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Invariance of Arens Irregularity under Extensions

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Abstract. Let m be a bounded bilinear mapping on Banach spaces. First we give a sufficient condition for strong irregularity of m and apply the result to extend some earlier results by Ülger. Next, we study invariance of irregularity for m under extension of its domain from left or right. As a consequence, we improve some known results about Arens regularity. Finally, we investigate invariance of irregularity for direct sum of a family of Banach algebras.

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1. Introduction and preliminaries

Let X, Y and Z be normed spaces and let $m : X \times Y \longrightarrow Z$ be a bounded bilinear map. In 1951, Richard Arens [2], (see also [1]), introduced two natural extensions of m and discussed the problem of coincidence of these maps. We begin with introducing these maps and the related concepts to establish our notation. Throughout the paper for a normed space X, we denote by X' the Banach dual of X and also x' for an arbitrary element of X'. The dual space X' with the weak*-topology is denoted by $(X', \sigma(X', X))$. We write X'' for $(X')'$ and so on. As usual, we identify X with its canonical image $\hat{X}$ in the second dual X'' by the standard way.

Let m be a bounded bilinear map as above. The *adjoint* of m is also a bounded bilinear map namely $m^* : Z' \times X \longrightarrow Y'$ which is defined via

$$\langle m^*(z', x), y\rangle = \langle z', m(x, y)\rangle \quad (x \in X,\ y \in Y,\ z' \in Z').$$

Clearly for each x in X, the mapping $z' \mapsto m^*(z', x)$ from Z' into Y' is weak* to weak* continuous. By repeating this process, we can define the second and third adjoints of m as $m^{**} : Y'' \times Z' \longrightarrow X'$ and $m^{***} : X'' \times Y'' \longrightarrow Z''$, respectively; the latter map extends m in the sense that $m^{***}|_{X\times Y} = m$. More precisely, m^{***} is the unique extension of m such that the mapping $x'' \mapsto m^{***}(x'', y'')$ from X'' into Z'' is weak* to weak* continuous for every y''

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in Y'', and also the mapping $y'' \mapsto m^{***}(x, y'')$ from Y'' into Z'' is weak* to weak* continuous for every x in X. Then the *first* or *left topological center* of m is defined by

$$\mathfrak{Z}_t^{(1)}(m) = \left\{x'' \in X'' : \ y'' \mapsto m^{***}(x'', y'') \text{ is weak}^* \text{ to weak}^* \text{ continuous}\right\}.$$

Now let $m^t : Y \times X \longrightarrow Z$ be the *transpose* of m defined by $m^t(y,x) = m(x,y)$, for all x in X and y in Y. In analogy to m, the map m^t can be extended to the (unique) bounded bilinear mapping $m^{t***} : Y'' \times X'' \longrightarrow Z''$ such that the mapping $y'' \mapsto m^{t***}(y'', x'')$ from Y'' into Z'' is weak* to weak* continuous for every x'' in X'', and also the mapping $x'' \mapsto m^{t***}(y, x'')$ from X'' into Z'' is weak* to weak* continuous, for every y in Y. Then the *second* or *right topological center* of m is defined by

$$\mathfrak{Z}_t^{(2)}(m) = \left\{y'' \in Y'' : \ x'' \mapsto m^{t***}(y'', x'') \text{ is weak}^* \text{ to weak}^* \text{ continuous}\right\}.$$

Equivalently, the first and the second topological centers of m can be defined as

$$\mathfrak{Z}_t^{(1)}(m) = \left\{x'' \in X'' : \ m^{***}(x'', y'') = m^{t***t}(x'', y''), \text{ for all } y'' \in Y''\right\}$$

and

$$\mathfrak{Z}_t^{(2)}(m) = \left\{y'' \in Y'' : \ m^{***}(x'', y'') = m^{t***t}(x'', y''), \text{ for all } x'' \in X''\right\}.$$

The bilinear map m is called *Arens regular,* if the equality $m^{***} = m^{t***t}$ holds. This is equivalent to the equality $\mathfrak{Z}_t^{(1)}(m) = X''$, and equivalently $\mathfrak{Z}_t^{(2)}(m) = Y''$. On the other hand, the bilinear map m is called *left strongly Arens irregular,* if $\mathfrak{Z}_t^{(1)}(m) = X$, and *right strongly Arens irregular* if $\mathfrak{Z}_t^{(2)}(m) = Y$. Also, m is called *strongly Arens irregular,* if the equalities $\mathfrak{Z}_t^{(1)}(m) = X$ and $\mathfrak{Z}_t^{(2)}(m) = Y$ both hold.

An alternative approach to the definition of the previous extensions can be given as follows. Let x'' and y'' be arbitrary elements in X'' and Y'', respectively. By the Goldstine's theorem [6, p. 425], there exist bounded nets $(x_\alpha)_\alpha$ in X and $(y_\beta)_\beta$ in Y with $x_\alpha \to x''$ in $(X'', \sigma(X'', X'))$ and $y_\beta \to y''$ in $(Y'', \sigma(Y'', Y'))$. Then one may easily verify that

$$m^{***}(x'', y'') = \text{weak}^* - \lim_\alpha \lim_\beta m(x_\alpha, y_\beta)$$

and

$$m^{t***t}(x'', y'') = \text{weak}^* - \lim_\beta \lim_\alpha m(x_\alpha, y_\beta).$$

Now let $\mathfrak{A}$ be a Banach algebra and X be a left Banach $\mathfrak{A}$-module with the left module action

$$\pi_\ell : \mathfrak{A} \times X \longrightarrow X.$$

Then a bounded net $(e_\alpha)_\alpha$ in $\mathfrak{A}$ is called a bounded left approximate identity (BLAI) for X, if $\lim_\alpha \pi_\ell(e_\alpha, x) = x$, for every $x \in X$. In the case that $X = \mathfrak{A}$, this net is simply called a BLAI. Similarly, one can define a bounded right approximate identity (BRAI) and a two sided bounded approximate identity (BAI) for a right Banach $\mathfrak{A}$-module (X, π_r) and a Banach $\mathfrak{A}$-bimodule (π_ℓ, X, π_r), respectively.

For a Banach algebra $\mathfrak{A}$ with the multiplication $\pi : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}$, we consider π as a bilinear map. It can then extend to the multiplications π^{***} and π^{t***t} on $\mathfrak{A}''$, which are called the first and the second Arens products and are denoted by $\Box$ and $\Diamond$, respectively. Moreover, $(\mathfrak{A}'', \Box)$ and $(\mathfrak{A}'', \Diamond)$ both are Banach algebras. The Banach algebra $\mathfrak{A}$ is called Arens regular, if the multiplication π is Arens regular; i.e., the two products $\Box$ and $\Diamond$ coincide on $\mathfrak{A}''$.

2. Arens irregularity and reflexivity

Let X, Y and Z be normed spaces. Recall that a bounded bilinear map m from $X\times Y$ into Z factors if m is onto, see [7] for example. In the following we define a strong version of this concept.

Definition 2.1. *Let $m:\ X\times Y\longrightarrow Z$ be a bounded bilinear map. We say that m factors strongly from the left by an element y in Y, if $m(X,y)=Z$. Similarly, m factors strongly from the right by x in X, if $m(x,Y)=Z$.*

Our first result presents a sufficient condition for the left strong Arens irregularity of a certain bilinear map. This is a generalization of [7, Theorem 2.2], where the same result was proved for a right module action.

Theorem 2.1. *Let $m:\ X\times Y\longrightarrow Z$ be a bounded bilinear map such that m^* and m^{**} factor strongly from the right and the left by an element $z'\in Z'$, respectively. Then m is Arens regular if and only if X is reflexive.*

Proof. If X is reflexive, then obviously $\mathfrak{Z}_t^{(1)}(m)=X$, i.e., m is left strongly Arens irregular. Conversely, assume that m is Arens regular and $x''\in X''$. We show that $x''\in(X',\sigma(X',X))'=X$. Let $x'_\alpha\to x'$ in $(X',\sigma(X',X))$. Since by the assumptions $m^{**}(Y'',z')=X'$, for each α there is y''_α in Y'' such that $m^{**}(y''_\alpha,z')=x'_\alpha$. Also we have $m^{**}(y'',z')=x'$, for some $y''\in Y''$. We claim that $y''_\alpha\to y''$ in $(Y'',\sigma(Y'',Y'))$. So let $y'\in Y'$ be arbitrary. Since $m^*(z',X)=Y'$, then we can choose x in X such that $m^*(z',x)=y'$. Now we can write

$$\begin{aligned}\lim_\alpha\langle y''_\alpha,y'\rangle &= \lim_\alpha\langle y''_\alpha,m^*(z',x)\rangle=\lim_\alpha\langle m^{**}(y''_\alpha,z'),x\rangle\\ &=\langle m^{**}(y'',z'),x\rangle=\langle y'',m^*(z',x)\rangle=\langle y'',y'\rangle,\end{aligned}$$

so that $y''_\alpha\to y''$ in $(Y'',\sigma(Y'',Y'))$ as claimed.

On the other hand, since $\mathfrak{Z}_t^{(1)}(m)=X''$, we have $x''\in\mathfrak{Z}_t^{(1)}(m)$ and hence $m^{***}(x'',y''_\alpha)\to m^{***}(x'',y'')$ in $(Z'',\sigma(Z'',Z'))$. Consequently we have

$$\begin{aligned}\lim_\alpha\langle x'',x'_\alpha\rangle &= \lim_\alpha\langle x'',m^{**}(y''_\alpha,z')\rangle=\lim_\alpha\langle m^{***}(x'',y''_\alpha),z'\rangle\\ &=\langle m^{***}(x'',y''),z'\rangle=\langle x'',m^{**}(y'',z')\rangle=\langle x'',x'\rangle.\end{aligned}$$

This proves that $x''\in(X',\sigma(X',X))'$ as required. ∎

Although the assumptions of Theorem 2.1 seems to be strong, there are some well known results related to Arens regularity and reflexivity which satisfy the same conditions. One of them is a result by Ülger which states that for a Banach algebra $\mathfrak{A}$ with a BRAI, the bilinear mapping $f:\mathfrak{A}'\times\mathfrak{A}\longrightarrow\mathfrak{A}'$ defined by $f(a',a)=a'\cdot a$ is Arens regular if and only if $\mathfrak{A}$ is reflexive, (see [9], Theorem 3.1 and the remarks before Corollary 3.4; see also [5, Proposition 4.5]). Using Theorem 2.1, we can extend this fact to every right $\mathfrak{A}$-module X.

Corollary 2.1. *Let (X,π_r) be a Banach right $\mathfrak{A}$-module such that $\mathfrak{A}$ has a* BRAI *for X. If $\pi_r^*:\ X'\times X\longrightarrow\mathfrak{A}'$ is Arens regular, then X is reflexive.*

Proof. Suppose that $(e_\alpha)_\alpha$ is a BRAI in $\mathfrak{A}$ for X. We can assume that $(e_\alpha)_\alpha$ is weak*-convergent to some e''_0 in $\mathfrak{A}''$. We show that π_r^{**} and π_r^{***} factor strongly from the right and the left by e''_0, respectively. Take $x\in X$ and $x'\in X'$. Then

$$\langle\pi_r^{**}(e''_0,x'),x\rangle=\lim_\alpha\langle\pi_r^{**}(e_\alpha,x'),x\rangle=\lim_\alpha\langle\pi_r^*(x',x),e_\alpha\rangle=\lim_\alpha\langle x',\pi_r(x,e_\alpha)\rangle=\langle x',x\rangle.$$

So that $\pi_r^{**}(e_0'', x') = x'$ and hence $\pi_r^{**}(e_0'', X') = X'$. Now for every $x'' \in X''$ and $x' \in X'$ we have

$$\langle \pi_r^{***}(x'', e_0''), x' \rangle = \langle x'', \pi_r^{**}(e_0'', x') \rangle = \langle x'', x' \rangle.$$

Thus $\pi_r^{***}(X'', e_0'') = X''$. By Theorem 2.1, X' is reflexive, and so is X. ∎

Another result, also due to Ülger, states that for a Banach space X, Arens regularity of the bilinear form $f: X' \times X \longrightarrow \mathbb{C}$ defined by

$$f(x', x) = \langle x', x \rangle \quad (x \in X,\, x' \in X'),$$

is equivalent to the reflexivity of X, see [9, Corollary 3.2]. In the following we show that this map also belongs to the class of bilinear maps, stated in Theorem 2.1. To do this, consider X as a Banach $\mathbb{C}$-module with the scalar multiplication. Then X is a unital $\mathbb{C}$-module; i.e., $\pi_\ell(1, x) = x$ and $\pi_r(x, 1) = x$, for all x in X. Moreover, we can identify $\mathbb{C}'$ with $\mathbb{C}$ in the sense that $\lambda \in \mathbb{C}$, as a functional, is the product by λ.

Corollary 2.2. [9, Corollary 3.2] *Let X be a Banach space. Then the bilinear form f : $X' \times X \longrightarrow \mathbb{C}$, $(x', x) \mapsto \langle x', x \rangle$ is Arens regular if and only if X is reflexive.*

Proof. Consider the bilinear map $\pi_r: X \times \mathbb{C} \longrightarrow X$, $(x, \lambda) \mapsto \lambda x$. Identifying $\mathbb{C}'$ with $\mathbb{C}$ as above, it is easy to see that $\pi_r^*(x', x) = \langle x', x \rangle$. Hence by Corollary 2.1, Arens regularity of $f = \pi_r^*$ implies that X is reflexive. The converse is clear. ∎

3. Invariance of Arens irregularity

Let $\mathfrak{A}$ be an Arens regular Banach algebra. We know that every closed subalgebra of $\mathfrak{A}$ is also Arens regular, but the converse is not the case; i.e., the Arens regularity of a subalgebra dose not guarantee this property for $\mathfrak{A}$, necessarily. Similarly, for a bilinear mapping m : $X \times Y \longrightarrow Z$, if m is Arens regular and X_1, Y_1 are arbitrary closed subspaces of X and Y, respectively, then the mapping $\tilde{m} := m|_{X_1 \times Y_1}$ is Arens regular. On the other hand, let G be an infinite, compact Hausdorff group. Then by [4, Proposition 11], the bounded bilinear mapping

$$m: L^1(G) \times L^\infty(G) \longrightarrow L^1(G)$$

defined by $m(f, g) = f * g$ is Arens regular, where

$$f * g(x) = \int_G f(y) g(y^{-1}x) dy \quad \left(x \in G,\, f \in L^1(G),\, g \in L^\infty(G)\right).$$

Nevertheless, the extension of m on $L^1(G) \times L^1(G)$, denoted by $\tilde{m}$, which is the same as convolution on $L^1(G)$, is not regular; in fact, $\tilde{m}$ is strongly Arens irregular [8]. These examples motivate us to think that irregularity often, not necessarily, (see Corollary 3.8 below), strengthen by enlarging algebras or spaces. The aim of this section is a partial studying of this situation.

Let Y be a closed linear subspace of a Banach space X. Consider the annihilator of Y in X' as

$$Y^0 = \left\{x' \in X' : x'|_Y = 0\right\},$$

then Y^0 is a (weak*) closed subspace of X'. It is known that the second dual of Y can be identified with the weak* closed subspace Y^{00} of X''. In fact in this identification, Y'' is equal to the closure of Y in $(X'', \sigma(X'', X'))$, i.e., $Y'' = \overline{Y}^{w^*}$ in X''. Furthermore, it is a standard fact that the weak*-topology of Y'' is the restriction of weak*-topology of X'' to Y''. When X is a Banach $\mathfrak{A}$-module and Y is a closed submodule, then we have the following lemma.

Lemma 3.1. *Suppose that $\mathfrak{A}$ is a Banach algebra, (π_ℓ, X) is a left Banach $\mathfrak{A}$-module and Y is a closed submodule of X. Then Y'' is a closed left $(\mathfrak{A}'', \Box)$-submodule of (π_ℓ^{***}, X'').*

Proof. Let $a'' \in \mathfrak{A}''$ and $y'' \in Y''$. Take bounded nets $(a_\alpha)_\alpha$ in $\mathfrak{A}$ and $(y_\beta)_\beta$ in Y such that $a_\alpha \to a''$ in $(\mathfrak{A}'', \sigma(\mathfrak{A}'', \mathfrak{A}'))$ and $y_\beta \to y''$ in $(Y'', \sigma(Y'', Y'))$. Then using the facts stated above

$$\pi_\ell^{***}(a'', y'') = \text{weak}^* - \lim_\alpha \lim_\beta \underbrace{\pi_\ell(a_\alpha, y_\beta)}_{\in Y} \in Y'',$$

since Y'' is weak* closed in X''. ∎

Theorem 3.1. *Let $\mathfrak{A}$ be a Banach algebra, (π_ℓ, X) is a left Banach $\mathfrak{A}$-module and Y a closed submodule of X. Put*

$$\tilde{\pi}_\ell := \pi_\ell|_{\mathfrak{A}\times Y} : \mathfrak{A} \times Y \longrightarrow Y.$$

Then $\mathfrak{Z}_t^{(1)}(\pi_\ell) \subseteq \mathfrak{Z}_t^{(1)}(\tilde{\pi}_\ell)$.

Proof. Suppose that $a'' \in \mathfrak{Z}_t^{(1)}(\pi_\ell)$ and $(y''_\alpha)_\alpha$ is a convergent net to some $y'' \in Y''$ in $(Y'', \sigma(Y'', Y'))$. Then we have $y''_\alpha \to y''$ in $(X'', \sigma(X'', X'))$. Since $a'' \in \mathfrak{Z}_t^{(1)}(\pi_\ell)$, then

$$\begin{aligned}\tilde{\pi}_\ell^{***}\left(a'', y''\right) = \pi_\ell^{***}\left(a'', y''\right) &= \lim_\alpha \pi_\ell^{***}(a'', y''_\alpha) \quad \text{in } \left(X'', \sigma\left(X'', X'\right)\right) \\ &= \lim_\alpha \tilde{\pi}_\ell^{***}(a'', y''_\alpha) \quad \text{in } \left(Y'', \sigma\left(Y'', Y'\right)\right).\end{aligned}$$

Consequently, $a'' \in \mathfrak{Z}_t^{(1)}(\tilde{\pi}_\ell)$ and the result follows. ∎

Corollary 3.1. *Let X, Y, π_ℓ and $\tilde{\pi}_\ell$ be as Theorem 3.1. If $\tilde{\pi}_\ell$ is left strongly Arens irregular, then so is π_ℓ. On the other hand, if π_ℓ is Arens regular, then so is $\tilde{\pi}_\ell$.*

The following two corollaries of Theorem 3.1 extend some results of [7], in the sense that $\mathfrak{A}$ does not necessarily have a BRAI.

Corollary 3.2. *Let $\mathfrak{A}$ be a Banach algebra, and suppose that the bounded bilinear mapping $\mathfrak{A} \times \mathfrak{A}^{(2n)} \longrightarrow \mathfrak{A}^{(2n)}$ is Arens regular for some $n \in \mathbb{N}$. Then $\mathfrak{A}$ is Arens regular.*

Corollary 3.3. *If a Banach algebra $\mathfrak{A}$ is left strongly Arens irregular, then the mapping $\mathfrak{A} \times \mathfrak{A}^{(2n)} \longrightarrow \mathfrak{A}^{(2n)}$ is left strongly Arens irregular for all $n \in \mathbb{N}$.*

For a locally compact group G, we denote by π the convolution in $L^1(G)$, i.e., $\pi(f, g) = f * g$ for all $f, g \in L^1(G)$. Then we know that $\mathfrak{Z}_t^{(1)}(\pi) = L^1(G)$. With this in mind, Theorem 3.1 will result in the following.

Corollary 3.4. *Let G be a locally compact group and $\pi_\ell : L^1(G) \times M(G) \longrightarrow M(G)$ be the convolution product in $M(G)$. Then $\mathfrak{Z}_t^{(1)}(\pi_\ell) = L^1(G)$.*

In the following, we make a partial study of Arens irregularity for a left module action, related to the Banach algebra and its subalgebras.

Proposition 3.1. *Let $\mathfrak{A}$ be a Banach algebra, (π_ℓ, X) a left Banach $\mathfrak{A}$-module and $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$. If $\tilde{\pi}_\ell$ is the restriction of π_ℓ to $\mathfrak{B} \times X$, then*

$$\mathfrak{Z}_t^{(1)}(\pi_\ell) \cap \mathfrak{B}'' \subseteq \mathfrak{Z}_t^{(1)}(\tilde{\pi}_\ell).$$

In particular, if $\tilde{\pi}_\ell$ is left strongly Arens irregular, then the equality holds, i.e., $\mathfrak{Z}_t^{(1)}(\pi_\ell) \cap \mathfrak{B}'' = \mathfrak{B}$.

Proof. Suppose that $b'' \in \mathfrak{Z}_t^{(1)}(\pi_\ell) \cap \mathfrak{B}''$ and $(x''_\alpha)_\alpha$ as an arbitrary net in X'' converging to some $x'' \in X''$ in $(X'', \sigma(X'', X'))$. Since $b'' \in \mathfrak{Z}_t^{(1)}(\pi_\ell)$, we have

$$\text{weak}^* - \lim_\alpha \pi_\ell^{***}(b'', x''_\alpha) = \pi_\ell^{***}(b'', x'').$$

Then the equality $\pi_\ell^{***}|_{\mathfrak{B}'' \times X''} = \tilde{\pi}_\ell^{***}$, implies that

$$\text{weak}^* - \lim_\alpha \tilde{\pi}_\ell{}^{***}(b'', x''_\alpha) = \tilde{\pi}_\ell^{***}(b'', x''),$$

and so $b'' \in \mathfrak{Z}_t^{(1)}(\tilde{\pi}_\ell)$. Finally, if $\tilde{\pi}$ is left strongly Arens irregular, then $\mathfrak{Z}_t^{(1)}(\tilde{\pi}_\ell) = \mathfrak{B}$ and consequently $\mathfrak{Z}_t^{(1)}(\pi_\ell) \cap \mathfrak{B}'' = \mathfrak{B}$. ∎

Note that Arens regularity of π obviously implies this property for $\tilde{\pi}$. On the other hand, we don't know if the left strong Arens irregularity of $\tilde{\pi}$ is induced to π. However, we give a partial answer to this question.

Corollary 3.5. *Let $\mathfrak{A}$ be a Banach algebra with the multiplication π. Suppose that $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$ for some reflexive subspace $\mathfrak{C}$ of $\mathfrak{A}$. If the bilinear mapping $\tilde{\pi} := \pi|_{\mathfrak{B} \times \mathfrak{A}}$ from $\mathfrak{B} \times \mathfrak{A}$ into $\mathfrak{A}$ is left strongly Arens irregular, then so is π.*

Proof. By reflexivity of $\mathfrak{C}$, it is easy to see that $\mathfrak{B}'' \cap \mathfrak{C} = \{0\}$ and $\mathfrak{A}'' = \mathfrak{B}'' \oplus \mathfrak{C}$. Suppose that $a'' \in \mathfrak{Z}_t^{(1)}(\pi)$, then $a'' = b'' + c$ for some $b'' \in \mathfrak{B}''$ and $c \in \mathfrak{C}$. Since $c \in \mathfrak{Z}_t^{(1)}(\pi)$ and $\mathfrak{Z}_t^{(1)}(\pi)$ is a linear space, then we have $b'' \in \mathfrak{Z}_t^{(1)}(\pi)$. Now using Proposition 3.1, we obtain $b'' \in \mathfrak{B}$ and hence $a'' \in \mathfrak{A}$; that is $\mathfrak{A}$ is left strongly Arens irregular. ∎

Corollary 3.6. *Let $\mathfrak{B}$ be a closed subalgebra in a Banach algebra $\mathfrak{A}$ with finite codimension. If $\mathfrak{B}$ is left strongly Arens irregular, then so is $\mathfrak{A}$.*

Proof. It is clear by Corollary 3.1 and Corollary 3.5. ∎

We conclude this paper by considering some cases of irregularity preserving in Banach algebras and bilinear maps. First let $m : X \times Y \longrightarrow Z$ be a bounded bilinear mapping, where X, Y and Z are Banach spaces. There is a Banach algebra construction corresponding to m due to P. G. Dixon, see [3, p. 384], as follows. Put $\mathfrak{A} = X \oplus Y \oplus Z$ and let the algebra multiplication and the norm of $\mathfrak{A}$ be given by

$$\|a\| = \|x\| + \|y\| + \|z\|$$

and $ab = (0, 0, m(x, y_1))$, where $a = (x, y, z)$ and $b = (x_1, y_1, z_1)$ are arbitrary elements in $\mathfrak{A}$. Then with pointwise addition, scalar multiplication and the norm and the algebra multiplication given above, $\mathfrak{A}$ becomes a Banach algebra. It is easy to verify that $\mathfrak{A}$ is Arens regular if and only if the bilinear mapping m is Arens regular. Moreover, for $a'' = (x'', y'', z'')$ and $b'' = (x''_1, y''_1, z''_1)$ in $\mathfrak{A}''$, a simple verification shows that

$$a'' \Box b'' = \left(0, 0, m^{***}\left(x'', y''_1\right)\right)$$

and

$$a'' \Diamond b'' = \left(0, 0, m^{t***t}\left(x'', y''_1\right)\right).$$

Thus we have

$$\begin{aligned}\mathfrak{Z}_t^{(1)}(\mathfrak{A}) &= \left\{a'' \in \mathfrak{A}'' : a'' \Box b'' = a'' \Diamond b'',\ b'' \in \mathfrak{A}''\right\} \\ &= \left\{a'' \in \mathfrak{A}'' : x'' \in \mathfrak{Z}_t^{(1)}(m)\right\} = \mathfrak{Z}_t^{(1)}(m) \oplus Y'' \oplus Z''.\end{aligned}$$

Consequently, for m and $\mathfrak{A}$ as above, we have the following.

Corollary 3.7. *The Banach algebra $\mathfrak{A}$ is left strongly Arens irregular if and only if the following assertions hold:*

(i) *the map m is left strongly Arens irregular;*
(ii) *the spaces Y and Z are both reflexive.*

Finally, we verify irregularity directly for some of Banach algebras. Let $\{\mathfrak{A}_i\}_{i\in I}$ be a family of Banach algebras and consider their direct sum

$$\mathfrak{A} = \bigoplus_{i\in I} \mathfrak{A}_i = \left\{ (a_i) : \ a_i \in \mathfrak{A}_i, \ \|(a_i)\| = \sum_{i\in I} \|a_i\| < \infty \right\}$$

as a Banach algebra with pointwise multiplication and the norm given above. If I is finite, then $\mathfrak{A}'' = \bigoplus_{i\in I} \mathfrak{A}_i''$ and so $\mathfrak{Z}_t^{(1)}(\mathfrak{A}) = \bigoplus_{i\in I} \mathfrak{Z}_t^{(1)}(\mathfrak{A}_i)$. On the other hand, if I is infinite, then $\mathfrak{A}'' = \mathfrak{B}' \oplus \mathfrak{B}^0$, where

$$\mathfrak{B} = \left\{ (a_i') \in \prod_{i\in I} \mathfrak{A}_i' : \ \|a_i'\| \to 0 \right\}$$

and $\mathfrak{B}^0$ is the annihilator of $\mathfrak{B}$ in $\mathfrak{A}''$, see [3, Proposition 5]. Also, by the proof of Theorem 6 in [3], for $a'' = (a_1'', a_2'')$ and $b'' = (b_1'', b_2'')$ in $\mathfrak{A}''$ with $a_1'', b_1'' \in \mathfrak{B}'$ and $a_2'', b_2'' \in \mathfrak{B}^0$, we have $a'' \Box b'' = a_1'' \Box b_1''$ and $a'' \Diamond b'' = a_1'' \Diamond b_1''$. Since I is infinite, then $\mathfrak{B} \neq 0$. Suppose that $a_i \in \mathfrak{A}_i$ for every $i \in I$ and $c'' \in \mathfrak{B}^0$ be nonzero, then we have

$$\left((a_i) + c''\right) \Box \left(b_1'', b_2''\right) = (a_i) \Box b_1'' = (a_i) \Diamond b_1'' = \left((a_i) + c''\right) \Diamond \left(b_1'', b_2''\right).$$

Therefore $((a_i) + c'') \in \mathfrak{Z}_t^{(1)}(\mathfrak{A})$ and consequently $\mathfrak{A}$ is not left strongly Arens irregular.

We summarize the previous discussions in the following.

Corollary 3.8. *Let $\{\mathfrak{A}_i\}_{i\in I}$ be a family of Banach algebras and $\mathfrak{A}$ be the Banach algebra of their direct sum. If I is finite, then $\mathfrak{A}$ is left strongly Arens irregular if and only if $\mathfrak{A}_i$ is left strongly Arens irregular, for every $i \in I$. On the other hand, if I is infinite, then $\mathfrak{A}$ is never left strongly Arens irregular even if each $\mathfrak{A}_i$ is left strongly Arens irregular.*

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