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Optimal Inequalities, Contact δ-Invariants and Their Applications

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Abstract. Associated with a k-tuple $(n_1,\dots,n_k)\in\mathscr{S}(2n+1)$ with $n\geq 1$, we define a contact δ-invariant, $\delta^c(n_1,\dots,n_k)$, on an almost contact metric $(2n+1)$-manifold M. For an arbitrary isometric immersion of M into a Riemannian manifold, we establish an optimal inequality involving $\delta^c(n_1,\dots,n_k)$ and the squared mean curvature of the immersion. Furthermore, we investigate isometric immersions of contact metric and K-contact manifolds into Riemannian space forms which verify the equality case of the inequality for some k-tuple.

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1. Introduction

According to the celebrated embedding theorem of J. F. Nash [20], every Riemannian manifold can be isometrically embedded in Euclidean spaces with sufficiently high codimension. The Nash theorem was established in the hope that if Riemannian manifolds could always be regarded as Riemannian submanifolds, this would then yield the opportunity to use extrinsic help. As observed by M. Gromov [17] in 1985, this hope had not been materialized however. The main reason for this is lack of controls of the extrinsic properties of the submanifold by the known intrinsic invariants.

In order to overcome such difficulties as well as to provide answers to an open question on minimal immersions, the first author introduced in the early 1990's new types of Riemannian invariants $\delta(n_1,\dots,n_k)$, known as the δ-invariants (also known as Chen invariants in the literature). At the same time he was able to establish general optimal inequalities involving the new intrinsic invariants and the main extrinsic invariants, the squared mean curvature, for arbitrary Riemannian submanifolds. The δ-invariants are very different in nature from the "classical" Ricci and scalar curvatures (see [5–7]).

After δ-invariants were invented and the corresponding inequalities were established in [5, 6], δ-invariants were investigated by many geometers in the last two decades. Such

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invariants have been applied to several areas, including spectral geometry, affine geometry and general relativity (see for instance [9–11, 18, 22]; in particular, see [7, 9, 14] for a recent survey on δ-invariants and their applications).

For an almost contact metric manifold M and an integer $q \geq 2$, the first author and Mihai defined in [12] a contact invariant $\delta^c(q)$. They also established an optimal inequality for isometric immersions of M into real space forms involving $\delta^c(q)$. Moreover, they investigated K-contact submanifolds in Riemannian space forms which satisfy the equality case of the inequality.

In this paper, we extend $\delta^c(q)$ to contact invariants $\delta^c(n_1,\dots,n_k)$ on almost contract metric manifolds. For an isometric immersion of an almost contract metric $(2n+1)$-manifold M into any Riemannian manifold, we establish in section 4 an optimal inequality relating $\delta^c(n_1,\dots,n_k)$ to the squared mean curvature. In section 5, we prove a minimality result for contact metric manifolds in Riemannian space forms. In section 6, we show that K-contact manifolds in Riemannian space forms satisfying the equality case of the inequality are Sasakian. In the last section, we prove that every K-contact hypersurface of a Riemannian space form satisfying the equality case of the inequality for a k-tuple $(n_1,\dots,n_k)$ with $\sum_{j=1}^{k} n_j \leq 2n$ is totally geodesic.

2. Preliminaries

We recall some general definitions and basic formulas which will be used later. For general background on almost contact metric manifolds and submanifolds, we recommend references [1] and [3] respectively.

2.1. Almost contact metric manifolds

An odd-dimensional Riemannian manifold (M,g) is called an *almost contact metric manifold* if there exist on M a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

for vector fields X,Y on M. On an almost contact metric manifold, we also have $\phi\xi = 0$ and $\eta \circ \phi = 0$. The vector field ξ is called the *structure vector field*.

By a *contact manifold* we mean a $(2n+1)$-manifold M together with a global 1-form η satisfying $\eta \wedge (d\eta)^n \neq 0$ on M. If η of an almost contact metric manifold (M,ϕ,ξ,η,g) is a contact form and if η satisfies $d\eta(X,Y) = g(X,\phi Y)$ for all vectors X,Y tangent to M, then M is called a *contact metric manifold*.

A contact metric manifold is called *K-contact* if its characteristic vector field ξ is a Killing vector field. It is well-known that a contact metric manifold is a K-contact manifold if and only if

$$\nabla_X \xi = -\phi X \tag{2.3}$$

holds for all vector fields X on M. In fact, an almost contact metric manifold satisfying condition (2.3) is also a K-contact manifold. Condition (2.3) is equivalent to

$$K(X,\xi) = 1 \tag{2.4}$$

for every unit tangent vector X orthogonal to ξ.

An almost contact metric structure of M is called *normal* if the Nijenhuis torsion $[\phi,\phi]$ of ϕ equals to $-2d\eta\otimes\xi$. A normal contact metric manifold is called a *Sasakian manifold*. It can be proved that an almost contact metric manifold is Sasakian if and only if the Riemann curvature tensor R satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.5}$$

for any vector fields X,Y on M. A Sasakian manifold is also K-contact but the converse is not true in general if $\dim M \geq 5$.

On a contact metric $(2n+1)$-manifold M, $\eta = 0$ defines a $2n$-dimensional distribution in TM, which is called the contact distribution. A submanifold N of M is called an *integral submanifold* if $\eta(X) = 0$ for every tangent vector $X \in TN$. On a contact manifold of dimension $2n+1$, there exist integral submanifolds of the contact distribution of dimension less than or equal to n, but of no higher dimension.

2.2. Basic formulas, equations and definitions

Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional almost contact metric manifold with $n\geq 1$ isometrically immersed in a Riemannian m-manifold $(\tilde{M}^m,\tilde{g})$. Let $\langle\ ,\ \rangle$ denote the inner product of $\tilde{M}^m$ as well as of M. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M and $\tilde{M}^m$ respectively. Let h,D and A be the second fundamental form, the normal connection, and shape operator of M, respectively.

The Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.6}$$

$$\tilde{\nabla}_X V = -A_V X + D_X V \tag{2.7}$$

for vector fields X,Y tangent to M and V normal to M.

Denote by R and $\tilde{R}$ the Riemann curvature tensors of M and $\tilde{M}^m$, respectively. Then the Gauss and Codazzi equations are given by

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = \langle \tilde{R}(X,Y)Z,W\rangle &+ \langle h(X,W),h(Y,Z)\rangle \\ &- \langle h(X,Z),h(Y,W)\rangle,\end{aligned} \tag{2.8}$$

$$(\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{2.9}$$

where X,Y,Z,W are tangent vectors of M, $(\tilde{R}(X,Y)Z)^{\perp}$ is the normal component of $\tilde{R}(X,Y)Z$, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z). \tag{2.10}$$

When the ambient space $\tilde{M}^m$ is a Riemannian space form of constant curvature c, equations (2.8) and (2.9) of Gauss and Codazzi reduce to

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = c\{&\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\} \\ &+ \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle,\end{aligned} \tag{2.11}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.12}$$

The mean curvature vector of M in $\tilde{M}^m$ is defined by $H = (\operatorname{trace} h)/\dim M$. The squared mean curvature H^2 is defined as $H^2 = \langle H,H\rangle$.

2.3. Twisted products

The notion of twisted products was introduced in [4, page 66] as follows:

Let B and F be Riemannian manifolds with Riemannian metrics g_B and g_F, respectively, and f a positive differentiable function on $B \times F$. Consider the product manifold $B \times F$ with its projection $\pi_B : B \times F \to B$ and $\pi_F : B \times F \to F$. The *twisted product* $B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$||X||^2 = ||\pi_{B*}(X)||^2 + f^2 ||\pi_{F*}(X)||^2$$

for any vector X tangent to $B \times_f F$. Thus, we have $g = g_B + f^2 g_F$.

The function f above is called the twisting function of the twisted product. When f depends only on B, the twisted product is a warped product and f is called the warping function.

For vector fields V, X tangent to B and F respectively, we have

$$\nabla_X V = \nabla_V X = (Xf)V, \tag{2.13}$$

where ∇ is the Levi-Civita connection of the twisted product $B \times_f F$.

3. Contact δ-invariants

Suppose that (M, ϕ, ξ, η, g) is an almost contact metric $(2n+1)$-manifold with $n \geq 1$. If $\{e_1, \ldots, e_r\}$ is an orthonormal basis of a linear r-space $L \subseteq T_pM$ at $p \in M$, we define the scalar curvature of L by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha, e_\beta).$$

The scalar curvature $\tau(p)$ at a point $p \in M$ is $\tau(p) = \tau(T_pM)$.

For an integer $k \geq 1$, let $\mathscr{S}(2n+1, k)$ be the set of k-tuples $(n_1, \ldots, n_k)$ of integers satisfying

$$2 \leq n_1, \cdots, n_k \leq 2n \text{ and } 2 \leq n_1 + \cdots + n_k \leq 2n+1.$$

We denote by $\mathscr{S}(2n+1)$ the union: $\cup_{k \geq 1} \mathscr{S}(2n+1, k)$.

For each k-tuple $(n_1, \ldots, n_k) \in \mathscr{S}(2n+1)$, the δ-invariant $\delta(n_1, \ldots, n_k)$ was introduced in [6]. Now, we define the contact version of the δ-invariant in the same spirit as in [6, 12].

Definition 3.1. Let M be an almost contact metric $(2n+1)$-manifold and $(n_1, \ldots, n_k) \in \mathscr{S}(2n+1)$. The *contact δ-invariant* $\delta^c(n_1, \ldots, n_k)$ is defined by

$$\delta^c(n_1, \ldots, n_k)(p) := \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where $L_1, L_2, \ldots, L_k$ run over all mutually orthogonal subspaces of T_pM so that L_1 contains the characteristic vector ξ and $\dim L_j = n_j$, $j = 1, \ldots, k$.

Definition 3.2. Let M be an almost contact metric manifold and let $L_1, \ldots, L_k$ be mutually orthogonal subspaces of T_pM with $\dim L_j \geq 2$, $j = 1, \ldots, k$. A plane section $\pi \subset T_pM$ is said to be *orthogonal to* $L_1, \ldots, L_k$ if there exists an orthonormal basis $\{\bar{e}_1, \bar{e}_2\}$ such that $\pi = \mathrm{Span}\{\bar{e}_1, \bar{e}_2\}$ and one of the following three cases occurs:

(1) $\bar{e}_1 \in L_i$ and $\bar{e}_2 \in L_j$ with $1 \leq i \neq j \leq k$;

(2) $\bar{e}_1 \in L_i$ for some $i \in \{1, \ldots, k\}$ and $\bar{e}_2 \perp L_1, \ldots, L_k$;

(3) $\bar{e}_1, \bar{e}_2 \perp L_1, \ldots, L_k$.

We call an orthonormal frame $\{e_1,\dots,e_{2n+1}\}$ an *orthonormal ξ-frame* if e_1 is parallel to ξ.

We need the following algebraic lemma from [5] for later use.

Lemma 3.1. *Let $a_1,\dots,a_p,\zeta$ be $p+1$ real numbers such that*

$$\left(\sum_{i=1}^{p} a_i\right)^2 = (p-1)\left(\zeta + \sum_{i=1}^{p} a_i^2\right).$$

Then $2a_1a_2 \geq \zeta$, with equality holding if and only if $a_1+a_2 = a_3 = \ldots = a_p$.

4. An optimal inequality

For each k-tuple $(n_1,\dots,n_k) \in \mathscr{S}(2n+1)$, let $c(n_1,\dots,n_k)$ and $b(n_1,\dots,n_k)$ be the positive numbers given by

$$c(n_1,\dots,n_k) = \frac{(2n+1)^2(2n+k-\sum_{j=1}^{k} n_j)}{2(2n+k+1-\sum_{j=1}^{k} n_j)},$$
$$b(n_1,\dots,n_k) = n(2n+1) - \frac{1}{2}\sum_{j=1}^{k} n_j(n_j-1).$$

Put

$$\Delta_1 = \{1,\dots,n_1\},\dots,\Delta_k = \{n_1+\cdots+n_{k-1}+1,\dots,n_1+\cdots+n_k\},$$
$$\Delta = \Delta_1 \cup \cdots \cup \Delta_k,\ \ \Delta^2 = (\Delta_1\times\Delta_1)\cup\cdots\cup(\Delta_k\times\Delta_k).$$

Now, we modify the proof of [8, Theorem 3.1] to obtain the following.

Theorem 4.1. *Let M be a $(2n+1)$-dimensional almost contact metric manifold isometrically immersed in a Riemannian m-manifold $\tilde{M}^m$. Then, for each point $p\in M$ and each k-tuple $(n_1,\dots,n_k)\in\mathscr{S}(2n+1)$, we have:*

$$\delta^c(n_1,\dots,n_k)(p) \leq c(n_1,\dots,n_k)H^2(p) + b(n_1,\dots,n_k)\max\tilde{K}(p), \tag{4.1}$$

where $\max\tilde{K}(p)$ is the maximum of the sectional curvature function of $\tilde{M}^m$ restricted to 2-plane sections of the tangent space T_pM.

Moreover, the equality case of inequality (4.1) *holds at p if and only if the following two conditions hold:*

(a) *There exists an orthonormal ξ-basis $\{e_1,\dots,e_{2n+1}\}$ of T_pM and an orthonormal basis $\{e_{2n+2},\dots,e_m\}$ of the normal space $T_p^{\perp}M$ such that the shape operator with respect to $\{e_1,\dots,e_m\}$ satisfies*

$$A_{e_r} = \begin{pmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^r & \\ & 0 & & \mu_r I \end{pmatrix}, \quad r = 2n+2,\dots,m, \tag{4.2}$$

where I is an identity matrix and each A_j^r is a symmetric $n_j\times n_j$ submatrix satisfying

$$\operatorname{trace}(A_1^r) = \cdots = \operatorname{trace}(A_k^r) = \mu_r. \tag{4.3}$$

(b) *There exist mutually orthogonal subspaces* $L_1,\dots,L_k$ *of* T_pM *with* $\xi\in L_1$ *and* $\delta^c(n_1,\dots,n_k)(p)=\tau(p)-\sum_{j=1}^k\tau(L_j)$ *such that any plane section* $\pi\subset T_pM$ *orthogonal to* $L_1,\dots,L_k$ *satisfies* $\tilde K(\pi)=\max\tilde K(p)$.

Proof. Let M be an almost contact metric $(2n+1)$-manifold isometrically immersed in $\tilde M^m$. Then, at a point $p\in M$, the equation of Gauss gives

$$2\tau(p)=(2n+1)^2H^2(p)-||h||^2(p)+2\tilde\tau(T_pM), \tag{4.4}$$

where $||h||^2$ is the squared norm of h and $\tilde\tau(T_pM)$ is the scalar curvature of the ambient space $\tilde M^m$ corresponding to $T_pM\subset T_p\tilde M^m$. Let us put

$$\eta=2\tau(p)-\frac{(2n+1)^2(2n+k-\sum_{j=1}^k n_j)}{2n+k+1-\sum_{j=1}^k n_j}H^2(p)-2\tilde\tau(T_pM). \tag{4.5}$$

Then (4.4) and (4.5) give

$$(2n+1)^2H^2(p)=\gamma\big(\eta+||h||^2(p)\big),\ \gamma=2n+k+1-n_1-\cdots-n_k. \tag{4.6}$$

Let us choose an orthonormal ξ-basis $\{e_1,\dots,e_{2n+1}\}$ of T_pM in such way that $e_{\alpha_i}\in L_i$ for each $\alpha_i\in\Delta_i$. We choose an orthonormal basis $\{e_{2n+2},\dots,e_m\}$ of $T_p^\perp M$ so that e_{2n+2} in the direction of H at p. Then (4.6) yields

$$\Big(\textstyle\sum_{A=1}^{2n+1}a_A\Big)^2=\gamma\Big[\eta+\displaystyle\sum_A a_A^2+\sum_{A\neq B}(h_{AB}^{2n+2})^2+\sum_{r=2n+3}^{m}\sum_{A,B=1}^{2n+1}(h_{AB}^r)^2\Big] \tag{4.7}$$

with $a_A=h_{AA}^{2n+2}$ for $1\le A,B\le 2n+1$. Equation (4.7) is equivalent to

$$\begin{aligned}\Big(\textstyle\sum_{i=1}^{\gamma+1}\bar a_i\Big)^2&=\gamma\Big[\eta+\displaystyle\sum_{i=1}^{\gamma+1}(\bar a_i)^2+\sum_{A\neq B}(h_{AB}^{2n+2})^2+\sum_{r=2n+3}^{m}\sum_{A,B=1}^{2n+1}(h_{AB}^r)^2\\&\quad-\sum_{2\le\alpha_1<\beta_1\le n_1}2a_{\alpha_1}a_{\beta_1}-\sum_{\alpha_2<\beta_2}2a_{\alpha_2}a_{\beta_2}-\cdots-\sum_{\alpha_k<\beta_k}2a_{\alpha_k}a_{\beta_k}\Big],\end{aligned} \tag{4.8}$$

where $\alpha_2,\beta_2\in\Delta_2,\ \dots,\ \alpha_k,\beta_k\in\Delta_k$ and

$$\begin{aligned}&\bar a_1=a_1,\ \bar a_2=a_2+\cdots+a_{n_1},\\&\bar a_3=a_{n_1+1}+\cdots+a_{n_1+n_2},\\&\cdots\\&\bar a_{k+1}=a_{n_1+\cdots+n_{k-1}+1}+\cdots+a_{n_1\cdots+n_k},\\&\bar a_{k+2}=a_{n_1\cdots+n_k+1},\dots,\bar a_{\gamma+1}=a_{2n+1}.\end{aligned}$$

Thus, by applying Lemma 3.1 to (4.8) we obtain

$$\begin{aligned}&\sum_{1\le\alpha_1<\beta_1\le n_1}a_{\alpha_1}a_{\beta_1}+\sum_{\alpha_2<\beta_2}a_{\alpha_2}a_{\beta_2}+\cdots+\sum_{\alpha_k<\beta_k}a_{\alpha_k}a_{\beta_k}\\&\ge\frac{\eta}{2}+\sum_{A<B}(h_{AB}^{2n+2})^2+\sum_{r=2n+3}^{m}\sum_{A,B=1}^{2n+1}\frac12(h_{AB}^r)^2,\end{aligned} \tag{4.9}$$

where $\alpha_i,\beta_i\in\Delta_i$, $i=1,\dots,k$. It also follows from Lemma 3.1 that the equality sign of (4.9) holds if and only if $a_1+\bar a_2=\bar a_3=\cdots=\bar a_{\gamma+1}$.

On the other hand, equation of Gauss implies that, for each $j \in \{1,\dots,k\}$,

$$\tau(L_j) = \sum_{r=2n+2}^{m} \sum_{\alpha_j<\beta_j} \Big(h^r_{\alpha_j\alpha_j} h^r_{\beta_j\beta_j} - (h^r_{\alpha_j\beta_j})^2 \Big) + \tilde{\tau}(L_j) \tag{4.10}$$

for $\alpha_j, \beta_j \in \Delta_j$, where $\tilde{\tau}(L_j)$ is the scalar curvature of L_j in $\tilde{M}^m$. Then, by combining (4.5), (4.9) and (4.10) we obtain

$$\begin{aligned} \tau(L_1)+\cdots+\tau(L_k) &\geq \frac{\eta}{2} + \frac{1}{2}\sum_{r=2n+2}^{m} \sum_{(\alpha,\beta)\notin\Delta^2} (h^r_{\alpha\beta})^2 \\ &+ \frac{1}{2}\sum_{r=2n+3}^{m}\sum_{j=1}^{k}\Big(\sum_{\alpha_j\in\Delta_j} h^r_{\alpha_j\alpha_j}\Big)^2 + \sum_{j=1}^{k}\tilde{\tau}(L_j) \geq \frac{\eta}{2} + \sum_{j=1}^{k}\tilde{\tau}(L_j) \\ &= \tau - \frac{(2n+1)^2(2n+k-\sum n_j)}{2(2n+k+1-\sum n_j)} H^2 - \tilde{\tau}(T_pM) + \sum_{j=1}^{k}\tilde{\tau}(L_j). \end{aligned} \tag{4.11}$$

From (4.11) we find

$$\tau - \sum_{j=1}^{k} \tau(L_j) \leq \frac{(2n+1)^2(2n+k-\sum n_j)}{2(2n+k+1-\sum n_j)} H^2 + \tilde{\tau}(T_pM) - \sum_{j=1}^{k}\tilde{\tau}(L_j), \tag{4.12}$$

which implies that

$$\delta^c(n_1,\dots,n_k) \leq \frac{(2n+1)^2(2n+k-\sum n_j)}{2(2n+k+1-\sum n_j)} H^2 + \widetilde{\delta}^c(n_1,\dots,n_k), \tag{4.13}$$

where

$$\widetilde{\delta}^c(n_1,\dots,n_k) := \tilde{\tau}(T_pM) - \inf\{\tilde{\tau}(\tilde{L}_1)+\cdots+\tilde{\tau}(\tilde{L}_k)\} \tag{4.14}$$

with $\tilde{L}_1,\dots,\tilde{L}_k$ run over all k mutually orthogonal subspaces of T_pM such that $\xi \in \tilde{L}_1$ and $\dim \tilde{L}_j = n_j$; $j = 1,\dots,k$. From (4.13) we obtain (4.1).

If the equality case of (4.1) holds at p, then all of the inequalities in (4.9) and (4.11) become equalities. Hence, we obtain condition (a). Moreover, it follows from (4.13) and (4.14) that condition (b) holds too.

The converse can be easily verified. ∎

As an immediate consequence of Theorem 4.1, we have the following.

Theorem 4.2. *Let M be an almost contact metric $(2n+1)$-manifold isometrically immersed in a Riemannian space form $R^m(c)$ of constant curvature c. Then, for any k-tuple $(n_1,\dots,n_k) \in \mathscr{S}(2n+1)$, we have:*

$$\delta^c(n_1,\dots,n_k) \leq c(n_1,\dots,n_k)H^2 + b(n_1,\dots,n_k)c. \tag{4.15}$$

The equality case of inequality (4.15) *holds at a point $p \in M$ if and only if there exists an orthonormal ξ-basis $\{e_1,\dots,e_{2n+1}\}$ of T_pM and an orthonormal basis $\{e_{2n+2},\dots,e_m\}$ of $T_p^{\perp}M$ such that the shape operator with respect to $\{e_1,\dots,e_m\}$ satisfies*

$$A_{e_r} = \begin{pmatrix} A^r_1 & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A^r_k & \\ & 0 & & \mu_r I \end{pmatrix}, \quad r = 2n+2,\dots,m, \tag{4.16}$$

where each A^r_j is a symmetric $n_j \times n_j$ submatrix satisfying

$$\operatorname{trace}(A^r_1) = \cdots = \operatorname{trace}(A^r_k) = \mu_r. \tag{4.17}$$

Definition 4.1. Let M be an almost contact metric $(2n+1)$-manifold immersed in $\tilde{M}^m$. If M satisfies the equality case of (4.1) for a k-tuple $(n_1,\dots,n_k)$, then an orthonormal ξ-frame $\{e_1,\dots,e_{2n+1}\}$ satisfying (4.2) and (4.3) is called an *adapted ξ-frame.*

5. Minimality

Let M be an almost contact metric $(2n+1)$-manifold isometrically immersed in a Riemannian m-manifold $\tilde{M}^m$. If M satisfies the equality case of (4.1) for some $(n_1,\dots,n_k) \in \mathscr{S}(2n+1)$. Then, with respect to an adapted ξ-frame $\{e_1,\dots,e_{2n+1}\}$, we have

$$h(e_\alpha, Y) = \frac{(2n+1)\langle e_\alpha, Y\rangle}{2n+k+1-\sum_{j=1}^k n_j} H, \quad \forall Y \in TM, \tag{5.1}$$

for $\alpha \in \{1+\sum_{j=1}^k n_j, \dots, 2n+1\}$.

Next, we prove the following minimality result.

Theorem 5.1. *Let M be a contact metric $(2n+1)$-manifold isometrically immersed in a Riemannian space form $R^m(c)$. If M satisfies the equality case of (4.15) for a k-tuple $(n_1,\dots,n_k) \in \mathscr{S}(2n+1)$ with $\sum_{j=1}^k n_j \le n$, then M is a minimal submanifold of $R^m(c)$*

Proof. Let M be a contact metric $(2n+1)$-manifold isometrically immersed in $R^m(c)$ such that the equality case of (4.15) is satisfied for some k-tuple $(n_1,\dots,n_k)$ with $\sum_{j=1}^k n_j \le n$. Then Theorem 4.2 implies that there exists an adapted ξ-basis $\{e_1,\dots,e_{2n+1}\}$ of T_pM and an orthonormal basis $\{e_{2n+2},\dots,e_m\}$ of $T_p^\perp M$ such that (4.16) and (4.17) hold.

Now, assume that M is non-minimal in $R^m(c)$, i.e. $H \neq 0$. In order to derive a contradiction, let us put

$$\mathscr{D}(p) = \left\{ X \in T_pM : h(X,Y) = \frac{(2n+1)\langle X,Y\rangle}{2n+k+1-\sum_{j=1}^k n_j} H, \forall Y \in T_pM \right\}.$$

It follows from (5.1) that $\dim \mathscr{D}(p) \ge 2n+1-\sum_{j=1}^k n_j$. Clearly, $\dim \mathscr{D}(p)$ is constant on some nonempty open submanifold, say U, of M. Since $H \neq 0$, it follows from the definition of adapted ξ-frame, (4.16) and (4.17) that, for any $X \in \mathscr{D}$, we have $\eta(X) = 0$. Hence, $\mathscr{D}$ is a contact distribution on $U \subset M$.

Let $\mathscr{D}^\perp$ be the orthogonal complementary distribution of $\mathscr{D}$. Then we have $h(\mathscr{D}, \mathscr{D}^\perp) = \{0\}$. Thus, for vector fields $X, Y \in \mathscr{D}$ and $Z \in \mathscr{D}^\perp$, we have $(\bar{\nabla}_X h)(Y,Z) = -h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$. Therefore, after applying the equation of Codazzi, we obtain

$$\begin{aligned}
h([X,Y],Z) &= h(\nabla_X Y, Z) - h(\nabla_Y X, Z) \\
&= -(\bar{\nabla}_X h)(Y,Z) - h(\nabla_Y X, Z) - h(Y, \nabla_X Z) \\
&= -(\bar{\nabla}_Y h)(X,Z) - h(\nabla_Y X, Z) - h(Y, \nabla_X Z) \\
&= h(X, \nabla_Y Z) - h(Y, \nabla_X Z) \\
&= \frac{2n+1}{2n+k+1-\sum_{j=1}^k n_j} (\langle X, \nabla_Y Z\rangle - \langle Y, \nabla_X Z\rangle) H \\
&= \frac{2n+1}{2n+k+1-\sum_{j=1}^k n_j} \langle [X,Y], Z\rangle H,
\end{aligned}$$

which implies $[X,Y] \in \mathscr{D}$. Hence, $\mathscr{D}$ is an involutive distribution on U whose leaves are integral submanifolds of M.

On the other hand, it is known that the maximal dimension of integral submanifolds is n. Hence, we get $2n+1-\sum_{j=1}^{k} n_j \leq n$, i.e. $\sum_{j=1}^{k} n_j \geq n+1$. This contradicts the assumption $\sum_{j=1}^{k} n_j \leq n$. ∎

Remark 5.1. The condition $\sum_{j=1}^{k} n_j \leq n$ in Theorem 5.1 is necessary. This can be seen from the following simple example.

Example 5.1. Consider $\psi : \mathbb{R} \times S^2(1) \to \mathbb{E}^4$ defined by

$$\psi(t,\theta,\varphi) = \big(t, \cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi\big),$$

where $\mathbb{E}^4$ is the Euclidean 4-space endowed with the standard flat metric.

Define a contact metric structure (ϕ,ξ,η,g) on $M := \mathbb{R} \times S^2(1)$ by

$$\begin{aligned}
&\eta = \cos\theta dt + \sin\theta d\varphi, \;\; \xi = \cos\theta \frac{\partial}{\partial t} + \sin\theta \frac{\partial}{\partial \varphi}, \\
&\phi\left(\frac{\partial}{\partial t}\right) = -\tan\theta \frac{\partial}{\partial \theta}, \;\; \phi\left(\frac{\partial}{\partial \varphi}\right) = \frac{\partial}{\partial \theta}, \\
&\phi\left(\frac{\partial}{\partial \theta}\right) = \cos\varphi\left(\sin\theta \frac{\partial}{\partial t} - \cos\theta \frac{\partial}{\partial \varphi}\right), \\
&g = dt^2 + d\varphi^2 + \cos^2\varphi d\theta^2.
\end{aligned}$$

Then $\eta \wedge d\eta = dt \wedge d\theta \wedge d\varphi \neq 0$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$ hold. This contact metric hypersurface is non-minimal in $\mathbb{E}^4$ and it satisfies the equality case of (4.15) with $k=1, n_1=2$ and $\dim M = 3$.

6. K-contact submanifold satisfying the equality

For a submanifold M of a Riemannian manifold $\tilde{M}$ with second fundamental form h, the subspace $\ker h_p$, $p \in M$, denoted by $\mathscr{N}_p$, is given by

$$\mathscr{N}_p = \{X \in T_pM : h(X,Y) = 0 \text{ for all } Y \in T_pM\}.$$

$\mathscr{N}_p$ is called the *relative nullity space* at p. The dimension ν_p of $\mathscr{N}_p$ is called the *relative nullity* at p.

Theorem 6.1. *Let M be a K-contact $(2n+1)$-manifold isometrically immersed in a Riemannian space form $R^m(c)$. If there exists a k-tuple $(n_1,\ldots,n_k) \in \mathscr{S}(2n+1)$ with $\sum_{j=1}^{k} n_j \leq 2n$ such that the equality case of inequality (4.15) holds, then $c \geq 1$. Moreover, if $c=1$, then M is a Sasakian manifold whose characteristic vector field ξ lies in relative nullity space, i.e. $\xi \in \mathscr{N}$.*

Proof. Let M be a K-contact manifold isometrically immersed in $R^m(c)$. Assume that the equality case of inequality (4.15) holds for some k-tuple $(n_1,\ldots,n_k) \in \mathscr{S}(2n+1)$ with $\sum_{j=1}^{k} n_j \leq 2n$. Then Theorem 4.2 implies that there exists an orthonormal ξ-basis $\{e_1,\ldots,e_{2n+1}\}$ of T_pM and an orthonormal basis $\{e_{2n+2},\ldots,e_m\}$ of $T_p^{\perp}M$ such that the shape operator with respect to $\{e_1,\ldots,e_m\}$ satisfy (4.16) and (4.17).

Since $\{e_1,\dots,e_{2n+1}\}$ is an orthonormal ξ-frame, e_1 is parallel to ξ. It follows from Gauss equation and (4.16) that, for any unit tangent vector e_j perpendicular to ξ, we have

$$K(\xi,e_j) = c+\langle h(e_1,e_1),h(e_j,e_j)\rangle - \langle h(e_1,e_j),h(e_1,e_j)\rangle \tag{6.1}$$
$$= \begin{cases} c+\langle h(e_1,e_1),h(e_j,e_j)\rangle - |h(e_1,e_j)|^2, & \text{if } j=2,\dots,n_1, \\ c+\langle h(e_1,e_1),h(e_j,e_j)\rangle, & \text{if } n_1+1\le j\le 2n+1. \end{cases}$$

On the other hand, since M is K-contact, we have $K(\xi,X)=1$ for any $X\perp\xi$. Hence, we obtain

$$\begin{aligned} n_1-2 &= \sum_{j=2}^{n_1} K(\xi,e_j) - K(\xi,e_{2n+1}) \\ &= (n_1-2)c + \langle h(e_1,e_1), \sum_{j=1}^{n_1} h(e_j,e_j)\rangle - \sum_{j=1}^{n_1} |h(e_1,e_j)|^2 \\ &\quad - \langle h(e_1,e_1),h(e_{2n+1},e_{2n+1})\rangle \end{aligned}$$

Therefore, after applying (4.16) and (4.17), we deduce that

$$(n_1-2)(c-1) = \sum_{j=1}^{n_1} |h(e_1,e_j)|^2 \ge 0. \tag{6.2}$$

If $n_1>2$, then (6.2) yields $c\ge 1$ immediately. If $n_1=2$, then (6.2) implies that $h(e_1,e_j)=0$ for $j=1,\dots,2n+1$. Combining this with (4.16) shows that $h(\xi,X)=0$ for $X\in TM$, since e_1 is parallel to ξ. Thus, we see from (6.1) and $K(\xi,X)=1$ that $c=1$. So, $c\ge 1$ holds for both cases.

Now, suppose that $c=1$ holds, then we find from (6.2) that $h(\xi,X)=0$, for all tangent vector X. Thus, ξ lies in the relative nullity space $\mathcal{N}$.

Finally, after applying $h(\xi,X)=0$ and the equation of Gauss, we obtain

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

which implies that M is Sasakian by (2.5). ∎

7. K-contact hypersurfaces satisfying the equality

Now, we study K-contact hypersurfaces satisfying the equality case of (4.15). To do so, we recall the following two results from [23]:

Theorem 7.1. *Let M^{2n+1} be a K-contact manifold isometrically immersed in a Riemannian space form $R^{2n+2}(c)$ with $c\ne 1$. Then $c<1$ and M^{2n+1} is of constant curvature one.*

Theorem 7.2. *Let M^{2n+1} be a K-contact manifold isometrically immersed in a Riemannian space form $R^{2n+2}(1)$ of constant curvature one. Then*
(a) *the rank of the shape operator A is ≤ 2, and*
(b) *M^{2n+1} is of constant curvature one if and only if its scalar curvature satisfies $\tau = n(2n+1)$.*

Finally, we prove the following result for K-contact hypersurfaces in Riemannian space forms.

Theorem 7.3. *Let M be a K-contact hypersurface of a Riemannian space form $R^{2n+2}(c)$. If M satisfies the equality case of inequality* (4.15) *for some k-tuple $(n_1,\dots,n_k) \in S(2n+1)$ with $\sum_{j=1}^{k} n_j \le 2n$, then $c=1$. Moreover, M is a Sasakian manifold of constant curvature one immersed in $R^{2n+2}(1)$ as a totally geodesic hypersurface.*

Proof. Let M be a K-contact hypersurface in $R^{2n+2}(c)$. Assume M satisfies the equality case of (4.15) for a k-tuple $(n_1,\dots,n_k)$ with $\sum_{j=1}^{k} n_j \le 2n$, then it follows from Theorem 6.1 that $c \ge 1$. Combining this with Theorem 7.1 yields $c=1$. Hence, by Theorem 6.1, M is a Sasakian manifold whose characteristic vector field ξ satisfies $h(\xi,X)=0$ for $X \in TM$. Moreover, according to Theorem 7.2, the rank of the shape operator satisfies $rk(A) \le 2$.

Because $c=1$, without loss of generality, we may assume that $R^{2n+2}(1)$ is the unit hypersphere $S^{2n+2}(1) \subset \mathbb{E}^{2n+3}$. Since the equality case of (4.15) is satisfied for $(n_1,\dots,n_k)$ with $\sum_{j=1}^{k} n_j \le 2n$, Theorem 6.1 shows that, with respect to an adapted ξ-frame, the shape operator takes the form:

$$A = \begin{pmatrix} A_1 & \dots & 0 & \\ \vdots & \ddots & \vdots & \mathbf{0} \\ 0 & \dots & A_k & \\ & \mathbf{0} & & \mu I \end{pmatrix} \tag{7.1}$$

with

$$trace(A_1) = \cdots = trace(A_k) = \mu. \tag{7.2}$$

Because e_1 is in the direction of ξ, it follows from $h(\xi,X)=0$ that the first row and the first column of A_1 are zero submatrices.

Case (a): $rk(A)=0$. In this case, M is a totally geodesic hypersurface.

Case (b): $rk(A)=1$. It follows from (7.1) and (7.2) that $\mu=0$, which is impossible since $rk(A)=1$.

Case (c): $rk(A)=2$. We divide this into three subcases:

Case (c.1): *There exist $A_i, A_j, i \ne j$,* with $rk(A_i)=rk(A_j)=1$. In this case, by using $\sum_{j=1}^{k} n_j \le 2n$ and $rk(A)=2$, we obtain $\mu=0$. But this is impossible due to (7.2).

Case (c.2): *There exists A_i with $rk(A_i)=1$ and $A_j=0$ for $i \ne j \in \{1,\dots,k\}$.* It follows from (7.2) and the condition $rk(A)=2$ that $k=1$, $n_1=2n$ and $trace(A_1)=\mu \ne 0$. So, we can choose an orthonormal $\{e_1,\dots,e_{2n+1},e_{2n+2}\}$ such that the second fundamental form satisfies

$$\begin{aligned} &h(e_{2n},e_{2n}) = h(e_{2n+1},e_{2n+1}) = \mu e_{2n+2}, \\ &h(e_i,e_j) = 0 \ \text{ otherwise}, \end{aligned} \tag{7.3}$$

where e_{2n+2} is a unit normal vector field.

Let ω_i^j be the connection 1-forms defined by

$$\nabla_{e_i} e_j = \sum_{\ell=1}^{2n+1} w_j^{\ell}(e_i) e_\ell, \ i,j = 1,\dots,2n+1.$$

From (7.3) and the equation of Codazzi, we find

$$\omega_r^{2n}(e_s) = \omega_s^{2n+1}(e_r) = 0, \tag{7.4}$$

$$\omega^r_{2n}(e_{2n+1}) = \omega^r_{2n+1}(e_{2n}) = 0, \tag{7.5}$$

$$e_r(\ln\mu) = \omega^r_{2n}(e_{2n}) = \omega^r_{2n+1}(e_{2n+1}),\ \ e_{2n}\mu = e_{2n+1}\mu = 0, \tag{7.6}$$

for $r,s = 1,\dots,2n-1$.

Let $\mathscr{F} = \mathrm{Span}\{e_{2n}, e_{2n+1}\}$ and $\mathscr{F}^\perp = \mathrm{Span}\{e_1,\dots,e_{2n-1}\}$. Then (7.4) implies that $\mathscr{F}^\perp$ is a totally geodesic distribution, i.e. $\mathscr{F}^\perp$ is an involutive distribution whose leaves are totally geodesic submanifolds of M. Furthermore, it follows from (7.5) and (7.6) that $\mathscr{F}$ is an involutive distribution whose leaves are totally umbilical. Hence, a result of [21] implies that M is locally a twisted product $B\times_f F$, where B and F are leaves of $\mathscr{F}^\perp$ and $\mathscr{F}$, respectively, and f is the twisting function.

Since the characteristic vector field ξ lies in $\mathscr{F}^\perp$, it is tangent to B. Therefore, by (2.13), we have $\nabla_{e_{2n}}\xi = (\xi f)e_{2n}$. On the other hand, it follows from (2.3) that we have $\nabla_{e_{2n}}\xi = -\phi(e_{2n})$. Thus, we get $\phi(e_{2n}) = -(\xi f)e_{2n}$, which is a contradiction. Consequently, this case is impossible.

Case (c.3): *There exists one A_i with $rk(A_i) = 2$.* In this case, we obtain $trace(A_i) = \mu = 0$. Hence, A_i has exactly two nonzero eigenvalues $\lambda, -\lambda$ with multiplicity one and the remaining eigenvalues are zero. Because $\xi \in \mathscr{N}$, we can choose an orthonormal $\{e_1,\dots,e_{2n+1},e_{2n+2}\}$ such that e_1 is in the direction of ξ and the second fundamental form satisfies

$$\begin{aligned} &h(e_2,e_2) = \lambda e_{2n+2},\ h(e_3,e_3) = -\lambda e_{2n+2},\\ &h(e_i,e_j) = 0 \ \text{ otherwise.}\end{aligned} \tag{7.7}$$

By applying (7.7) and the equation of Codazzi, we find

$$\omega^2_r(e_s) = \omega^3_r(e_s) = 0, \tag{7.8}$$

$$\omega^r_2(e_3) = -\omega^r_3(e_2) = 2\omega^3_2(e_r), \tag{7.9}$$

$$e_r(\ln\lambda) = \omega^r_2(e_2) = \omega^r_3(e_3), \tag{7.10}$$

$$e_2(\ln\lambda) = 2\omega^2_3(e_3),\ e_3(\ln\lambda) = 2\omega^3_2(e_2) \tag{7.11}$$

for $r,s = 1,4,\dots,2n+1$.

Let W denote the open subset of M consisting of all non-totally geodesic points. Then W is an open dense subset of M due to minimality of M. Since W has relative nullity $2n-1$, we know that there is a non-totally geodesic, minimal isometric immersion $\psi : B^2 \to S^{2n+2}(1)$ of a surface B^2 into $S^{2n+2}(1)$ such that M is an open subset of the unit normal bundle NB^2 defined by (see, e.g. [13, Theorem 2])

$$N_pB^2 = \left\{ v \in T_{\psi(p)}S^{2n+2}(1) : \langle v,v\rangle = 1 \text{ and } \left\langle v, \psi_*(T_pB^2)\right\rangle = 0 \right\}.$$

Let $p \in B^2$ and (x,y) be an isothermal coordinate system on a neighborhood of p so that the metric tensor of B^2 is given by $g_B = E^2(x,y)(dx^2+dy^2)$ for some function $E > 0$. If $\{\xi_3(x,y),\dots,\xi_{2n+2}(x,y)\}$ is a local orthonormal frame of B^2 in $S^{2n+2}(1)$, then the immersion of NB^2 in $S^{2n+2}(1) \subset \mathbb{E}^{2n+3}$ can be parametrized by

$$F(x,y,u_3,\dots,u_{2n+1}) = \sum_{i=3}^{2n+2} y_i\xi_i(x,y), \tag{7.12}$$

where $y_3 = \cos u_3, y_4 = \sin u_3\cos u_4,\dots,$ $y_{2n+1} = \sin u_3 \cdots \sin u_{2n}\cos u_{2n+1}$ and $y_{2n+2} = \sin u_3\cdots\sin u_{2n+1}$.

Since B^2 is minimal in $S^{2n+2}(1)$ and (x,y) is an isothermal coordinate system, there exist functions λ_i, μ_i such that

$$\tilde{A}_{\xi_i}\left(\frac{\partial}{\partial x}\right) = \lambda_i \frac{\partial}{\partial x} + \mu_i \frac{\partial}{\partial y}, \ \tilde{A}_{\xi_i}\left(\frac{\partial}{\partial y}\right) = \mu_i \frac{\partial}{\partial x} - \lambda_i \frac{\partial}{\partial y}, \tag{7.13}$$

for $i \in \{3,\dots,2n+2\}$, where $\tilde{A}$ is the shape operator of $\psi : B^2 \to S^{2n+2}(1)$. Because B^2 does not contain any totally geodesic points, the functions λ_i, μ_i, $i = 3,\dots,2n+2$, do not vanish simultaneously. It follows from $|F| = 1$ and (7.13) that there exist functions $\alpha_i, \beta_i, i = 3,\dots,2n+1$, on NB^2 such that

$$F_*\left(\frac{\partial}{\partial x}\right) = \sum_{i=3}^{2n+1} \alpha_i F_*\left(\frac{\partial}{\partial u_i}\right) - \lambda\, \psi_*\left(\frac{\partial}{\partial x}\right) - \mu\, \psi_*\left(\frac{\partial}{\partial y}\right),$$
$$F_*\left(\frac{\partial}{\partial y}\right) = \sum_{i=3}^{2n+1} \beta_i F_*\left(\frac{\partial}{\partial u_i}\right) - \mu\, \psi_*\left(\frac{\partial}{\partial x}\right) + \lambda\, \psi_*\left(\frac{\partial}{\partial y}\right),$$

where $\lambda = \sum_{i=3}^{2n+2} \lambda_i y_i, \mu = \sum_{i=3}^{2n+2} \mu_i y_i$. Therefore, F is an immersion on an open dense subset of NB^2 and that on this open subset the space spanned by $\{F, F_*(\frac{\partial}{\partial x}), F_*(\frac{\partial}{\partial y}), F_*(\frac{\partial}{\partial u_3}), \dots, F_*(\frac{\partial}{\partial u_{2n+1}})\}$ coincides with the space spanned by $\{F_*(\frac{\partial}{\partial x}), F_*(\frac{\partial}{\partial y}), \xi_3, \dots, \xi_{2n+2}\}$. So, the tangent vector fields

$$F_*\left(\frac{\partial}{\partial u_3}\right), \dots, F_*\left(\frac{\partial}{\partial u_{2n+1}}\right)$$

of NB^2 are normal vector fields of B^2 in $S^{2n+2}(1)$. It is easy to verify that the second fundamental form h of NB^2 in $S^{2n+2}(1)$ satisfies

$$h\left(F_*\left(\frac{\partial}{\partial u_i}\right), \psi_*(X)\right) = 0,\ i = 3,\dots,2n+1,\ \ \forall X \in T(NB^2).$$

Therefore, the vector fields $\psi_x = \psi_*\left(\frac{\partial}{\partial x}\right)$, $\psi_y = \psi_*\left(\frac{\partial}{\partial y}\right)$ of NB^2 are perpendicular to $\mathscr{N} = \ker h$ which is spanned by $\{F_*(\frac{\partial}{\partial u_3}), \dots, F_*(\frac{\partial}{\partial u_{2n+1}})\}$. Since the characteristic vector field ξ lies in $\mathscr{N}$, ξ is a normal vector field of B^2 in $S^{2n+2}(1)$. Hence, we obtain from (2.7) that

$$\langle \nabla_{\psi_x} \xi, \psi_y \rangle = -\langle \tilde{A}_\xi \psi_x, \psi_y \rangle = -\langle \tilde{A}_\xi \psi_y, \psi_x \rangle = \langle \nabla_{\psi_y} \xi, \psi_x \rangle. \tag{7.14}$$

Therefore, if we put $e_1 = \xi/|\xi|$, $e_2 = \psi_x/E$, $e_3 = \psi_y/E$, then (7.14) gives $\omega_1^3(e_2) = \omega_1^2(e_3)$. Combining this with (7.9) with $r = 1$ gives

$$\omega_1^2(e_3) = \omega_1^3(e_2) = \omega_2^3(e_1) = 0. \tag{7.15}$$

On the other hand, it follows from (7.8) with $r = 1$ that

$$\nabla_{e_s} e_1 \in \mathrm{Span}\{e_4, \dots, e_{2n+1}\},\ s = 4, \dots, 2n-1. \tag{7.16}$$

Since e_1 is in the direction of ξ and $\nabla_X \xi = -\phi(X)$ for $X \in TM$, (7.16) implies that $\phi(e_s) \in \mathrm{Span}\{e_4, \dots, e_{2n+1}\}$. Hence, $\mathrm{Span}\{e_4, \dots, e_{2n+1}\}$ is ϕ-invariant, i.e. it is invariant under the action of ϕ. Therefore, $\mathrm{Span}\{e_2, e_3\}$ is also ϕ-invariant. Thus, we obtain $\nabla_{e_2} \xi = -\phi(e_2) \in \mathrm{Span}\{e_3\}$. Because $\phi(e_2) \neq 0$, we have $\omega_1^3(e_2) \neq 0$. This contradicts (7.15). Consequently, this case is also impossible. ∎

Remark 7.1. We shall point out that the results obtained in this article are quite different from those in [2, 15, 16, 19], since the target spaces in this article are Riemannian manifolds (without Sasakian structure), in contrast to [2, 15, 16, 19] in which Sasakian space forms are the target spaces.

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