phi Y),nh(\phi X,Y)) + g(mh(X,\phi Y),mh(\phi X,Y)) \\ &= R(X,\phi X,\phi Y,Y) - g(nh(X,Y),nh(X,Y)) + g(mh(X,Y),mh(X,Y)) \\ &\quad - g(nh(X,\phi Y),nh(X,\phi Y)) + g(\phi mh(X,Y),\phi mh(X,Y)) \\ &= R(X,\phi X,\phi Y,Y) - \|nh(X,Y)\|^2 + 2\|mh(X,Y)\|^2 - \|nh(X,\phi Y)\|^2. \end{aligned} \tag{3.1}$$

Now by putting $Y = \phi X$, $Z = \phi Y$, $W = Y$ in (2.11) it follows

$$\begin{aligned} R(X,\phi X,\phi Y,Y) &= R^*(X_*,\phi' X_*,\phi' Y_*,Y_*) - g(C(\phi X,\phi Y),C(X,Y)) \\ &\quad + g(C(X,\phi Y),C(\phi X,Y)) + 2g(C(X,\phi X),C(\phi Y,Y)) \\ &= R^*(X_*,\phi' X_*,\phi' Y_*,Y_*) - g(C(\phi X,\phi Y),C(X,Y)) \end{aligned}$$

$$-g(C(X,\phi Y),C(Y,\phi X))-2g(C(X,\phi X),C(Y,\phi Y)). \tag{3.2}$$

Applying ϕ to equation $\phi C(X,Y)=nh(X,\phi Y)$, we get $\phi^2 C(X,Y)=\phi nh(X,\phi Y)$. This gives

$$-C(X,Y)+\eta(C(X,Y))\xi=\phi nh(X,\phi Y)$$

or

$$C(X,Y)=-\phi nh(X,\phi Y).$$

Using the above relation in (3.2), we conclude

$$\begin{aligned} R(X,\phi X,\phi Y,Y) &= R^*(X_*,\phi' X_*,\phi' Y_*,Y_*)-\|nh(X,Y)\|^2 \\ &\quad -\|nh(X,\phi Y)\|^2-2g(nh(X,X),nh(Y,Y)). \end{aligned} \tag{3.3}$$

Put this value of $R(X,\phi X,\phi Y,Y)$ in (3.1) we obtain

$$\begin{aligned} \bar{R}(X,\phi X,\phi Y,Y) &= R^*(X_*,\phi' X_*,\phi' Y_*,Y_*)-\|nh(X,Y)\|^2-\|nh(X,\phi Y)\|^2 \\ &\quad -2g(nh(X,X),nh(Y,Y))-\|nh(X,Y)\|^2+2\|mh(X,Y)\|^2-\|nh(X,\phi Y)\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \bar{B}(X,Y) &= B'(X_*,Y_*)-2\|nh(X,Y)\|^2-2\|nh(X,\phi Y)\|^2 \\ &\quad -2g(nh(X,X),nh(Y,Y))+2\|mh(X,Y)\|^2. \end{aligned}$$

∎

Corollary 3.1. *Let $\pi: M \longrightarrow M'$ be a submersion of semi-invariant submanifold of a trans-Sasakian manifold $\bar{M}$ onto an almost contact metric manifold. Then the ϕ-sectional curvature of $\bar{M}$ and M' are related by*

$$\bar{H}(X)=H'(X_*)-4\|nh(X,X)\|^2+2\|mh(X,X)\|^2,$$

where $X\in(D\oplus\{\xi\})$.

Proof. Putting $X=Y$ in the above expression of $\bar{B}(X,Y)$ allow us to obtain

$$\begin{aligned} \bar{B}(X,X)=\bar{H}(X) &= H'(X_*)-2\|nh(X,X)\|^2-2\|nh(X,\phi X)\|^2 \\ &\quad -2g(nh(X,X),nh(X,X))+2\|mh(X,X)\|^2 \\ &= H'(X_*)-4\|nh(X,X)\|^2-2\|nh(X,\phi X)\|^2+2\|mh(X,X)\|^2. \end{aligned}$$

Putting $Y=X$ in (2.14) of Proposition 2.1

$$nh(X,\phi X)=\phi C(X,X)=0.$$

Thus we get

$$\bar{H}(X)=H'(X_*)-4\|nh(X,X)\|^2+2\|mh(X,X)\|^2.$$

∎

4. Cohomology of submersion of semi-invariant submanifolds of trans-Sasakian manifolds

In this section, we discuss how the submersion $\pi: M \longrightarrow M'$ of a semi-invariant submanifold M with minimal horizontal distribution $(D\oplus\{\xi\})$ effects the topology of M. Let M be a semi-invariant submanifold of a trans-Sasakian manifold $\bar{M}$ with almost contact metric structure (ϕ,ξ,η,g). Assume that $\dim(D\oplus\{\xi\})=2p+1$ and $\dim M=m$. We choose a local orthonormal frame $\{e_1,e_2,...,e_p,\phi e_1,\phi e_2,....,\phi e_p,\ e_{2p+1}=\xi,e_{2p+2},...,e_m\}$ on M such that $\{e_1,e_2,...,e_p,\phi e_1,\phi e_2,....,\phi e_p,e_{2p+1}=\xi\}$ is a local orthonormal frame of $(D\oplus\{\xi\})$ and $\{e_{2p+2},e_{2p+3},...,e_m\}$ is that of $D^\perp$. Let $\{\omega^1,\omega^2,....,\omega^{2p+1},\omega^{2p+2},......,\omega^m\}$ be the dual frame of 1-forms to the above local orthonormal frame. Define a $2p+1$-form Ω on M by

$$\Omega = \omega^1 \wedge \omega^2 \wedge .... \wedge \omega^{2p+1}, \tag{4.1}$$

which is globally defined on M.

Definition 4.1. *Let S be a q-dimensional distribution on a Riemannian manifold M. If $\sum_{i=1}^{q} \nabla_{e_i} e_i \in S$, then the distribution S is said to be minimal, where ∇ is the Riemannian connection on M and $\{e_1, e_2, ..., e_q\}$is a local orthonormal frame of S.*

Theorem 4.1. *Let $\bar{M}$ be a trans-Sasakian manifold and M be a closed semi-invarinat submanifold of $\bar{M}$ with minimal $(D \oplus \{\xi\})$. Let M' be a almost contact metric manifold and $\pi : M \longrightarrow M'$ a submersion. Then the $2p+1$-form Ω is closed which defines a canonical de Rham cohomology class $[\Omega] \in H^{2p+1}(M,R)$, where $2p+1 = \dim(D \oplus \{\xi\})$. Moreover the cohomology group $H^{2p+1}(M,R)$ is non-trivial if $D^{\perp}$ is minimal.*

Proof. From definition (4.1) of Ω, we have

$$d\Omega = \sum_{i=1}^{2p+1} (-1)^{i-1} \omega^1 \wedge ... \wedge d\omega^i \wedge ..... \wedge \omega^{2p+1}.$$

From the above equation it follows that $d\Omega = 0$ if and only if [8]

$$d\Omega(Z,W,E_1,....,E_{2p}) = 0 \quad \text{and} \quad d\Omega(Z,E_1,......,E_{2p+1}) = 0 \tag{4.2}$$

for $Z,W \in D^{\perp}$ and $E_1,......,E_{2p+1} \in (D \oplus \{\xi\})$. Choosing the vectors $E_1,......,E_{2p+1} \in (D \oplus \{\xi\})$ as a local orthonormal frame $\{e_1, e_2, ..., e_p, \phi e_1, \phi e_2, ..., \phi e_p, e_{2p+1} = \xi\}$ of $(D \oplus \{\xi\})$ to which $\{\omega^1, \omega^2, ..., \omega^{2p+1}\}$ works as dual frame of 1-forms, we get by a straightforward computation that the first equation in (4.2) holds if and only if $D^{\perp}$ is integrable; and the second equation in (4.2) holds if and only if $(D \oplus \{\xi\})$ is minimal. However, from the definition of submersion it follows that $D^{\perp}$ is integrable. The hypothesis of theorem gives that $(D \oplus \{\xi\})$ is minimal. Hence the form Ω is closed, and it defines a de Rham cohomology class $[\Omega] \in H^{2p+1}(M,R)$.

Now suppose that $D^{\perp}$ is minimal and we proceed to show that in this case

$$H^{2p+1}(M,R) \neq 0.$$

To accomplish this we show that the form Ω is harmonic which would then make the cohomology class $[\Omega]$ non-trivial. Define a $(m-2p-1)$-form $\Omega^{\perp}$ on M by setting

$$\Omega^{\perp} = \omega^{2p+2} \wedge .... \wedge \omega^m,$$

where $\{\omega^{2p+2}, ..., \omega^m\}$ is the dual frame to the local orthonormal frame $\{e_{2p+1}, ..., e_m\}$ of $D^{\perp}$. Then with the similar argument for Ω, it follows that $d\Omega^{\perp} = 0$ if $(D \oplus \{\xi\})$ is integrable and $D^{\perp}$ is minimal. It should be noted that minimality of $(D \oplus \{\xi\})$ implies its integrability. Since both conditions are met, we have $d\Omega^{\perp} = 0$. This proves that the $2p+1$-form Ω is co-closed, that is $\delta\Omega = 0$. Since $d\Omega = \delta\Omega = 0$ and M is closed submanifold, we get that Ω is harmonic $2p+1$-form; and this completes the proof. ∎

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