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Numerical Solution of Nonlinear Volterra Integral Equations with Nonincreasing Kernel and an Application

K. MALEKNEJAD AND E. NAJAFI
School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16846 13114, Iran
Maleknejad@iust.ac.ir, E_najafi@iust.ac.ir

Abstract. Employing the quasilinearization technique to solve the nonlinear Volterra integral equations when the kernel of equation is nonincreasing with respect to the unknown function, yields two coupled sequences of linear Volterra integral equations where the solutions of these two sequences converge monotonically to the solution of nonlinear equation. We use collocation method and solve these coupled linear equations numerically, and obtain two sequences of successive approximations convergent to the solution of nonlinear equation. Error analysis is performed and an application to a boundary-layer theory problem and examples illustrating the results are presented.

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1. Introduction

The idea of the method of quasilinearization as developed by Bellman and Kalaba [2,3] is to provide pointwise lower and upper estimates for the solution of nonlinear problems whenever the nonlinear function involved in convex or concave. The lower and upper estimates are the solutions of the corresponding linear problems that converge quadratically to the solution of the given nonlinear problem. This fruitful idea goes back to Chaplygin [5,12]. It is well known that the method of quasilinearization is an effective tool to obtain lower or upper bounds for the solutions of nonlinear differential equations. In order to describe this method, consider the initial value problem (IVP)

$$x' = f(x),\ x(0) = x_0, \tag{1.1}$$

on $J = [0,T]$. If f is convex, we can find a function $g(x,y)$ which is linear in x such that

$$f(x) = \max_y g(x,y).$$

Employing $g(x,y)$, provided a suitable initial approximation, we can construct a monotone sequence that converges quadratically to the unique solution of (1.1). Moreover, the sequence provides good lower bounds for the solution. If f is concave, on the other hand, a

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dual result holds that offers monotone approximations having similar properties and good upper bounds. This method has been generalized later by Lakshmikantham [13–15] and applied to a variety of problems [1, 8, 10, 18] such as high order nonlinear differential equations, reaction diffusion equations, the Falkner-Skan equation and so on. In the area of integral equations consider the following nonlinear equation:

$$u(t) = h(t) + \int_0^t k(t,s,u(s))ds. \tag{1.2}$$

Equation (1.2) has been the subject of several works such as [9, 16, 17]. If we apply numerical methods to discretize this equation directly, arisen algebraic system is nonlinear. Equation (1.2) has been studied in [7] using multistep rules with quadrature formulas and in [4] using collocation method and in both of them the integral terms are discretized to nonlinear algebraic systems. But these nonlinear systems need some conditions to have a unique solution and require an iteration method (In many cases the Newton's iteration) and a suitable starting point to converge to the solution. When we use the step-by-step methods, the process of solving these nonlinear algebraic systems is repeated in each step and this process causes a lot of computational costs and additional works.

The method of quasilinearization is generalized in [19] to solve (1.2). When k is nonincreasing in u and satisfies a Lipschitz condition, this technique offers coupled monotonic sequences of linear iterations

$$v_p(t) = h(t) + \int_0^t \Big(k(t,s,w_{p-1}(s)) + k_u(t,s,w_{p-1}(s))\big(w_p(s) - w_{p-1}(s)\big)\Big)ds,$$
$$w_p(t) = h(t) + \int_0^t \Big(k(t,s,v_{p-1}(s)) + k_u(t,s,w_{p-1}(s))\big(v_p(s) - v_{p-1}(s)\big)\Big)ds,$$

uniformly and quadratically convergent to the unique solution of (1.2) where $v_0(t)$ and $w_0(t)$ are coupled lower and upper solutions of (1.2) defined in following. The purpose of this paper is to employ numerical methods to solve these two coupled linear integral equations in a piecewise continuous polynomials space and to combine it with the iterative schemes (where with respect to their linearity and quadratically convergent is more rapid in convergence) to approximate the unique solution of the nonlinear integral Equation (1.2), under some conditions on k as mentioned above. This method avoid to face with nonlinear algebraic systems and discussion on their existence and uniqueness and also finding their solution that arise when we discretize nonlinear problems. As an application of this work, is solving the boundary-layer theory problem [20, 21]. When this problem is converted into a Volterra integral equation, has a nonincreasing and convex kernel. In the section of Numerical Examples this problem is considered. The structure of this paper is as follows. Section 2 contains a general framework for the idea of quasilinearization used to solve the nonlinear integral equations. Section 3 shows using of collocation method in a piecewise polynomials space to approximate the solution of the coupled linear integral equations. In Section 4, discretization to a linear algebraic system and the convergence of the method are discussed. Some numerical examples are given in Section 5.

2. Volterra integral inequalities and generalized quasilinearization

For $T \in \mathbb{R}$ and $T > 0$ let $J = [0,T]$ and $D = \{(t,s) \in J \times J : s \leq t\}$. Consider the nonlinear integral equation

$$u(t) = h(t) + \int_0^t k(t,s,u(s))ds, \tag{2.1}$$

where $h \in C[J,\mathbb{R}]$ and $k \in C[D \times \mathbb{R}, \mathbb{R}]$.

Definition 2.1. A function $v \in C[J,\mathbb{R}]$ is called a lower solution of (2.1) on J if

$$v(t) \leq h(t) + \int_0^t k(t,s,v(s))ds,\ t \in J,$$

and an upper solution, if the reversed inequality holds. If

$$w(t) \geq h(t) + \int_0^t k(t,s,v(s))ds,\ v(t) \leq h(t) + \int_0^t k(t,s,w(s))ds,\ t \in J,$$

then v and w are said to be coupled lower and upper solutions of (2.1) on J.

There are interesting consequences about coupled lower and upper solutions of (2.1) where the following two theorems in [19] state them:

Theorem 2.1. *Assume that $h \in C[J,\mathbb{R}]$, $k \in C[D \times \mathbb{R}, \mathbb{R}]$ and $v_0(t), w_0(t) \in C[J,\mathbb{R}]$ are coupled lower and upper solutions of (2.1) and $k(t,s,u)$ is nonincreasing in u for each fixed pair $(t,s) \in D$ and satisfies one-sided Lipschitz condition*

$$k(t,s,\alpha) - k(t,s,\beta) \geq -L(\alpha - \beta),\ \alpha \leq \beta,\ L > 0.$$

Then $v_0(0) \leq w_0(0)$ implies

$$v_0(t) \leq w_0(t),\ t \in J.$$

Let

$$\Omega = \{(t,s,u) \in D \times \mathbb{R}\ ;\ v_0(t) \leq u \leq w_0(t),\ t \in J\},$$

then if Theorem 2.1 holds and $k \in C[\Omega,\mathbb{R}]$, it is shown in [19] that there exists a unique solution $u(t)$ of (2.1) such that

$$v_0(t) \leq u(t) \leq w_0(t),\ t \in J.$$

By letting $\|u\| = max_{t\in J}|u(t)|$ and defining two iterative schemes as coupled system of linear integral equations

$$v_p(t) = h(t) + \int_0^t \Big(k(t,s,w_{p-1}(s)) + k_u(t,s,w_{p-1}(s))\big(w_p(s) - w_{p-1}(s)\big)\Big)ds, \tag{2.2}$$

$$w_p(t) = h(t) + \int_0^t \Big(k(t,s,v_{p-1}(s)) + k_u(t,s,w_{p-1}(s))\big(v_p(s) - v_{p-1}(s)\big)\Big)ds, \tag{2.3}$$

for $p = 1,2,\cdots$, and $v_0(t), w_0(t) \in C[J,\mathbb{R}]$, coupled lower and upper solutions of (2.1), the following theorem shows the quadratically convergence of two coupled sequences $\{v_p(t)\}$ and $\{w_p(t)\}$ derived from (2.2) and (2.3) to the unique solution of (2.1). It is needed to use vectorial inequalities which are understood to mean the same inequalities hold between their corresponding components.

Theorem 2.2. *Suppose that*
(H_1) $v_0, w_0 \in C[J,\mathbb{R}]$, $v_0(t) \le w_0(t)$ *on J, are coupled lower and upper solutions of (2.1) on J respectively.*
(H_2) $k \in C^2[\Omega,\mathbb{R}]$, $k_u(t,s,u) \le 0$ *and the sign of* $k_{uu}(t,s,u)$ *does not change on J for* $(t,s,u) \in \Omega$.
Then the two coupled iterative schemes (2.2) and (2.3) define a nondecreasing sequence $\{v_p(t)\}$ *and a nonincreasing sequence* $\{w_p(t)\}$ *in* $C[J,\mathbb{R}]$ *such that* $v_p \longrightarrow u$ *and* $w_p \longrightarrow u$ *uniformly on J, and the following quadratic convergent estimate holds:*

$$\|r_p(t)\| \le \Big(TQ + \frac{T^2}{2} PQ \exp(PT)\Big) \|r_{p-1}(t)\|^2,$$

where

$$\|r_p(t)\|^i = \left[\begin{array}{c} \|u(t) - v_p(t)\|^i \\ \|w_p(t) - u(t)\|^i \end{array} \right], \; i = 1,2,$$

$$Q = \left[\begin{array}{cc} 0 & M_1 \\ 2M_1 & M_1 \end{array} \right], \; P = \left[\begin{array}{cc} 0 & M_2 \\ M_2 & 0 \end{array} \right],$$

and $M_1 = \max_\Omega k_{uu}$, $M_2 = \max_\Omega k_u$. *Also these two sequences satisfy the relation*

$$v_0 \le v_1 \le \cdots \le v_p \le w_p \le \cdots \le w_1 \le w_0.$$

The following lemma distinguished as Newmann Lemma in [6] is needed later. In the whole of the work we refer to $\|.\|$ as the maximum norm of the functions or matrices.

Lemma 2.1. *Suppose E is a matrix such that* $\|E\| < 1$. *Then* $(I - E)$ *is nonsingular and*

$$\|(I-E)^{-1}\| \le (1 - \|E\|)^{-1}.$$

3. Collocation method in the piecewise polynomials space

Suppose $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ is a partition on J and let $h_n = (t_{n+1} - t_n)$, $n = 0, \cdots, N-1$, and $h = \max_n h_n$, then the above partition is denoted by J_h.

Definition 3.1. Suppose that J_h is a given partition on J. The piecewise polynomials space $\mathbb{S}_\mu^{(d)}(J_h)$ with $\mu \ge 0$, $-1 \le d \le \mu$ is defined by

$$\mathbb{S}_\mu^{(d)}(J_h) = \{q(t) \in C^d[J,\mathbb{R}^2] : q|_{\sigma_n} \in \pi_\mu \; \times \pi_\mu \; ; 0 \le n \le N-1\}.$$

In this definition $\sigma_n = (t_n, t_{n+1}]$, and π_μ denotes the space of polynomials of degree not exceeding μ and it is easy to see that $\mathbb{S}_\mu^{(d)}(J_h)$ is a real vector space and its dimension is given by

$$dim(\mathbb{S}_\mu^{(d)}(J_h)) = 2(N(\mu - d) + d + 1).$$

Now, consider the coupled system of linear integral Equations (2.2) and (2.3). They may be shown in the form of

$$v_p(t) = H_p(t) + \int_0^t k_p(t,s) w_p(s) ds, \tag{3.1}$$

$$w_p(t) = G_p(t) + \int_0^t k_p(t,s) v_p(s) ds, \tag{3.2}$$

where

$$H_p(t) = h(t) + \int_0^t \Big(k(t,s,w_{p-1}(s)) - k_u(t,s,w_{p-1}(s))w_{p-1}(s)\Big)ds, \tag{3.3}$$

$$G_p(t) = h(t) + \int_0^t \Big(k(t,s,v_{p-1}(s)) - k_u(t,s,w_{p-1}(s))v_{p-1}(s)\Big)ds, \tag{3.4}$$

$$k_p(t,s) = k_u(t,s,w_{p-1}(s)),\ (t,s) \in D. \tag{3.5}$$

By letting

$$x_p(t) = \begin{bmatrix} v_p(t) \\ w_p(t) \end{bmatrix},\ y_p(t) = \begin{bmatrix} H_p(t) \\ G_p(t) \end{bmatrix},\ f_p(t,s) = \begin{bmatrix} 0 & k_p(t,s) \\ k_p(t,s) & 0 \end{bmatrix},$$

the coupled equations (3.1) and (3.2) may be written in the following form of the system of equations:

$$x_p(t) = y_p(t) + \int_0^t f_p(t,s)x_p(s)ds. \tag{3.6}$$

The solution of coupled system (3.6) will be approximated by collocation method in the piecewise continuous polynomials space

$$\mathbb{S}_{m-1}^{(-1)}(J_h) = \{q(t) \in C^{-1}[J,\mathbb{R}^2] : q|_{\sigma_n} \in \pi_{m-1} \times \pi_{m-1}; 0 \le n \le N-1\},$$

corresponding to the choice $\mu = m-1$ and $d = -1$ in Definition 3.1 which the polynomials are allowed to have finite jumps in the partition points. The collocation solution is denoted by

$$\hat{x}_p(t) = \begin{bmatrix} \hat{v}_p(t) \\ \hat{w}_p(t) \end{bmatrix} \in \mathbb{S}_{m-1}^{(-1)}(J_h),$$

and defined by the system of collocation equations

$$\hat{x}_p(t) = y_p(t) + \int_0^t f_p(t,s)\hat{x}_p(s)ds,\ t \in X_h, \tag{3.7}$$

where X_h contains the collocation points

$$X_h = \{t_n + c_i h_n :\ 0 \le c_1 \le \cdots \le c_m \le 1;\ 0 \le n \le N-1\}, \tag{3.8}$$

is determined by the points of the partition J_h and the given collocation parameters $\{c_i\} \in [0,1]$.

4. Discretization and error analysis

In this section the Lagrange polynomials are used to discretize the system (3.6) to an algebraic system of equations. When this polynomials are chosen as basis functions in each subinterval σ_n for the space $\mathbb{S}_{m-1}^{(-1)}(J_h)$ with respect to the collocation parameters $\{c_i\}$, a comfort computational form of the collocation equations system (3.7) is obtained.
Lagrange polynomials in σ_n can be written as

$$L_j(z) = \prod_{k \neq j}^{m} \frac{z - c_k}{c_j - c_k},\ z \in [0,1],\ j = 1, \cdots, m, \tag{4.1}$$

where belong to π_{m-1}. Also set

$$X_{n,j}^p = \hat{x}_p(t_n + c_j h_n) = \begin{bmatrix} \hat{v}_p(t_n + c_j h_n) \\ \hat{w}_p(t_n + c_j h_n) \end{bmatrix} = \begin{bmatrix} V_{n,j}^p \\ W_{n,j}^p \end{bmatrix},\ j = 1, \cdots, m. \tag{4.2}$$

The collocation solution $\hat{x}_p(t) \in \mathbb{S}_{m-1}^{(-1)}(J_h)$ when is restricted to the subinterval σ_n is as

$$\hat{x}_p(t) = \hat{x}_p(t_n + zh_n) = \sum_{j=1}^{m} L_j(z) X_{n,j}^p = \begin{bmatrix} \sum_{j=1}^{m} L_j(z) V_{n,j}^p \\ \\ \sum_{j=1}^{m} L_j(z) W_{n,j}^p \end{bmatrix}, \; z \in (0,1]. \tag{4.3}$$

Now, by letting $t = t_{n,i} = t_n + c_i h_n$, the collocation equations system (3.7) has the following form:

$$\hat{x}_p(t_{n,i}) = y_p(t_{n,i}) + \int_0^{t_n} f_p(t_{n,i}, s)\hat{x}_p(s)ds + h_n \int_0^{c_i} f_p(t_{n,i}, t_n + sh_n)\hat{x}_p(t_n + sh_n)ds,$$

and by employing (4.2) and (4.3), it is written as follows:

$$X_{n,i}^p = y_p(t_{n,i}) + F_p^n(t_{n,i}) + h_n \sum_{j=1}^{m} \Big(\int_0^{c_i} f_p(t_{n,i}, t_n + sh_n) L_j(s) ds \Big) X_{n,j}^p, \tag{4.4}$$

for $i = 1, \cdots, m$, where

$$F_p^n(t) = \int_0^{t_n} f_p(t,s)\hat{x}_p(s)ds = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 f_p(t, t_\ell + sh_\ell)\hat{x}_p(t_\ell + sh_\ell)ds. \tag{4.5}$$

$F_p^n(t)$ is known with respect to the collocation solution on $[0, t_n]$ and by applying representation (4.3) and setting $t = t_{n,i}$ in (4.5), it has the form:

$$F_p^n(t_{n,i}) = \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} \Big(\int_0^1 f_p(t_{n,i}, t_\ell + sh_\ell) L_j(s) ds \Big) X_{\ell,j}^p,$$

and by defining

$$B_p^{\ell,n} = \begin{bmatrix} \int_0^1 f_p(t_{n,i}, t_\ell + sh_\ell) L_j(s) ds \\ i,j = 1, \cdots, m \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \int_0^1 k_p(t_{n,i}, t_\ell + sh_\ell) L_j(s) ds \\ \\ \int_0^1 k_p(t_{n,i}, t_\ell + sh_\ell) L_j(s) ds & 0 \\ \\ & i,j = 1, \cdots, m \end{bmatrix},$$

for $0 \le \ell < n \le N-1$, as a $(2m) \times (2m)$ matrix, equation (4.5) is reduced to

$$F_p^n(t_{n,i}) = \sum_{\ell=0}^{n-1} h_\ell \sum_{j=1}^{m} \left[B_p^{\ell,n} \right]_{ij} X_{\ell,j}^p, \; i = 1, \cdots, m,$$

where $\left[B_p^{\ell,n} \right]_{ij}$ shows the (i,j)'th (2×2) block of the matrix $B_p^{\ell,n}$. Now, by letting

(4.6)

$$X_p^n = \Big([X_{n,1}^p]^T, \cdots, [X_{n,m}^p]^T \Big)^T = \Big(V_{n,1}^p, W_{n,1}^p, V_{n,2}^p, W_{n,2}^p, \cdots, V_{n,m}^p, W_{n,m}^p \Big)^T,$$

$$y_p = \Big([y_p(t_{n,1})]^T, \cdots, [y_p(t_{n,m})]^T \Big)^T = \Big(H_p(t_{n,1}), G_p(t_{n,1}), \cdots, H_p(t_{n,m}), G_p(t_{n,m}) \Big)^T,$$

$$\mathbb{F}_p^n = \Big([F_p^n(t_{n,1})]^T, \cdots, [F_p^n(t_{n,m})]^T\Big)^T = \sum_{\ell=0}^{n-1} h_\ell B_p^{\ell,n} X_p^\ell,$$

and defining the $(2m) \times (2m)$ matrix

$$B_p^n = \begin{bmatrix} \int_0^{c_i} f_p(t_{n,i}, t_n + sh_n) L_j(s) ds \\ i, j = 1, \cdots, m \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \int_0^{c_i} k_p(t_{n,i}, t_n + sh_n) L_j(s) ds \\ \int_0^{c_i} k_p(t_{n,i}, t_n + sh_n) L_j(s) ds & 0 \\ & i, j = 1, \cdots, m \end{bmatrix},$$

for $0 \le n \le N-1$, the collocation equation (3.7) is transformed to the algebraic system of linear equations

$$(I_{2m} - h_n B_p^n) X_p^n = y_p + \mathbb{F}_p^n,\ 0 \le n \le N-1,\ p = 1, 2, \cdots. \tag{4.7}$$

Here I_{2m} denotes the $(2m) \times (2m)$ identity matrix. The existence and uniqueness of the solution for the system (4.7) or the collocation solution for (3.7) in $\mathbb{S}_{m-1}^{(-1)}(J_h)$ is considered in the following theorem.

Theorem 4.1. *If $y_p(t)$ and $f_p(t,s)$ in the Volterra integral equation (3.6) are continuous on their domains J and D respectively, then there exists an $\bar{h} > 0$ such that for any partition J_h with partition diameter h, $0 < h < \bar{h}$, the linear algebraic system (4.7) has a unique solution X_p^n for $0 \le n \le N-1$ and $p = 1, 2, \cdots$.*

Proof. The continuity of $y_p(t)$ and $f_p(t,s)$ is obvious. Since the components of $f_p(t,s)$ are continuous and its domain is compact, then the components of the matrix B_p^n for $0 \le n \le N-1$ and $p = 1, 2, \cdots$, are all bounded. These implies if h_n's are chosen sufficiently small, the inequality $h_n \|B_p^n\| < 1$ holds and by Lemma 2.1 the inverse of the matrix $(I_m - h_n B_p^n)$ exists. In other words, there is an $\bar{h} > 0$ so that for any partition J_h with $h = \max\{h_n;\ 0 \le n \le N-1\} < \bar{h}$ the matrix $(I_m - h_n B_p^n)$ has a uniformly bounded inverse. Then the collocation equation (3.7) has a unique solution.
By letting $d = \frac{1}{m-1}$ and $c_i = (i-1)d$ for $i = 1, \cdots, m$, one can obtain a bound for $\|B_p^n\|$ and an estimation for $\bar{h}$ as follows:
Consider the components of the matrix B_p^n. Since

$$k_p(t,s) = k_u(t, s, w_{p-1}(s)) \le 0,\ (t,s) \in D,$$

then the kernel $k_p(t,s)$ has constant sign on J and the mean-value theorem of integral calculus gives

$$\int_0^{c_i} k_p(t_{n,i}, t_n + sh_n) L_j(s) ds = L_j(\xi) \int_0^{c_i} k_p(t_{n,i}, t_n + sh_n) ds, \tag{4.8}$$

for some $\xi \in [0, c_i]$. By attention to the continuity of $k_p(t,s)$ on D, the phrase $|\int_0^{c_i} k_p(t_{n,i}, t_n + sh_n) ds|$ has a finite bound as $K = \max |k_u(t,s,u)|$ on Ω. To obtain a bound for Lagrange polynomials, let $c_i \le s \le c_{i+1}$, then $(i-1) \le \frac{s}{d} \le i$ for $i = 1, \cdots, m-1$. This implies

$$\begin{aligned}|L_j(s)| = |\prod_{k\neq j}^{m} \frac{s-c_k}{c_j-c_k}| &= |\prod_{k\neq j}^{m} \frac{\frac{s}{d}-(k-1)}{j-k}| \\ &\le \frac{i(i-1)\cdots 2\cdot 1\cdot 1\cdot 2\cdots(m-i)}{c(j-1)(j-2)\cdots 2\cdot 1\cdot 1\cdot 2\cdots(m-j)} \\ &\le \max_i \frac{i!(m-i)!}{c(j-1)!(m-j)!} \\ &= \max_i \frac{j\begin{pmatrix} m \\ j \end{pmatrix}}{c\begin{pmatrix} m \\ i \end{pmatrix}} = Q_{jm},\ j = 1, \cdots, m,\end{aligned}$$

where

$$c = \begin{cases} i-j+1, & j \le i, \\ j-i, & j \ge i+1. \end{cases}$$

Now, relation (4.8) concludes

$$|\int_0^{c_i} k_p(t_{n,i}, t_n + sh_n) L_j(s) ds| \le KQ_{jm},\ j = 1, \cdots, m.$$

When $m = 2, 3$, the maximum values of Q_{jm}, are $Q_{2m} = 1, 2$, respectively, that are occurred for $i = 1$. For $m \ge 4$, the maximum values are occurred when $i = m-1$ for $j = 2, \cdots, m-1$, and the bounds for Lagrange polynomials are as following:

$$|L_j(s)| \le Q_{jm} = \frac{1}{m-j}\begin{pmatrix} m-1 \\ j-1 \end{pmatrix},\ j = 2, \cdots, m-1,$$

and for $j = 1, m$,

$$|L_1(s)| \le Q_{1m} = \max_i \frac{1\begin{pmatrix} m \\ 1 \end{pmatrix}}{i\begin{pmatrix} m \\ i \end{pmatrix}} = 1,$$

$$|L_m(s)| \le Q_{mm} = \max_i \frac{m\begin{pmatrix} m \\ m \end{pmatrix}}{(m-i)\begin{pmatrix} m \\ i \end{pmatrix}} = 1.$$

On the other hand, for estimation a bound for $\|B_p^n\|$ the maximum value of Q_{jm} for $j = 1, \cdots, m$ is needed and it is occurred when $j = m - [\frac{m}{2}] + 1$, for $m \ge 4$. Then by setting

$$Q_m = \begin{cases} 1, & m = 2, \\ 2, & m = 3, \\ \frac{1}{[\frac{m}{2}]-1}\begin{pmatrix} m-1 \\ m-[\frac{m}{2}] \end{pmatrix}, & m \ge 4, \end{cases}$$

the bound for $\|B_p^n\|$ is as

$$\|B_p^n\| \le KQ_m,$$

and estimation $\bar{h} = \frac{1}{KQ_m}$ is obtained.
When the unknown vector X_p^n is computed from (4.7), the collocation solution for $t = t_n + zh_n \in \bar{\sigma}_n = [t_n, t_{n+1}]$ is given by

$$\hat{x}_p(t) = y_p(t) + F_p^n(t) + h_n \sum_{j=1}^{m} \Big(\int_0^{c_i} f_p(t, t_n + sh_n) L_j(s) ds \Big) X_{n,j}^p.$$

The following theorem shows that the solution of system of collocation equations (3.7) is convergent to the solution of system of linear integral equations (3.6). The proof of Theorem when the approximated function is 1-dimensional, is showed in [4]. But its proof with some changes and definitions like (4.6) may be extended for 2-dimensional vector functions, thereby for the following theorem.

Theorem 4.2. *Suppose that in (3.6), $f_p \in C^i[D, \mathbb{R}^2]$ and $y_p \in C^i[J, \mathbb{R}^2]$, where $1 \le i \le m$, and $\hat{x}_p \in \mathbb{S}_{m-1}^{(-1)}(J_h)$ is the collocation solution of (3.6) with $h \in (0, \bar{h})$, defined by (3.7). If $x_p(t)$ is the exact solution of (3.6), then*

$$\|x_p - \hat{x}_p\| = \begin{bmatrix} \|v_p - \hat{v}_p\| \\ \|w_p - \hat{w}_p\| \end{bmatrix} \le C \|x_p^{(i)}\| h^i = C \begin{bmatrix} \|v_p^{(i)}\| \\ \|w_p^{(i)}\| \end{bmatrix} h^i,$$

holds on J, for any collocation points X_h. The constant C depends on the parameters $\{c_i\}$ but not on h.

The above argument provides an approximation solution $\hat{x}_p(t)$ to the unique solution of the system of linear integral equations (3.6) in the space $\mathbb{S}_{m-1}^{(-1)}(J_h)$ and the iterative schemes (2.2) and (2.3) produce two sequences $\{v_p(t)\}$ and $\{w_p(t)\}$ that are quadratically convergent to the unique solution of nonlinear integral equation (2.1). The inequality

$$\begin{bmatrix} \|u - \hat{v}_p\| \\ \|\hat{w}_p - u\| \end{bmatrix} \le \begin{bmatrix} \|u - v_p\| \\ \|w_p - u\| \end{bmatrix} + \begin{bmatrix} \|v_p - \hat{v}_p\| \\ \|w_p - \hat{w}_p\| \end{bmatrix}, \tag{4.9}$$

and Theorems 2.2 and 4.2 show that the sequences of the collocation solutions $\{\hat{v}_p(t)\}$ and $\{\hat{w}_p(t)\}$ is convergent to the unique solution $u(t)$ of nonlinear integral equation (2.1). It is easy to see that the first term in the right hand side of (4.9) is quadratically convergent and the second term has a convergent of order $O(h^i), 1 \le i \le m$.

5. Numerical examples

For numerical experiments, the presented method is applied to solve three different examples of the integral equation (2.1). The first two examples have known solutions but the solution of the third example is unknown. In each three examples the subintervals and the collocation parameters are chosen such that

$$h = h_n = \frac{T}{N},\ n = 0, \ldots, N-1,$$
$$c_i = \frac{i-1}{m-1},\ i = 1, \ldots, m.$$

Example 5.1. The first example is the following integral equation:

$$u(t) = (-\frac{1}{2} + t^2 + \frac{1}{2} e^{-t^3}) + \int_0^t ts\, e^{-tu(s)} ds, \tag{5.1}$$

where $0 \le t \le 1$ and $k(t,s,u) = ts\ e^{-tu}$. Then, k is nonincreasing and convex in u on $\Omega = [0,1]\times[0,1]\times[0,2]$ and $v_0(t) = 0$ and $w_0(t) = 2$ are coupled lower and upper solutions of (5.1) on $[0,1]$. On the other hand, the exact solution is $u(t) = t^2$ where with respect to the conclusion of Theorem 2.1 satisfies the relation $v_0(t) = 0 \le u(t) = t^2 \le w_0(t) = 2$ on $J = [0,1]$ and according to the Theorem 2.2 the solutions of the coupled iterative schemes

$$v_p(t) = (-\frac{1}{2} + t^2 + \frac{1}{2}e^{-t^3}) + \int_0^t \Big(ts\ e^{-tw_{p-1}(s)} + t^2 s\ e^{-tw_{p-1}(s)} w_{p-1}(s)\Big)ds \\ - \int_0^t t^2 s\ e^{-tw_{p-1}(s)} w_p(s)ds,$$

$$w_p(t) = (-\frac{1}{2} + t^2 + \frac{1}{2}e^{-t^3}) + \int_0^t \Big(ts\ e^{-tv_{p-1}(s)} + t^2 s\ e^{-tw_{p-1}(s)} v_{p-1}(s)\Big)ds \\ - \int_0^t t^2 s\ e^{-tw_{p-1}(s)} v_p(s)ds,$$

are convergent to the exact solution of (5.1). To employ the given numerical procedure in approximating the solutions of these two coupled linear integral equations, two values $m = 3,5$ are chosen. These two values for m give the values $\bar{h} = \frac{1}{2}, \frac{1}{4}$ regards to $K = \max_\Omega |k(t,s,u)| = 1$. Then the values $h = \frac{1}{4}, \frac{1}{5}$ and consequently $N = 4,5$ can be used, respectively. The absolute values of errors for equation (5.1) are shown in Tables 1 and 2 and Figure 1 shows the convergence of the sequences $\{\hat{v}_p(t)\}$ and $\{\hat{w}_p(t)\}$ to the exact solution.

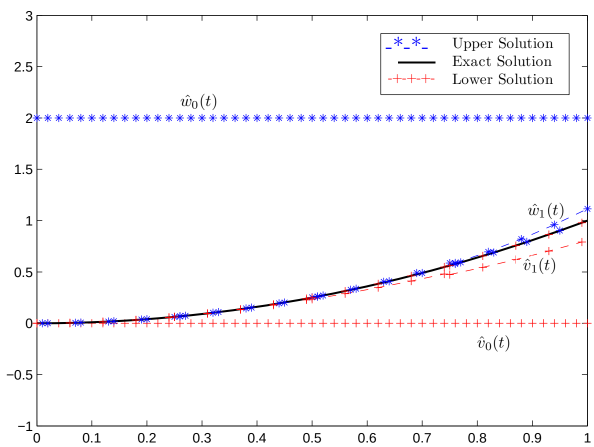

Figure 1. The convergence of the sequences $\{\hat{v}_p(t)\}$ and $\{\hat{w}_p(t)\}$ to the exact solution of Example 5.1.

Example 5.2. The second example,

$$u(t) = \frac{\sin(t)}{\sin(t)+\cos(t)} + \int_0^t \frac{2se^{s-t}\cos(u(s))ds}{\sin(t)+\cos(t)},$$

has the exact solution $u(t) = t$, the lower solution $v_0(t) = \frac{t}{2}$ and the upper solution $w_0(t) = 2t$. The kernel of this equation satisfies the necessary conditions and as in Example 5.1 the method is applied to this equation and the absolute values of the errors are presented in Tables 3 and 4 and Figure 2.

Table 1: Absolute errors in Example 5.1 for Lower Solutions: $|u(t_i) - \hat{v}_p(t_i)|$.

	$m=3$, $N=4$		$m=5$, $N=5$		
t_i	$p=2$	$p=3$	$p=2$	$p=3$	$p=4$
0.000	0	0	0	0	0
0.250	7.8670 E -10	1.1102 E -16	1.2257 E -11	2.7756 E -17	2.7756 E -17
0.375	5.2260 E -10	9.1593 E -16	2.4667 E -10	1.4337 E -12	8.3267 E -17
0.500	1.5301 E -06	2.0340 E -10	1.4198 E -07	2.7415 E -12	8.3267 E -17
0.750	2.0390 E -04	3.6184 E -07	2.4670 E -05	8.3782 E -09	2.7089 E -14
0.875	1.7093 E -04	5.1957 E -07	4.2982 E -04	9.2744 E -07	2.0145 E -10
1.000	6.1487 E -04	7.1934 E -07	7.9141 E -04	1.3315 E -06	2.8970 E -10

Table 2: Absolute errors in Example 5.1 for Upper Solutions: $|u(t_i) - \hat{w}_p(t_i)|$.

	$m=3$, $N=4$		$m=5$, $N=5$		
t_i	$p=2$	$p=3$	$p=2$	$p=3$	$p=4$
0.000	0	0	0	0	0
0.250	1.6526 E -10	4.4756 E -16	8.6362 E -12	2.7756 E -17	2.7756 E -17
0.375	5.1178 E -09	6.6613 E -16	2.4895 E -09	1.4337 E -12	8.3267 E -17
0.500	1.1585 E -06	2.0340 E -10	2.1528 E -07	2.7416 E -12	5.5511 E -17
0.750	1.3040 E -04	3.6182 E -07	4.9169 E -05	8.3620 E -09	2.7089 E -14
0.825	1.7093 E -04	5.1879 E -07	8.3059 E -05	9.2687 E -07	2.0145 E -10
1.000	1.5447 E -03	6.7661 E -07	1.4003 E -03	1.2910 E -06	2.8970 E -10

Table 3: Absolute errors in Example 5.2 for Lower Solutions: $|u(t_i) - \hat{v}_p(t_i)|$.

	$m=3$, $N=5$		$m=5$, $N=10$		
t_i	$p=2$	$p=3$	$p=2$	$p=3$	$p=4$
0.0	0	0	0	0	0
0.2	6.8433 E -07	1.7060 E -07	6.1123 E -07	6.4161 E -10	6.6761 E -10
0.3	5.0677 E -07	2.2447 E -07	5.2886 E -06	4.5856 E -09	8.7809 E -10
0.4	5.4805 E -05	2.3597 E -07	1.9110 E -05	9.1943 E -08	9.3511 E -10
0.6	4.1518 E -04	3.0574 E -06	6.4763 E -05	2.4610 E -06	2.0606 E -08
0.7	3.4584 E -04	2.5430 E -06	8.3015 E -05	7.5724 E -06	1.1895 E -07
0.8	1.3823 E -03	4.8038 E -05	1.2741 E -04	2.0492 E -05	4.7716 E -07
1.0	7.7669 E -04	3.2720 E -05	1.4905 E -04	4.2855 E -05	1.2962 E -06

Table 4: Absolute errors in Example 5.2 for Upper Solutions: $|u(t_i) - \hat{w}_p(t_i)|$.

	$m=3$, $N=5$		$m=5$, $N=10$		
t_i	$p=2$	$p=3$	$p=2$	$p=3$	$p=4$
0.0	0	0	0	0	0
0.2	6.8651 E -07	1.7060 E -07	6.1297 E -07	6.4161 E -10	6.6761 E -10
0.3	5.7861 E -07	2.2447 E -07	5.3635 E -06	4.5856 E -09	8.7809 E -10
0.4	5.5827 E -05	2.3597 E -07	2.0106 E -05	9.1943 E -08	9.3511 E -10
0.6	4.4783 E -04	3.0575 E -06	9.6635 E -05	2.4610 E -06	2.0606 E -08
0.7	4.5533 E -04	2.5431 E -06	1.9174 E -04	7.5726 E -06	1.1895 E -07
0.8	1.6888 E -03	4.8040 E -05	4.2211 E -04	2.0494 E -05	4.7716 E -07
1.0	2.0780 E -03	3.2770 E -05	1.3981 E -03	4.2901 E -05	1.2962 E -06

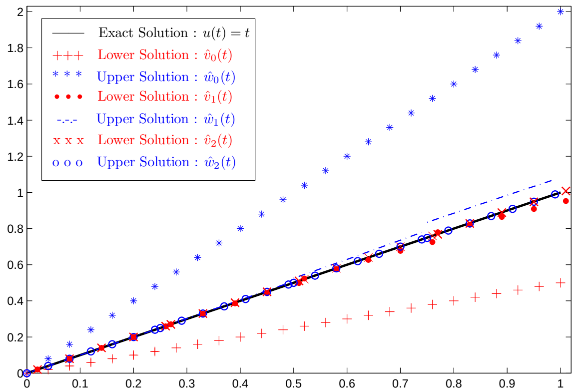

Figure 2. The convergence of the sequences $\{\hat{v}_p(t)\}$ and $\{\hat{w}_p(t)\}$ to the exact solution of Example 5.2.

Example 5.3. The third order ordinary differential equation

$$y''' + 2yy'' = 0,\ 0 \le t < \infty,\ y(0) = y'(0) = 0,\ y''(0) = 1, \tag{5.2}$$

arises in connection with the boundary-layer theory of fluid flow [20, 21]. If we use the transformation $u = -\ln y''$, we acquire

$$y'''(t) = -u'(t)e^{-u(t)},$$

$$y(t) = \int_0^t (t-s)e^{-u(s)}ds,$$

and letting in the equation (5.2) reduces it to the following Volterra integral equation

$$u(t) = \int_0^t (t-s)^2 e^{-u(s)}ds, \tag{5.3}$$

Table 5: Differences of Upper and Lower Solutions in Example 5.3: $|\hat{w}_p(t_i) - \hat{v}_p(t_i)|$.

	$m=3$, $N=4$		$m=5$, $N=5$		
t_i	$p=2$	$p=3$	$p=2$	$p=3$	$p=4$
0.000	0	0	0	0	0
0.125	3.4694 E -18	8.6736 E -19	2.1684 E -19	4.3368 E -19	6.5052 E -19
0.250	1.7347 E -18	3.4694 E -18	2.6021 E -18	1.7347 E -18	1.7348 E -18
0.375	1.3010 E -18	1.0842 E -18	1.0408 E -17	3.4694 E -18	1.7347 E -18
0.500	6.9389 E -18	1.7347 E -18	6.9389 E -18	6.9389 E -18	6.9389 E -18
0.625	1.3878 E -16	2.7756 E -17	6.5226 E -16	6.9389 E -17	2.7755 E -17
0.750	1.3906 E -14	5.5511 E -17	1.2657 E -14	5.5511 E -17	2.7756 E -17
0.825	2.8191 E -13	2.7756 E -17	5.2686 E -13	2.7756 E -17	5.5511 E -17
1.000	4.3689 E -12	1.3878 E -17	4.4167 E -12	2.2204 E -16	1.1102 E -16

with the kernel $k(t,s,u) = (t-s)^2 e^{-u}$. This kernel is nonincreasing and convex in u. Equation (5.3) has the lower solution $v_0(t) = 0$ and the upper solution $w_0(t) = \frac{t^3}{3}$. Then according to the Theorem 2.1 and its conclusion this equation has a unique solution $u(t)$ where satisfies in the relation $v_0(t) = 0 \le u(t) \le w_0(t) = \frac{t^3}{3}$ and the solutions of the coupled iterative schemes 2.2 and 2.3 for this equation are convergent to the exact solution $u(t)$. For the error representation, the absolute values of the differences between $\hat{v}_p(t)$ and $\hat{w}_p(t)$ for $p = 1, 2, \dots,$ in some points of $[0, 1]$ are computed and shown in Table 5 and Figure 3.

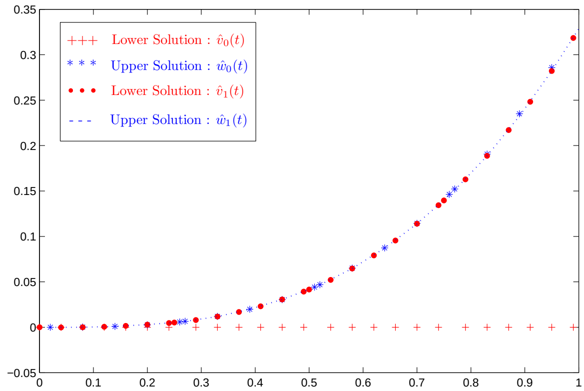

Figure 3. The convergence of the sequences $\{\hat{v}_p(t)\}$ and $\{\hat{w}_p(t)\}$ to the unknown solution of Example 5.3

Conclusion

The method presented in this article has been applied on the nonlinear equations. This method gives two numerical approximations for the solution and this is a preference for the method. The occurred algebraic systems are all linear while solving nonlinear equations directly lead to nonlinear algebraic systems and their problems. The accuracy of this method is comparatively good. The weakness of this method is in its assumptions that the nonlinear section of the problems must satisfies them.

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