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Linear Preservers of Extremes of Matrix Pairs Over Nonbinary Boolean Algebra

SEOK-ZUN SONG AND MUN-HWAN KANG
Department of Mathematics (and RIBS), Jeju National University, Jeju 690-756, Korea
szsong@jejunu.ac.kr, gudeok@megapass.net

Abstract. The $m \times n$ Boolean matrix A is said to be of Boolean rank r if there exist $m \times r$ Boolean matrix B and $r \times m$ Boolean matrix C such that $A = BC$ and r is the smallest positive integer that such a factorization exists. We characterize linear operators that preserve the sets of matrix ordered pairs which satisfy extremal properties with respect to Boolean rank inequalities of matrices over nonbinary Boolean algebras.

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1. Introduction

There are many papers on linear operators that preserve some properties of matrices [2]-[12]. We call such a topic of research "Linear Preserver Problems". These linear preserver problems have been studied for various characterizations of matrices during a century. In 1896, Frobenius characterized the linear operators that preserve the determinant of matrices over the real field, which was the first result on linear preserver problems. After his result, many researchers have studied linear operators that preserve some matrix functions, say, rank and the permanent of matrices [11]. Beasley and Guterman [1] investigated rank inequalities of matrices over semirings. And they characterized the equality cases for some inequalities in [2]. These characterization problems are open even over fields (see [3]). The structure of matrix varieties which arise as extremal cases in these inequalities is far from being understood over fields, as well as over semirings. A usual way to generate elements of such a variety is to find a pair of matrices which belongs to it and to act on this pair by various linear operators that preserve this variety. The investigation of the corresponding problems over semirings for the column rank function was done in [3]. The complete classification of linear operators that preserve equality cases in matrix inequalities over fields was obtained in [5]. For details on linear operators preserving matrix invariants one can see [10] and [11]. Almost all research on linear preserver problems over semirings have dealt with those semirings without zero-divisors to avoid the difficulties of multiplication arithmetic

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for the elements in those semirings [2]-[7]. But nonbinary Boolean algebra is not the case. That is, all elements except 0 and 1 in most nonbinary Boolean algebras are zero-divisors. So there are few results on the linear preserver problems for the matrices over nonbinary Boolean algebra [8], [9] and [12]. Kirkland and Pullman characterized the linear operators that preserve rank of matrices over nonbinary Boolean algebra in [9].

In this paper, we characterize the linear operators that preserve the sets of matrix ordered pairs which satisfy extremal properties with respect to Boolean rank inequalities of matrices over nonbinary Boolean algebra.

2. Preliminaries and basic results

Definition 2.1. *A semiring S consists of a set S with two binary operations, addition and multiplication, such that:*

- *S is an Abelian monoid under addition (the identity is denoted by 0);*
- *S is a monoid under multiplication (the identity is denoted by 1, $1 \neq 0$);*
- *multiplication is distributive over addition on both sides;*
- *$s0 = 0s = 0$ for all $s \in S$.*

Definition 2.2. *A semiring S is called antinegative if the zero element is the only element with an additive inverse.*

Definition 2.3. *A semiring S is called a Boolean algebra if S is equivalent to a set of subsets of a given set M, the sum of two subsets is their union, and the product is their intersection. The zero element is the empty set and the identity element is the whole set M.*

Let $S_k = \{a_1, a_2, \cdots, a_k\}$ be a set of k-elements, $\mathscr{P}(S_k)$ be the set of all subsets of S_k and $\mathbb{B}_k$ be a Boolean algebra of subsets of $S_k = \{a_1, a_2, \cdots, a_k\}$, which is a subset of $\mathscr{P}(S_k)$. It is straightforward to see that a Boolean algebra $\mathbb{B}_k$ is a commutative and antinegative semiring. If $\mathbb{B}_k$ consists of only the empty subset and M then it is called a *binary Boolean algebra*. If $\mathbb{B}_k$ is not binary Boolean algebra then it is called a *nonbinary Boolean algebra*. Let $\mathbb{M}_{m,n}(\mathbb{B}_k)$ denote the set of $m \times n$ matrices with entries from the Boolean algebra $\mathbb{B}_k$. If $m = n$, we use the notation $\mathbb{M}_n(\mathbb{B}_k)$ instead of $\mathbb{M}_{n,n}(\mathbb{B}_k)$.

Throughout the paper, we assume that $m \leq n$ and $\mathbb{B}_k$ denotes the nonbinary Boolean algebra, which contains at least 3 elements. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones and $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J and O, respectively. The matrix $E_{i,j}$, which is called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the $(i,j)^{th}$ entry. A *weighted cell* is any nonzero scalar multiple of a cell, that is, $\alpha E_{i,j}$ is a weighted cell for any $0 \neq \alpha \in \mathbb{B}_k$. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere. We denote by $\#(A)$ the number of nonzero entries in the matrix A. We denote by A[i,j|r,s] the 2×2 submatrix of A which lies in the intersection of the i^{th} and j^{th} rows with the r^{th} and s^{th} columns.

Definition 2.4. *The matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ is said to be of Boolean rank r if there exist matrices $B \in \mathbb{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathbb{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$ and r is the smallest positive integer that such a factorization exists. We denote $b(A) = r$.*

By definition, the unique matrix with Boolean rank equal to 0 is the zero matrix O.

If S is a field, then there is the usual rank function $\rho(A)$ for any matrix $A \in \mathbb{M}_{m,n}(S)$. It is well-known that the behavior of the function ρ with respect to matrix addition and multiplication is given by the following inequalities [4]:

- the rank-sum inequalities:

$$|\rho(A)-\rho(B)| \leq \rho(A+B) \leq \rho(A)+\rho(B),$$

- Sylvester's laws:

$$\rho(A)+\rho(B)-n \leq \rho(AB) \leq \min\{\rho(A),\rho(B)\},$$

where A, B are conformal matrices with entries from a field.

The arithmetic properties of Boolean rank are restricted by the following list of inequalities [1], since the Boolean algebra is antinegative:

(1) $\mathrm{b}(A+B) \leq \mathrm{b}(A) + \mathrm{b}(B)$;
(2) $\mathrm{b}(AB) \leq \min\{\mathrm{b}(A), \mathrm{b}(B)\}$.
(3) $\mathrm{b}(A + B) \geq \begin{cases} \mathrm{b}(A) & \text{if } B = O, \\ \mathrm{b}(B) & \text{if } A = O, \\ 1 & \text{if } A \neq O \text{ and } B \neq O; \end{cases}$

Below, we use the following notation in order to denote sets of matrices that arise as extremal cases in the inequalities listed above:

$\mathscr{R}_{SA}(\mathbb{B}_k) = \{\, (X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 |\ \mathrm{b}(X+Y) = \mathrm{b}(X)+\mathrm{b}(Y)\,\}$,
$\mathscr{R}_{S1}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 |\ \mathrm{b}(X+Y) = 1\}$.

In this paper, we characterize the linear operators that preserve $\mathscr{R}_{SA}(\mathbb{B}_k)$ and $\mathscr{R}_{S1}(\mathbb{B}_k)$.

Definition 2.5. *We say that an operator T preserves a set $\mathscr{P}$ if $X \in \mathscr{P}$ implies that $T(X) \in \mathscr{P}$ or if $\mathscr{P}$ is the set of ordered pairs such that $(X,Y) \in \mathscr{P}$ implies $((T(X),T(Y)) \in \mathscr{P}$.*

Definition 2.6. *An operator T strongly preserves a set $\mathscr{P}$ if $X \in \mathscr{P}$ if and only if $T(X) \in \mathscr{P}$ or if $\mathscr{P}$ is the set of ordered pairs such that $(X,Y) \in \mathscr{P}$ if and only if $(T(X),T(Y)) \in \mathscr{P}$.*

Definition 2.7. *For $X,Y \in \mathbb{M}_{m,n}(S)$, the matrix $X \circ Y$ denotes the Hadamard or Schur product, i.e., the $(i,j)^{th}$ entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.*

Definition 2.8. *An operator T is called a (P,Q,B)-operator if there exist permutation matrices P and Q and a matrix $B \in \mathbb{M}_{m,n}(S)$ with no zero entries such that $T(X) = P(X \circ B)Q$ for all $X \in \mathbb{M}_{m,n}(S)$ or if for $m = n$, $T(X) = P(X \circ B)^T Q$ for all $X \in \mathbb{M}_{m,n}(S)$. A (P,Q,B)-operator is called a (P,Q)-operator if $B = J$, the matrix of all ones.*

Definition 2.9. *An operator $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ is called linear if it satisfies $T(X+Y) = T(X)+T(Y)$ and $T(\alpha X) = \alpha T(X)$ for all $X,Y \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $\alpha \in \mathbb{B}_k$.*

Definition 2.10. *A line of a matrix A is a row or a column of the matrix A.*

Definition 2.11. *We say that the matrix A dominates the matrix B if and only if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.*

Lemma 2.1. *Let P and Q be permutation matrices of m-square and n-square respectively. If $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ is defined by $T(X) = PX$ or $T(X) = XQ$ for any $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$. Then T preserves Boolean rank. That is $b(T(X)) = b(X)$.*

Proof. Let $A,B \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and P be an $m \times m$ permutation matrix. Since $\mathrm{b}(AB) \leq \min\{\mathrm{b}(A), \mathrm{b}(B)\}$, we have $\mathrm{b}(PX) \leq \min\{\mathrm{b}(P), \mathrm{b}(X)\} \leq \mathrm{b}(X)$. And $\mathrm{b}(X) = \mathrm{b}(IX) = \mathrm{b}((P^TP)X) = \mathrm{b}(P^T(PX)) \leq \mathrm{b}(PX)$. Hence $\mathrm{b}(PX) = \mathrm{b}(X)$. Similarly $\mathrm{b}(XQ) = \mathrm{b}(X)$ for all $n \times n$ permutation matrix Q. ∎

Lemma 2.2. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{2,2}(\mathbb{B}_k)$. *Then b(A)=1 if and only if* $ad = bc$.

Proof. Suppose that b(A) = 1. Then there exist vectors $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{y} = [y_1, y_2]$ such that $A = \mathbf{xy}$. Thus

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{bmatrix}.$$

Hence $ad = x_1x_2y_1y_2 = bc$.
Conversely, assume that $ad = bc$. Then

$$\begin{bmatrix} a+b \\ c+d \end{bmatrix} \begin{bmatrix} a+c & b+d \end{bmatrix} = \begin{bmatrix} (a+b)(a+c) & (a+b)(b+d) \\ (c+d)(a+c) & (c+d)(b+d) \end{bmatrix}$$

$$= \begin{bmatrix} aa+ba+ac+bc & ab+bb+ad+bd \\ ca+cc+da+dc & cb+db+cd+dd \end{bmatrix} = \begin{bmatrix} a+bc & b+ad \\ c+ad & d+bc \end{bmatrix}$$

$$= \begin{bmatrix} a+ad & b+bc \\ c+bc & d+ad \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A,$$

where the 3rd and 5th equalities come from the definitions of addition and multiplication of Boolean algebra and the 4th equality comes from assumption. Thus $b(A) = 1$. ∎

Lemma 2.3. *Let* $A \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, *where* $m, n \geq 2$. *b(A) = 1 if and only if b(*A'*) = 1 for any* 2×2 *submatrix* A' *of* A.

Proof. $\Rightarrow$) Suppose that $b(A) = 1$, then there exist vectors $\mathbf{a} = [a_1, a_2, \ldots, a_m]^T$ and $\mathbf{b} = [b_1, b_2, \ldots, b_n]$ such that $A = \mathbf{ab}$. i.e., $a_{i,j} = a_ib_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus for any 2×2 submatrix

$$A' = A[i, j|r, s] = \begin{bmatrix} a_ib_r & a_ib_s \\ a_jb_r & a_jb_s \end{bmatrix} = \begin{bmatrix} a_i \\ b_j \end{bmatrix} \begin{bmatrix} b_r & b_s \end{bmatrix}.$$

That is, b(A') = 1.

$\Leftarrow$) Suppose that b(A) $= r > 1$. Then there exist matrices $B \in \mathbb{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathbb{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$. Thus there exist matrices $B' \in \mathbb{M}_{m,2}(\mathbb{B}_k)$ and $C' \in \mathbb{M}_{2,n}(\mathbb{B}_k)$ such that $A' = B'C'$ has Boolean rank 2. Therefore there exist matrices $B'' \subset B'$ and $C'' \subset C'$ in $\mathbb{M}_{2,2}(\mathbb{B}_k)$ such that $A'' = B''C'' \in \mathbb{M}_{2,2}(\mathbb{B}_k)$ with b(A'') = 2, a contradiction. ∎

Theorem 2.1. *Let* $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \rightarrow \mathbb{M}_{m,n}(\mathbb{B}_k)$ *be a linear operator. Then the following conditions are equivalent:*

(a) *T is bijective;*
(b) *T is surjective;*
(c) *T is injective;*
(d) *there exists a permutation σ on $\{(i, j)|i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Proof. (a), (b) and (c) are equivalent since $\mathbb{M}_{m,n}(\mathbb{B}_k)$ is a finite set.

(d)$\Rightarrow$(b) For any $D \in \mathbb{M}_{m,n}(\mathbb{B}_k)$, we may write

$$D = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{i,j}E_{i,j}.$$

Since σ is a permutation, there exist $\sigma^{-1}(i,j)$ and

$$D' = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)}E_{\sigma^{-1}(i,j)}$$

such that

$$T(D') = T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma^{-1}(i,j)}E_{\sigma^{-1}(i,j)}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{\sigma\sigma^{-1}(i,j)}E_{\sigma\sigma^{-1}(i,j)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} d_{i,j}E_{i,j} = D.$$

(a)⇒(d) We assume that T is bijective. Suppose that $T(E_{i,j}) \neq E_{\sigma(i,j)}$ where σ be a permutation on $\{(i,j)|i=1,2,\ldots,m; j=1,2,\ldots,n\}$. Then there exist some pairs (i,j) and (r,s) such that $T(E_{i,j}) = \alpha E_{r,s}(\alpha \neq 1)$ or some pairs $(i,j),(r,s)$ and (u,v) $((r,s)\neq(u,v))$ such that $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z(\alpha \neq 0, \beta \neq 0, Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r,s)^{th}$ and $(u,v)^{th}$ entries of Z are zeros.

Case 1. Suppose that there exist some pairs (i,j) and (r,s) such that $T(E_{i,j}) = \alpha E_{r,s}(\alpha \neq 1)$. Since T is bijective, there exist $X_{r,s} \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(X_{r,s}) = E_{r,s}$. Then $\alpha T(X_{r,s}) = \alpha E_{r,s} = T(E_{i,j})$, and hence $\alpha X_{r,s} = E_{i,j}$, which contradicts the fact that $\alpha \neq 1$.

Case 2. Suppose that there exist some pairs $(i,j),(r,s)$ and (u,v) such that $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z(\alpha \neq 0, \beta \neq 0, Z \in \mathbb{M}_{m,n}(\mathbb{B}_k))$, where the $(r,s)^{th}$ and $(u,v)^{th}$ entries of Z are zeros. Since T is bijective, there exist $X_{r,s}$, $X_{u,v}$ and $Z' \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(X_{r,s}) = \alpha E_{r,s}$, $T(X_{u,v}) = \beta E_{u,v}$, and $T(Z') = Z$. Thus $T(E_{i,j}) = \alpha E_{r,s} + \beta E_{u,v} + Z = T(X_{r,s}) + T(X_{u,v}) + T(Z') = T(X_{r,s} + X_{u,v} + Z')$. So $E_{i,j} = X_{r,s} + X_{u,v} + Z'$, a contradiction. ∎

Remark 2.1. One can easily verify that if $m = 1$ or $n = 1$, then all operators under consideration are (P,Q,B)-operators and if $m = n = 1$, then all operators under consideration are (P,P^T,B)-operators.

Henceforth we will always assume that $m,n \geq 2$.

Lemma 2.4. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator which maps a line to a line and T be defined by the rule $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j)|i=1,2,\ldots,m; j=1,2,\ldots,n\}$ and $b_{i,j} \in \mathbb{B}_k$ are nonzero elements for $i = 1,2,\ldots,m; j=1,2,\ldots,n$. Then T is a (P,Q,B)-operator.*

Proof. Since no combination of p rows and q columns can dominate J for any nonzero p and q with $p+q=m$, we have that either the image of each row is a row and the image of each column is a column, or $m=n$ and the image of each row is a column and the image of each column is a row. Thus there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$, $T(C_j) \leq PC_jQ$ or, if $m=n$, $T(R_i) \leq P(R_i)^TQ$, $T(C_j) \leq P(C_j)^TQ$. Since each nonzero entry of a cell lies in the intersection of a row and a column and T maps nonzero cells into nonzero (weighted) cells, it follows that $T(E_{i,j}) = Pb_{i,j}E_{i,j}Q = P(E_{i,j} \circ B)Q$, or, if $m=n$, $T(E_{i,j}) = P(b_{i,j}E_{i,j})^TQ = P(E_{i,j} \circ B)^TQ$ where $B = (b_{i,j})$ is defined by the action of T on the cells. ∎

Lemma 2.5. *If $T(X) = X \circ B$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and b(B) = 1 then there exist diagonal matrices D and E such that $T(X) = DXE$ for all $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$.*

Proof. Since b(B) = 1, there exist vectors $\mathbf{d} = [d_i, d_2, \ldots, d_m]^T \in \mathbb{M}_{m,1}$ and $\mathbf{e} = [e_i, e_2, \ldots, e_n] \in \mathbb{M}_{1,n}$ such that $B = \mathbf{de}$ or $b_{i,j} = d_i e_j$. Let $D = diag\{d_1, d_2, \ldots, d_m\}$ and $E = diag\{e_1, e_2, \ldots, e_n\}$. Now the $(i,j)^{th}$ entry of $T(X)$ is $b_{i,j}x_{i,j}$ and the $(i,j)^{th}$ entry of DXE is $d_i x_{i,j} e_j = b_{i,j}x_{i,j}$. Hence $T(X) = DXE$. ∎

Example 2.1. Consider the linear operator $T : \mathbb{M}_{3,3}(\mathbb{B}_3) \to \mathbb{M}_{3,3}(\mathbb{B}_3)$ defined by $T(X) = X \circ B$ for all $X \in \mathbb{M}_{3,3}(\mathbb{B}_3)$ with $\mathbb{B}_3 = \mathscr{P}(\{a,b,c\})$. Then b($B$) = 1 but T does not preserve the Boolean rank.

$$\text{Consider } X = \begin{bmatrix} \{a,b\} & \{a,b,c\} & \{a,b\} \\ \{a,c\} & \{a,b\} & \{a,c\} \\ \{a\} & \{b,c\} & \{a,b,c\} \end{bmatrix} \text{ and } B = \begin{bmatrix} \{a\} & \{b\} & \{a\} \\ \{a\} & \{b\} & \{a\} \\ \{a\} & \{b\} & \{a\} \end{bmatrix}.$$

Then b(X) = 3, but

$$T(X) = X \circ B = \begin{bmatrix} \{a\} & \{b\} & \{a\} \\ \{a\} & \{b\} & \{a\} \\ \{a\} & \{b\} & \{a\} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \{a\} & \{b\} & \{a\} \end{bmatrix}.$$

That is, b($T(X)$) = b($X \circ B$) $= 1 \neq 3 =$b(X). Thus b(B) = 1 but T does not preserve Boolean rank.

3. Linear preservers of $\mathscr{R}_{SA}(\mathbb{B}_k)$

Recall that

$$\mathscr{R}_{SA}(\mathbb{B}_k) = \{\, (X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 \,|\, \mathrm{b}(X+Y) = \mathrm{b}(X) + \mathrm{b}(Y)\,\}.$$

We begin with some general observations on linear operators of special types that preserve $\mathscr{R}_{SA}(\mathbb{B}_k)$.

Lemma 3.1. *Let σ be a permutation of the set $\{(i,j) | i = 1,2,\ldots, m; j = 1,2,\ldots,n\}$, and $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator defined by $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some nonzero scalars $b_{i,j}$, $1 \le i \le m$ and $1 \le j \le n$. If T preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$, then T is a (P,Q,B)-operator.*

Proof. We examine the action of T on rows and columns of a matrix. Suppose that the image of two cells are in the same line, but the cells are not, say, E and F are cells such that b(E + F) = 2 and b($T(E+F)$) = 1. Then $(E,F) \in \mathscr{R}_{SA}(\mathbb{B}_k)$ but $(T(E), T(F)) \notin \mathscr{R}_{SA}(\mathbb{B}_k)$, a contradiction since T preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$. Thus T maps any line to a line. By Lemma 2.4, we obtain that T is a (P,Q,B)-operator. ∎

Lemma 3.2. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. If for some B = ($b_{i,j}$), where $b_{i,j}$ are nonzero scalars for all $1 \le i \le m$ and $1 \le j \le n$, $T(X) = X \circ B$ preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$, then b($B$) = 1. Moreover, $T(X) = DXE$ for diagonal matrices D and E of appropriate sizes.*

Proof. If b(B) $\ge$ 2, then by Lemma 2.3, there is a 2×2 submatrix $B[i,j|r,s]$ such that b($B[i,j|r,s]$) = 2. Let $Y = E_{i,r} + E_{j,r} + E_{i,s} + E_{j,s}$. Thus $T(Y) = b_{i,r}E_{i,r} + b_{j,r}E_{j,r} + b_{i,s}E_{i,s} + b_{j,s}E_{j,s} = Z$ has Boolean rank 2 from b($B[i,j|r,s]$) = 2. Then for $q \neq r,s$, we have b($E_{i,q} + Y$) = 2 = b($E_{i,q}$) + b(Y), so that $(E_{i,q}, Y) \in \mathscr{R}_{SA}(\mathbb{B}_k)$, while b($T(E_{i,q} + Y)$) = b($b_{i,q}E_{i,q} + Z$) = 2 $\neq$ b($b_{i,q}E_{i,q}$) + b(Z) = 1 + 2 = 3, a contradiction since T preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$. Thus b($B$) = 1. Moreover, by Lemma 2.5, there exist diagonal matrices D and E such that $T(X) = DXE$. ∎

Theorem 3.1. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear operator. If T preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$, then T is a (P,Q)-operator.*

Proof. If T is surjective, then by Theorem 2.1, we have that T is defined by a permutation σ on the set $\{(i,j)|i = 1,2,\ldots,m; j = 1,2,\ldots,n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. By Lemma 3.1, we have that T is a (P,Q,J)-operator. Thus T is a (P,Q)-operator. ∎

Corollary 3.1. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear operator. The operator T preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$ if and only if T is a (P,Q)-operator.*

Proof. Suppose that T preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$. Then T is a (P,Q)-operator by Theorem 3.1.
Conversely, assume that T is a (P,Q)-operator. For any $(X,Y) \in \mathscr{R}_{SA}(\mathbb{B}_k)$, we have b$(X+Y)$ = b(X) + b(Y). Thus

$$\begin{aligned} b(T(X)+T(Y)) &= b(T(X+Y)) = b(P(X+Y)Q) = b(X+Y) \\ &= b(X)+b(Y) = b(PXQ)+b(PYQ) = b(T(X))+b(T(Y)). \end{aligned}$$

Hence the operator T preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$. ∎

Lemma 3.3. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. Then there is a power of T which is idempotent.*

Proof. Since $\mathbb{B}_k$ is finite, there are only finitely many linear operators from $\mathbb{M}_{m,n}(\mathbb{B}_k)$ into $\mathbb{M}_{m,n}(\mathbb{B}_k)$. Thus the sequence $\{T, T^2, T^3, \ldots, T^m, \ldots\}$ is finite for sufficiently large n. That is, there exist integers $N \geq 1$ and $d \geq 1$ such that for $m,n \geq N$ with $m \equiv n \pmod d$, $T^m = T^n$. Let $p = Nd$. Then $2p \equiv p (\mathrm{mod}\ d)$. Hence $(T^p)^2 = T^{2p} = T^p$. That is, T^p is idempotent. ∎

Theorem 3.2. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator. Then T strongly preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$ if and only if T is a (P,Q)-operator.*

Proof. By Lemma 3.3, there is a power of T which is idempotent. Say $L = T^p$ with $L^2 = L$. If $X \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ and $(X,X) \in \mathscr{R}_{SA}(\mathbb{B}_k)$, then b$(X)$ = b$(X+X)$ = b(X) + b(X). Thus b(X) = 0, $X = O_{m,n}$. Similarly, if $(T(X),T(X)) \in \mathscr{R}_{SA}(\mathbb{B}_k)$, then $T(X) = O_{m,n}$. Thus $(X,X) \in \mathscr{R}_{SA}(\mathbb{B}_k)$ if and only if $(L(X),L(X)) \in \mathscr{R}_{SA}(\mathbb{B}_k)$ since T strongly preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$. So, b$(X)$ = 0 if and only if b$(L(X))$ = 0. That is, $X = O_{m,n}$ if and only if $L(X) = O_{m,n}$. Hence, if $A \neq O$, then we have $L(A) \neq O$ since T strongly preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$. We examine the action of L on rows and columns. Assume that $L(R_i)$ is not dominated by R_i. Then there is some (r,s) such that $E_{r,s} \leq L(R_i)$ while $E_{r,s} \not\leq R_i$. Then it is easy to see that

$$(R_i, aE_{r,s}) \in \mathscr{R}_{SA}(\mathbb{B}_k). \tag{3.1}$$

Since $E_{r,s} \leq L(R_i)$, we can find a matrix $X = (x_{i,j}) \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ with $x_{r,s} = 0$ such that $L(R_i) = aE_{r,s} + X$ for nonzero a in $\mathbb{B}_k$. We have $L(R_i + aE_{r,s}) = L(R_i) + L(aE_{r,s}) = L^2(R_i) + L(aE_{r,s}) = L(aE_{r,s} + X) + L(aE_{r,s}) = L(X) + L(aE_{r,s}) + L(aE_{r,s}) = L(X) + L(aE_{r,s}) = L(X + aE_{r,s}) = L(L(R_i)) = L^2(R_i) = L(R_i)$. That is,

$$b(L(R_i) + L(aE_{r,s})) = b(L(R_i + aE_{r,s})) = b(L(R_i)). \tag{3.2}$$

But if b$(L(R_i))$ + b$(L(aE_{r,s}))$ = b$(L(R_i) + L(aE_{r,s}))$ = b$(L(R_i))$, then b$(L(aE_{r,s}))$ = 0. Then $L(aE_{r,s}) = 0$ and $aE_{r,s} = 0$; which is impossible. Thus $(L(R_i)), L(aE_{r,s}) \notin \mathscr{R}_{SA}(\mathbb{B}_k)$, contradiction from (3.1), since T and L strongly preserves $\mathscr{R}_{SA}(\mathbb{B}_k)$. Therefore we have established that $L(R_i) \leq R_i$ for all i. Similarly, $L(C_j) \leq C_j$ for all j. By considering that $E_{i,j}$

is dominated by both R_i and C_j, we have that $L(E_{i,j}) \le R_i$ and $L(E_{i,j}) \le C_j$, and hence $L(E_{i,j}) \le E_{i,j}$. Since $\mathbb{B}_k$ is antinegative, T also maps a cell to a weighted cell and $T(J)$ has all nonzero entries. So, T induces a permutation σ on the set $\{(i,j)|i=1,2,\ldots,m; j=1,2,\ldots,n\}$. That is, $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for some nonzero scalar $b_{i,j}$ in $\mathbb{B}_k$. By Lemma 3.1, T is a (P,Q,B)-operator.

It remains to show that $B=J$. On the contrary assume that $B \ne J$ say $b_{i,j} \ne 1$. Consider $Y = R_k + b_{i,j}R_i$ for some $k \ne i$. Then $b(Y)=1$ and $(Y, E_{i,j}) \in \mathscr{R}_{SA}(\mathbb{B}_k)$ while $(T(Y), T(E_{i,j})) \notin \mathscr{R}_{SA}(\mathbb{B}_k)$ since $b(T(Y)+T(E_{i,j})) = b(P(Y \circ B)Q + P(E_{i,j} \circ B)Q) = b(P(Y \circ B)Q) = b(T(Y)) \ne b(T(Y)) + b(T(E_{i,j}))$. This fact implies that T does not preserve $\mathscr{R}_{SA}(\mathbb{B}_k)$.

The converse is obvious. ∎

4. Linear preservers of $\mathscr{R}_{S1}(\mathbb{B}_k)$

Recall that

$$\mathscr{R}_{S1}(\mathbb{B}_k) = \{(X,Y) \in \mathbb{M}_{m,n}(\mathbb{B}_k)^2 |\ \mathrm{b}(X+Y)=1\}.$$

Theorem 4.1. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a surjective linear operator. Then T preserves $\mathscr{R}_{S1}(\mathbb{B}_k)$ if and only if T is a (P,Q)-operator.*

Proof. If T is a surjective linear operator, by Theorem 2.1, we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for all $1 \le i \le m$ and $1 \le j \le n$. It is easy to see that the weighted cells $\alpha E_{i,j}$ and $\beta E_{r,s}$ are in the same line if and only if $\mathrm{b}(\alpha E_{i,j} + \beta E_{r,s}) = 1$ if and only if $(\alpha E_{i,j}, \beta E_{r,s}) \in \mathscr{R}_{S1}(\mathbb{B}_k)$. If T preserves $\mathscr{R}_{S1}(\mathbb{B}_k)$, then $(T(\alpha E_{i,j}),\ T(\beta E_{r,s})) \in \mathscr{R}_{S1}(\mathbb{B}_k)$ for $(\alpha E_{i,j}, \beta E_{r,s}) \in \mathscr{R}_{S1}(\mathbb{B}_k)$. And hence $\mathrm{b}(T(\alpha E_{i,j}) + T(\beta E_{r,s})) = 1$ which implies $T(\alpha E_{i,j})$ and $T(\beta E_{r,s})$ are weighted cells in the same line. Thus lines are mapped to lines by T, and we have that T is a (P,Q,B)-operator by Lemma 2.4. Here we have $B=J$ from $T(E_{i,j}) = E_{\sigma(i,j)}$. Thus T be a (P,Q)-operator.

Conversely let T be a (P,Q)-operator and consider any $(X,Y) \in \mathscr{R}_{S1}(\mathbb{B}_k)$. Then $\mathrm{b}(X+Y)=1$. Thus $\mathrm{b}(T(X)+T(Y)) = \mathrm{b}(T(X+Y)) = \mathrm{b}(P(X+Y)Q) = \mathrm{b}(X+Y) = 1$. That is, $(T(X), T(Y)) \in \mathscr{R}_{S1}(\mathbb{B}_k)$. Hence T preserves $\mathscr{R}_{S1}(\mathbb{B}_k)$. ∎

Theorem 4.2. *Let $T : \mathbb{M}_{m,n}(\mathbb{B}_k) \to \mathbb{M}_{m,n}(\mathbb{B}_k)$ be a linear operator preserving $\mathscr{R}_{S1}(\mathbb{B}_k)$. Then the following conditions are equivalent:*

(a) *T is bijective;*
(b) *T is injective;*
(c) *T is surjective;*
(d) *T strongly preserves $\mathscr{R}_{S1}(\mathbb{B}_k)$;*
(e) *T is a (P,Q)-operator.*

Proof. (a), (b) and (c) are equivalent by Theorem 2.1.

(c)⇒(e) If T is a surjective linear operator preserving $\mathscr{R}_{S1}(\mathbb{B}_k)$, then T is a (P,Q)-operator by Theorem 4.1.

(e)⇒(d) Assume that T is a (P,Q)-operator. Then $(X,Y) \in \mathscr{R}_{S1}(\mathbb{B}_k)$ if and only if $\mathrm{b}(X+Y)=1$ if and only if $\mathrm{b}(P(X+Y)Q)=1$ if and only if $\mathrm{b}(T(X+Y))=1$ if and only if $\mathrm{b}(T(X)+T(Y))=1$ if and only if $(T(X),T(Y)) \in \mathscr{R}_{S1}(\mathbb{B}_k)$. That is T strongly preserves $\mathscr{R}_{S1}(\mathbb{B}_k)$.

(d)⇒(c) Suppose T strongly preserves $\mathscr{R}_{S1}(\mathbb{B}_k)$. We claim that T is surjective. Assume that T is not surjective. Then T is not injective by (b) and hence T is not injective on the set of all mn cells in $\mathbb{M}_{m,n}(\mathbb{B}_k)$. Therefore there exists two distinct cells $E_{i,j}, E_{h,l} \in \mathbb{M}_{m,n}(\mathbb{B}_k)$ such that $T(E_{i,j}) = T(E_{h,l}) = E_{r,s}$. Then we have 3 cases as follows:

Case 1. Two cells in distinct lines are mapped to a cell. That is $T(E_{i,j}) = E_{r,s} = T(E_{h,l})$ with $i \neq h,\ j \neq l$. Let $X = E_{i,j}$, $Y = E_{h,l}$. Then b$(X+Y)$ = 2, but b$(T(X)+T(Y))$ = b$(E_{r,s})$ = 1; contradicts the fact that T strongly preserves $\mathscr{R}_{S1}(\mathbb{B}_k)$.

Case 2. Two cells in a row are mapped to a cell. That is $T(E_{i,j}) = E_{r,s} = T(E_{i,l})$ with $j \neq l$. Since T strongly preserves $\mathscr{R}_{S1}(\mathbb{B}_k)$, i^{th} row are mapped to r^{th} row (or s^{th} column) and j^{th} column are mapped to s^{th} column (or r^{th} row) under T. Say $T(E_{u,j}) = E_{v,s}$ with $i \neq u$. Let $X = E_{i,j} + E_{i,l}$ and $Y = E_{u,j}$. Then b$(X+Y)$ = 2, but b$(T(X)+T(Y))$ = b$(E_{r,s}+E_{v,s})$ = 1; contradicts the fact that T strongly preserves $\mathscr{R}_{S1}(\mathbb{B}_k)$.

Case 3. Two cells in a column are mapped to a cell. We have a similar contradiction as Case 2.

Therefore these 3 cases implies that T is injective and hence T is surjective by the equivalence of (a)$\sim$(c). ∎

As a concluding remark, we have characterized the linear operators that preserve the extreme sets of matrix ordered pairs over nonbinary Boolean algebra which come from certain Boolean rank inequalities.

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