BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(2) (2013), 401–418*

On the Definition of Atanassov's Intuitionistic Fuzzy Subrings and Ideals

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Abstract. On the basis of the concept of grades of a fuzzy point to belongingness ($\in$) or quasi-coincident (q) or belongingness and quasi-coincident ($\in \wedge q$) or belongingness or quasi-coincident ($\in \vee q$) in an intuitionistic fuzzy set of a ring, the notion of a (α,β)-intuitionistic fuzzy subring and ideal is introduced by applying the Lukasiewicz 3-valued implication operator. Using the notion of fuzzy cut set of an intuitionistic fuzzy set, the support and α-level set of an intuitionistic fuzzy set are defined and it is established that, for $\alpha \neq \in \wedge q$, the support of a (α,β)-intuitionistic fuzzy ideal of a ring is an ideal of the ring. It is also established that the level sets of an intuitionistic fuzzy ideal with thresholds (s,t) of a ring is an ideal of the ring. We investigate that an intuitionistic fuzzy set A of a ring is a $(\in,\in)$ (or $(\in,\in \vee q)$ or $(\in \wedge q,\in)$)-intuitionistic fuzzy ideal of the ring if and only if A is an intuitionistic fuzzy ideal with thresholds $(0,1)$ (or $(0,0.5)$ or $(0.5,1)$) of the ring respectively. We also establish that A is a $(\in,\in)$ (or $(\in,\in \vee q)$ or $(\in \wedge q,\in)$)-intuitionistic fuzzy ideal of the ring if and only if for any $a \in (0,1]$ (or $a \in (0,0.5]$ or $a \in (0.5,1]$), A_a is a fuzzy ideal of the ring. Finally, we investigate that an intuitionistic fuzzy set of a ring is an intuitionistic fuzzy ideal with thresholds (s,t) of the ring if and only if for any $a \in (s,t]$, the cut set A_a is a fuzzy ideal of R.

2010 Mathematics Subject Classification: 08A72, 16D25.

Keywords and phrases: Intuitionistic fuzzy set, fuzzy subring, fuzzy ideal, intuitionistic fuzzy subring, intuitionistic fuzzy ideal, Lukasiewicz implication operator.

1. Introduction

Since the introduction of fuzzy sets by Zadeh [26] in 1965, the researchers have been carrying out research in various concepts of abstract algebra in fuzzy setting. Fuzzy subgroups of a group was introduced by Rosenfeld [19] in 1971. Since then many generalization of this fundamental concept have been done. A self contained survey of the state of art of the fuzzy binary relations and some of their applications has been provided by Beg and Ashraf in [4]. Bhakat and Das in [5, 6], redefined fuzzy subgroups of a group using the notion of belongings to ($\in$) and quasi-coincident (q) of a fuzzy point to a fuzzy set of the

Communicated by Lee See Keong.
Received: April 8, 2011; *Revised:* August 13, 2011.

group. In [7], fuzzy subring and ideal are redefined. Davvaz *et al.* in [9,10], generalized the concept to H_v-submodules and redefined fuzzy H_v-submodules by applying many valued implication operators. In [14] the notion of interval valued fuzzy k-ideals of semirings is introduced, which is a generalization of a fuzzy k-ideal. As a generalization of fuzzy set, intuitionistic fuzzy set was introduced by Atanassov [1], also see [2,3]. Since then various concepts of fuzzy setting have been generalized to intuitionistic fuzzy set, for example see [8,11–13,15,24]. Fuzzy aspects of ordered semigroups have been studied by many researchers as seen in [16,20,21]. Characterization of different types of (α,β)-intuitionistic fuzzy subgroups A of a group using the notions of grades of a fuzzy point belongs to A or quasi-coincident with A or belongs to and quasi-coincident ($\in \wedge q$) or belongs to or quasi-coincident ($\in \vee q$) has been done in [23]. Intuitionistic fuzzy ideal with thresholds (s,t) of a ring was introduced in [22]. In this paper, using the notions of grades of a fuzzy point x_a belongs to an intuitionistic fuzzy set A, in a ring R or quasi-coincident with A or belongs to and quasi-coincident ($\in \wedge q$) or belongs to or quasi-coincident ($\in \vee q$), a (α,β)-intuitionistic fuzzy subring and ideal is defined by applying the Lukasiewicz 3-valued implication operator, see [17]. The support and α-level set of an intuitionistic fuzzy set is defined based on fuzzy cut set and grades of belongs to respectively. It is established that, for $\alpha \neq\in \wedge q$, the support of a (α,β)-intuitionistic fuzzy ideal of a ring is an ideal of the ring. We investigate that the level sets of an intuitionistic fuzzy ideal with thresholds (s,t) of a ring is an ideal of the ring. We obtain necessary and sufficient conditions between (α,β)-intuitionistic fuzzy ideal and intuitionistic fuzzy ideal with thresholds (s,t). It is established that an intuitionistic fuzzy set A of a ring is a $(\in,\in)$ (or $(\in,\in\vee q)$ or $(\in\wedge q,\in)$)-intuitionistic fuzzy ideal of the ring if and only if A is an intuitionistic fuzzy ideal with thresholds $(0,1)$ (or $(0,0.5)$ or $(0.5,1)$) of the ring respectively. We also establish that A is a $(\in,\in)$ (or $(\in,\in\vee q)$ or $(\in\wedge q,\in)$-intuitionistic fuzzy ideal of the ring if and only if for any $a\in(0,1]$ (or $a\in(0,0.5]$ or $a\in(0.5,1]$), A_a is a fuzzy ideal of the ring respectively. Finally, we investigate that an intuitionistic fuzzy set of a ring is an intuitionistic fuzzy ideal with thresholds (s,t) of the ring if and only if for any $a\in(s,t]$, the cut set A_a is a fuzzy ideal of R.

2. Basic definitions and notations

A ring is a non-empty set R having two binary operations addition $(+)$ and multiplication $(\cdot)$, where $(R,+)$ is a commutative group, $(R,\cdot)$ is a semigroup and addition is distributive with respect to multiplication. By zero (0) we mean the additive identity of R. A non-empty subset I of R is called an ideal of R, if for any $x,y\in I$ and $r\in R$, we have $x-y, rx, xr\in I$. A fuzzy set on a non-empty set was introduced by Zadeh [26] in 1965 and was defined as follows:

By a fuzzy set of a ring R, we mean any mapping μ from R to $[0,1]$. By $[0,1]^R$ we will denote the set of all fuzzy subsets of R. For each fuzzy set μ in R and any $\alpha\in[0,1]$, we define two sets

$$U(\mu,\alpha)=\{x\in R \mid \mu(x)\geq\alpha\} \text{ and } L(\mu,\alpha)=\{x\in R \mid \mu(x)\leq\alpha\},$$

which are called an upper level cut and a lower level cut of μ, respectively. The complement of μ, denoted by μ^c, is the fuzzy set on R defined by $\mu^c(x)=1-\mu(x)$.

Let $x\in R$ and $t\in(0,1]$, then a fuzzy subset $\mu\in[0,1]^R$ is called a fuzzy point if

$$\mu(y)=\begin{cases} t, & \text{if } y=x \\ 0, & \text{if } y\neq x \end{cases}$$

and it is denoted by x_t.

Definition 2.1. [5] *Let* μ *be a fuzzy subset of R and* x_a *be a fuzzy point. Then*

(1) *If* $\mu(x) \geq a$, *then we say* x_a *belongs to* μ, *and it is denoted by* $x_a \in \mu$.
(2) *If* $\mu(x) + a > 1$, *then we say* x_a *is quasi-coincident with* μ, *and it is denoted by* $x_a q \mu$.
(3) $x_a \in \wedge q\mu \Leftrightarrow x_a \in \mu$ *and* $x_a q \mu$.
(4) $x_a \in \vee q\mu \Leftrightarrow x_a \in \mu$ *or* $x_a q \mu$.

The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold. Let $\mu, \sigma \in [0,1]^R$. Then, the intersection and union of μ and σ are given by the fuzzy sets $\mu \cap \sigma$ and $\mu \cup \sigma$ respectively and are defined as follows:

(1) $(\mu \cap \sigma)(x) = \mu(x) \wedge \sigma(x)$;
(2) $(\mu \cup \sigma)(x) = \mu(x) \vee \sigma(x)$,
where $\mu(x) \wedge \sigma(x) = \min\{\mu(x), \sigma(x)\}$ and $\mu(x) \vee \sigma(x) = \max\{\mu(x), \sigma(x)\}$.

Definition 2.2. [18] *Let R be a ring and* μ *be a fuzzy subset in R. Then,* μ *is called a fuzzy subring of R if and only if for every* $x, y \in R$ *the following conditions are satisfied:*

(1) $\mu(x+y) \geq \mu(x) \wedge \mu(y)$;
(2) $\mu(-x) \geq \mu(x)$;
(3) $\mu(xy) \geq \mu(x) \wedge \mu(y)$.

Definition 2.3. [18] *Let R be a ring and* μ *be a fuzzy subset in R. Then,* μ *is called a fuzzy ideal of R if and only if for every* $x, y \in R$ *the following conditions are satisfied:*

(1) $\mu(x+y) \geq \mu(x) \wedge \mu(y)$;
(2) $\mu(-x) \geq \mu(x)$;
(3) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

An intuitionistic fuzzy set (abbreviated as IFS) introduced by Atanassov in [1] was defined as follows: An intuitionistic fuzzy set in a ring R, is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in R\}$, where μ_A and ν_A are fuzzy sets in R and denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in R$ to the set A respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in R$. By IFS(R) we denote the set of all IFSs of R.

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs of R. Then

(1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in R$;
(2) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x)), \nu_A(x) \vee \nu_B(x)) \mid x \in R\}$;
(3) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)) \mid x \in R\}$.

For our convenience we shall use the notation $A(x) \geq B(x)$, when $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for all $x \in R$.

Definition 2.4. [22] *Let* $A = (\mu_A, \nu_A)$ *be an intuitionistic fuzzy set in R. Then, A is said to be an intuitionistic fuzzy ideal with thresholds* (α, β) *of R, if it satisfies the following properties:*

(1) $\mu_A(x+y) \vee \alpha \geq (\mu_A(x) \wedge \mu_A(y)) \wedge \beta$;
(2) $\mu_A(-x) \vee \alpha \geq \mu_A(x) \wedge \beta$;
(3) $\mu_A(xy) \vee \alpha \geq (\mu_A(x) \vee \mu_A(y)) \wedge \beta$;
(4) $\nu_A(x+y) \wedge (1-\alpha) \leq (\nu_A(x) \vee \nu_A(y)) \vee (1-\beta)$;

(5) $\nu_A(-x)\wedge(1-\alpha)\leq \nu_A(x)\vee(1-\beta)$;
(6) $\nu_A(xy)\wedge(1-\alpha)\leq(\nu_A(x)\wedge\nu_A(y))\vee(1-\beta)$.

for all $x,y\in R$, where $\alpha,\beta\in[0,1]$.

Definition 2.5. [25] *Let $A=(\mu_A,\nu_A)$ be an IFSs of R, and $a\in[0,1]$. Then*

(1)

$$A_a(x)=\begin{cases}1, & \text{if } \mu_A(x)\geq a\\ \frac{1}{2}, & \text{if } \mu_A(x)<a\leq 1-\nu_A(x)\\ 0, & \text{for } a>1-\nu_A(x)\end{cases}$$

and

$$A_{\underline{a}}(x)=\begin{cases}1, & \text{if } \mu_A(x)>a\\ \frac{1}{2}, & \text{if } \mu_A(x)\leq a<1-\nu_A(x)\\ 0, & \text{for } a\geq 1-\nu_A(x)\end{cases}$$

are called the a-upper cut set and a- strong upper cut set of A, respectively.

(2)

$$A^a(x)=\begin{cases}1, & \text{if } \nu_A(x)\geq a\\ \frac{1}{2}, & \text{if } \nu_A(x)<a\leq 1-\mu_A(x)\\ 0, & \text{for } a>1-\mu_A(x)\end{cases}$$

and

$$A^{\underline{a}}(x)=\begin{cases}1, & \text{if } \nu_A(x)>a\\ \frac{1}{2}, & \text{if } \nu_A(x)\leq a<1-\mu_A(x)\\ 0, & \text{for } a\geq 1-\mu_A(x)\end{cases}$$

are called the a-lower cut set and a- strong lower cut set of A, respectively.

(3)

$$A_{[a]}(x)=\begin{cases}1, & \text{if } \mu_A(x)+a\geq 1\\ \frac{1}{2}, & \text{if } \nu_A(x)\leq a<1-\mu_A(x)\\ 0, & \text{for } a<\nu_A(x)\end{cases}$$

and

$$A_{[\underline{a}]}(x)=\begin{cases}1, & \text{if } \mu_A(x)+a>1\\ \frac{1}{2}, & \text{if } \nu_A(x)<a\leq 1-\mu_A(x)\\ 0, & \text{for } a\leq\nu_A(x)\end{cases}$$

are called the a-upper Q-cut set and a- strong upper Q-cut set of A, respectively.

(4)

$$A^{[a]}(x)=\begin{cases}1, & \text{if } \nu_A(x)+a\geq 1\\ \frac{1}{2}, & \text{if } \mu_A(x)\leq a<1-\nu_A(x)\\ 0, & \text{for } a<\mu_A(x)\end{cases}$$

and

$$A^{[\underline{a}]}(x) = \begin{cases} 1, & \text{if } \nu_A(x) + a > 1 \\ \frac{1}{2}, & \text{if } \mu_A(x) < a \le 1 - \nu_A(x) \\ 0, & \text{for } a \le \mu_A(x) \end{cases}$$

are called the a-lower Q-cut set and a-strong lower Q-cut set of A, respectively.

Definition 2.6. [23] *Let $A = (\mu_A, \nu_A)$ be an IFSs of R, and $a \in [0,1], x \in R$. Then*

(1) *The grades of $x_a \in A$ and $x_a qA$ denoted by $[x_a \in A]$ and $[x_a qA]$ respectively are given by the following relations:*

$$[x_a \in A] = A_a(x) \text{ and } [x_a qA] = A_{[\underline{a}]}(x).$$

(2) *The grades of $x_a \in \wedge qA$ and $x_a \in \vee qA$ denoted by $[x_a \in \wedge qA]$ and $[x_a \in \vee qA]$ respectively are given by the following relations:*

$$[x_a \in \wedge qA] = [x_a \in A] \wedge [x_a qA] = A_a(x) \wedge A_{[\underline{a}]}(x)$$

and

$$[x_a \in \vee qA] = [x_a \in A] \vee [x_a qA] = A_a(x) \vee A_{[\underline{a}]}(x).$$

(3) *The grades of $x_a \overline{\in} A$ and $x_a \overline{q} A$ denoted by $[x_a \overline{\in} A]$ and $[x_a \overline{q} A]$ respectively are given by the following relations:*

$$[x_a \overline{\in} A] = A^a(x) \text{ and } [x_a \overline{q} A] = A^{[\underline{a}]}(x).$$

(4) *The grades of $x_a \overline{\in \wedge q} A$ and $x_a \overline{\in \vee q} A$ denoted by $[x_a \overline{\in \wedge q} A]$ and $[x_a \overline{\in \vee q} A]$ respectively are given by the following relations:*

$$[x_a \overline{\in \wedge q} A] = [x_a \overline{\in} \vee \overline{q} A] = [x_a \overline{\in} A] \vee [x_a \overline{q} A] = A^a(x) \vee A^{[\underline{a}]}(x)$$

and

$$[x_a \overline{\in \vee q} A] = [x_a \overline{\in} \wedge \overline{q} A] = [x_a \overline{\in} A] \wedge [x_a \overline{q} A] = A^a(x) \wedge A^{[\underline{a}]}(x).$$

Table 1. The table of truth value of Lukasiewicz implication.

$\to$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

As in [23] we have

(1) $[x_a \overline{\in} A] = [x_a \in A^c]$, $[x_a \overline{q} A] = [x_a qA^c]$.
(2) $[x_a \overline{\in} \wedge \overline{q} A] = [x_a \in \wedge qA^c]$, $[x_a \overline{\in} \vee \overline{q} A] = [x_a \in \vee qA^c]$.
(3) $[x_a \in (\bigcap_{t \in T} A_t)] = \bigwedge_{t \in T} [x_a \in A]$, $[x_a q(\bigcup_{t \in T} A_t)] = \bigvee_{t \in T} [x_a qA]$.
(4) $[x_a \overline{\in} (\bigcup_{t \in T} A_t)] = \bigwedge_{t \in T} [x_a \overline{\in} A]$, $[x_a \overline{q} (\bigcap_{t \in T} A_t)] = \bigvee_{t \in T} [x_a \overline{q} A]$.

In the next section we present our main results.

3. Main results

Let R be a ring and $\alpha, \beta \in \{\in, q, \in \wedge q, \in \vee q\}$. Then, for $a \in [0,1]$, $x \in R$, x_a is a fuzzy point and $[x_a \alpha A], [x_a \beta A] \in \{0, 1/2, 1\}$.

Definition 3.1. *Let R be a ring and $A = (\mu_A, \nu_A)$ be an IF set in R. If for any $\alpha, \beta \in \{\in, q, \in \wedge q, \in \vee q\}$, $s, t \in (0,1]$, and $x, y \in R$, the following conditions are satisfied:*

(1) $([x_s\alpha A]\wedge[y_t\alpha A]\rightarrow[(x_s+y_t)\beta A])=1$;
(2) $([x_s\alpha A]\rightarrow[-x_s\beta A])=1$;
(3) $([x_s\alpha A]\wedge[y_t\alpha A]\rightarrow[(x_sy_t)\beta A])=1$; *then A is called a* (α,β)*-intuitionistic fuzzy subring of R, where* $(x_s+y_t)=(x+y)_{s\wedge t}$, $-x_s=(-x)_s$, *and* $(x_sy_t)=(xy)_{s\wedge t}$.

It is to note that, for $p,q\in\{0,1/2,1\}$, we have from Table1, $(p\rightarrow q)=1\Leftrightarrow q\geq p$. Therefore, Definition 3.1 is equivalent to the following definition.

Definition 3.2. *Let R be a ring and* $A=(\mu_A,\nu_A)$ *be an IF set in R. If for any* $\alpha,\beta\in\{\in,q,\in\wedge q,\in\vee q\}$, $s,t\in(0,1]$, *and* $x,y\in R$, *the following conditions are satisfied*

(1) $[(x_s+y_t)\beta A]\geq[x_s\alpha A]\wedge[y_t\alpha A]$;
(2) $[-x_s\beta A]\geq[x_s\alpha A]$;
(3) $[(x_sy_t)\beta A]\geq[x_s\alpha A]\wedge[y_t\alpha A]$;

then A is called a (α,β)*- intuitionistic fuzzy subring of R, where* $(x_s+y_t)=(x+y)_{s\wedge t}$, $-x_s=(-x)_s$, *and* $(x_sy_t)=(xy)_{s\wedge t}$.

Definition 3.3. *Let R be a ring and* $A=(\mu_A,\nu_A)$ *be an IF set in R. If for any* $\alpha,\beta\in\{\in,q,\in\wedge q,\in\vee q\}$, $s,t\in(0,1]$, *and* $x,y\in R$, *the following conditions are satisfied*

(1) $([x_s\alpha A]\wedge[y_t\alpha A]\rightarrow[(x_s+y_t)\beta A])=1$;
(2) $([x_s\alpha A]\rightarrow[-x_s\beta A])=1$;
(3) $([x_s\alpha A]\vee[y_t\alpha A]\rightarrow[(x_sy_t)\beta A])=1$;

then A is called a (α,β)*- intuitionistic fuzzy ideal of R, where* $(x_s+y_t)=(x+y)_{s\wedge t}$, $-x_s=(-x)_s$, *and* $(x_sy_t)=(xy)_{s\vee t}$.

This is equivalent to:

Definition 3.4. *Let R be a ring and* $A=(\mu_A,\nu_A)$ *be an IF set in R. If for any* $\alpha,\beta\in\{\in,q,\in\wedge q,\in\vee q\}$, $s,t\in(0,1]$, *and* $x,y\in R$, *the following conditions are satisfied:*

(1) $[(x_s+y_t)\beta A]\geq[x_s\alpha A]\wedge[y_t\alpha A]$;
(2) $[-x_s\beta A]\geq[x_s\alpha A]$;
(3) $[(x_sy_t)\beta A]\geq[x_s\alpha A]\vee[y_t\alpha A]$;

then A is called a (α,β)*- intuitionistic fuzzy ideal of R, where* $(x_s+y_t)=(x+y)_{s\wedge t}$, $-x_s=(-x)_s$, *and* $(x_sy_t)=(xy)_{s\vee t}$.

Example 3.1. Consider the ring $R=\mathbb{Z}_4=\{0,1,2,3\}$, where operations are addition modulo 4 and multiplication modulo 4. Let $A=\{0,2\}$. Then, A is an ideal of R. We consider the following IFS of R

$$\mu_A(x)=\begin{cases}0.4, & \text{if } x\in A\\ 0.2, & \text{for } x\notin A\end{cases}$$

and

$$\nu_A(x)=\begin{cases}0.2, & \text{if } x\in A\\ 0.7, & \text{for } x\notin A.\end{cases}$$

Then, we can verify that $A=(\mu_A,\nu_A)$ is both $(\in,\in)$ and $(\in,\in\vee q)$-IF ideal of R. Also, we consider A, defined as follows:

$$\mu_A(x)=\begin{cases}0.7, & \text{if } x\in A\\ 0.2, & \text{for } x\notin A\end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.2, & \text{if } x \in A \\ 0.6, & \text{for } x \notin A. \end{cases}$$

Then, it can be easily verified that $A = (\mu_A, \nu_A)$ is a $(\in \wedge q, \in)$-IF ideal of R. However, $A = (\mu_A, \nu_A)$ is not a (q,q)-IF ideal of R, because if take $x \in A$, $y \notin A$ and $s = 0.4, t = 0.85$, then $x+y \notin A$ and $[x_s qA] \wedge [y_t qA] = 1$ but $[(x_s + y_t)qA] < 1$. Again, if we take $\mu_A(x) = 0.4$ and $\nu_A(x) = 0.6$ for all $x \in R$, then $A = (\mu_A, \nu_A)$ is a (q,q)-IF ideal of R. We note that, in this case A is not a $(\in, \in)$-IF ideal of R.

Example 3.2. Consider the ring $R = \{0, a, b, c\}$ with addition and multiplication operations defined as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

and

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	b
c	0	0	b	b

Take $\mu_A(0) = r$, $\mu_A(a) = r$, $\mu_A(b) = s$, $\mu_A(c) = s$ and $\nu_A(0) = 1-t$, $\nu_A(a) = 1-t$, $\nu_A(b) = w$, $\nu_A(c) = w$, where $0 < s < t < 1$, $r \in [0, s)$ and $w \in [0, 1-t]$. Then, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideal with thresholds (s,t) of R. However, if we take $x = b, y = b, \alpha = \in, \beta = \in$ and let $p, q \in [0,1]$ be such that $[x_p \alpha A] \wedge [y_q \alpha A] = 1$, then we have $s \geq p, s \geq q$. Thus, $s \geq p \wedge q$. Since $x + y = 0$ so we have $\mu_A(x+y) = r < s$. Now if A is a $(\in, \in)$-intuitionistic fuzzy ideal of R, then $[(x_p + y_q)\beta A] \geq [x_p \alpha A] \wedge [y_q \alpha A]$ implies $r \geq p \wedge q$, which will lead to a contradiction if we choose $r < p, q < s$. Therefore, A is not a $(\in, \in)$-IF ideal of R. Here, we note that A is not an intuitionistic fuzzy ideal of R with thresholds $(0,1)$.

Definition 3.5. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in R. Then, by the support of A, we mean a crisp subset, A^* of R, and it is defined as follows:*

$$A^* = \{x \in R \mid \mu_A(x) \vee (1 - \nu_A(x)) > 0\}$$

That is , $A^* = \{x \in R \mid A_{\underline{0}}(x) > 0\}$.

Definition 3.6. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in R and $\alpha \in [0,1]$. Then, by a α-level set of A, we mean a crisp subset, $A_{\overline{\alpha}}$ of R, and it is defined as follows:*

$$A_{\overline{\alpha}} = \{x \in R \mid [x_\alpha \in A] > 0\}$$

Theorem 3.1. *Let $A = (\mu_A, \nu_A)$ be a non-zero (i.e.$A \neq (0,1)$) (α, β)-intuitionistic fuzzy ideal of R. If $\alpha \neq \in \wedge q$, then $A_{\underline{0}}$ is a fuzzy ideal of R.*

Proof. We show

(1) $A_{\underline{0}}(x+y) \geq A_{\underline{0}}(x) \wedge A_{\underline{0}}(y)$,
(2) $A_{\underline{0}}(-x) \geq A_{\underline{0}}(x)$,

(3) $A_{\underline{0}}(xy) \geq A_{\underline{0}}(x) \vee A_{\underline{0}}(y)$.

Since $(R,+)$ is a group so, (1) and (2) follow from Theorem 4.1 of [23], because A is also a (α,β)-intuitionistic fuzzy subgroup of $(R,+)$.

(I) For (3), first we claim that, $A_{\underline{0}}(x) \vee A_{\underline{0}}(y) = 1 \Rightarrow A_{\underline{0}}(xy) = 1$. Let $A_{\underline{0}}(x) \vee A_{\underline{0}}(y) = 1$. Then, $A_{\underline{0}}(x) = 1$ or $A_{\underline{0}}(y) = 1$, $\Rightarrow \mu_A(x) > 0$ or $\mu_A(y) > 0$. Put $t = \mu_A(x) \vee \mu_A(y)$, then $t > 0$. Therefore, we must have $s \in (0,1)$ such that $0 < 1-s < t = \mu_A(x) \vee \mu_A(y)$. Now, we have

$$
\begin{aligned}
& t = \mu_A(x) \vee \mu_A(y), \\
& \Rightarrow \text{ either } \mu_A(x) = t \text{ or } \mu_A(y) = t, \\
& \Rightarrow \text{ either } A_t(x) = 1 \text{ or } A_t(y) = 1, \\
& \Rightarrow \text{ either } [x_t \in A] = 1 \text{ or } [y_t \in A] = 1, \text{ and} \\
& 1-s < t = \mu_A(x) \vee \mu_A(y), \\
& \Rightarrow \text{ either } 1-s < \mu_A(x) \text{ or } 1-s < \mu_A(y), \\
& \Rightarrow \text{ either } A_{[\underline{s}]}(x) = 1 \text{ or } A_{[\underline{s}]}(y) = 1, \\
& \Rightarrow \text{ either } [x_s q A] = 1 \text{ or } [y_s q A] = 1.
\end{aligned}
$$

Now,

(i) if $\alpha = \in$, then for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from (3) of Definition 3.3
$$1 \geq [(x_t y_t)\beta A] \geq [x_t \alpha A] \vee [y_t \alpha A] = [x_t \in A] \vee [y_t \in A] = 1,$$
because $[x_t \in A] = 1$ or $[y_t \in A] = 1$. Therefore, $[(xy)_t \beta A] = 1 \Rightarrow$ either $A_t(xy) = 1$ or $A_{[\underline{t}]}(xy) = 1 \Rightarrow$ either $\mu_A(xy) \geq t > 0$ or $\mu_A(xy) > 1-t \geq 0 \Rightarrow \mu_A(xy) > 0 \Rightarrow A_{\underline{0}}(xy) = 1$.

(ii) if $\alpha = \in \vee q$, then for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from (3) of Definition 3.3 $1 \geq [(x_t y_t)\beta A] \geq [x_t \alpha A] \vee [y_t \alpha A] = [x_t \in \vee q A] \vee [y_t \in \vee q A] = [x_t \in A] \vee [x_t q A] \vee [y_t \in A] \vee [y_t q A] = 1$, because $[x_t \in A] = 1$ or $[y_t \in A] = 1$. Therefore, $[(xy)_t \beta A] = 1$,
$\Rightarrow$ *either* $A_t(xy) = 1$ or $A_{[\underline{t}]}(xy) = 1$;
$\Rightarrow$ *either* $\mu_A(xy) \geq t > 0$ or $\mu_A(xy) > 1-t \geq 0$;
$\Rightarrow \mu_A(xy) > 0 \Rightarrow A_{\underline{0}}(xy) = 1$.

(iii) if $\alpha = q$, then for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from (3) of Definition 3.3 $1 \geq [(x_s y_s)\beta A] \geq [x_s \alpha A] \vee [y_s \alpha A] = [x_s q A] \vee [y_s q A] = 1$, because $[x_a q A] = 1$ or $[y_s q A] = 1$. Therefore, $[(xy)_s \beta A] = 1 \Rightarrow$ either $A_s(xy) = 1$ or $A_{[\underline{s}]}(xy) = 1 \Rightarrow$ either $\mu_A(xy) \geq s > 0$ or $\mu_A(xy) > 1-s \geq 0 \Rightarrow \mu_A(xy) > 0 \Rightarrow A_{\underline{0}}(xy) = 1$.

(II) Next we show, $A_{\underline{0}}(x) \vee A_{\underline{0}}(y) = 1/2 \Rightarrow A_{\underline{0}}(xy) \geq 1/2$. Let $A_{\underline{0}}(x) \vee A_{\underline{0}}(y) = 1/2$. Then, $A_{\underline{0}}(x) = 1/2$ or $A_{\underline{0}}(y) = 1/2 \Rightarrow \nu_A(x) < 1$ or $\nu_A(y) < 1 \Rightarrow \nu_A(x) \wedge \nu_A(y) < 1$. So, there exists $s,t \in (0,1)$ such that $\nu_A(x) \wedge \nu_A(y) < 1-t < s < 1$. Then

$$
\begin{aligned}
& 0 < t < 1 - \nu_A(x) \wedge \nu_A(y) = (1-\nu_A(x)) \vee (1-\nu_A(y)), \\
& \Rightarrow \text{ either } \mu_A(x) = 0 < t < 1-\nu_A(x) \text{ or } \mu_A(y) = 0 < t < 1-\nu_A(y), \\
& \Rightarrow \text{ either } A_t(x) = 1/2 \text{ or } A_t(y) = 1/2, \\
& \Rightarrow \text{ either } [x_t \in A] = 1/2 \text{ or } [y_t \in A] = 1/2, \text{ and} \\
& \nu_A(x) \wedge \nu_A(y) < s < 1, \\
& \Rightarrow \text{ either } \nu_A(x) < s \leq 1 = 1-0 = 1-\mu_A(x) \text{ or } \nu_A(y) < s \leq 1 = 1-0 = 1-\mu_A(y),
\end{aligned}
$$

$\Rightarrow$ either $A_{[\underline{s}]}(x) = 1/2$ or $A_{[\underline{s}]}(y) = 1/2$,

$\Rightarrow$ either $[x_s qA] = 1/2$ or $[y_s qA] = 1/2$.

Now,

(i) if $\alpha = \in$, then for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$ we have from (3) of Definition 3.3

$$[(x_t y_t)\beta A] \geq [x_t \alpha A] \vee [y_t \alpha A] = [x_t \in A] \vee [y_t \in A] = 1/2,$$

because $[x_t \in A] = 1/2$ or $[y_t \in A] = 1/2$. Therefore, $[(xy)_t \beta A] \geq 1/2 \Rightarrow$ either $A_t(xy) \geq 1/2$ or $A_{[\underline{t}]}(xy) \geq 1/2 \Rightarrow$ either $\nu_A(xy) \leq 1-t < 1-0$ or $\nu_A(xy) < t < 1-0$ $\Rightarrow \nu_A(xy) < 1-0 \Rightarrow A_{\underline{0}}(xy) \geq 1/2$.

(ii) if $\alpha = \in \vee q$, then for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, we have from (3) of Definition 3.3

$$\begin{aligned}[(x_t y_t)\beta A] \geq [x_t \alpha A] \vee [y_t \alpha A] &= [x_t \in \vee qA] \vee [y_t \in \vee qA] \\ &= [x_t \in A] \vee [x_t qA] \vee [y_t \in A] \vee [y_t qA] \geq 1/2,\end{aligned}$$

because $[x_t \in A] = 1/2$ or $[y_t \in A] = 1/2$. Therefore, $[(xy)_t \beta A] \geq 1/2$ whence $A_{\underline{0}}(xy) \geq 1/2$.

(iii) if $\alpha = q$, then for $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, we have from (3) of Definition 3.3

$$[(x_s y_s)\beta A] \geq [x_s \alpha A] \vee [y_s \alpha A] = [x_s qA] \vee [y_s qA] = 1/2,$$

because $[x_a qA] = 1/2$ or $[y_s qA] = 1/2$. Therefore, $[(xy)_s \beta A] \geq 1/2 \Rightarrow$ either $A_s(xy) \geq 1/2$ or $A_{[\underline{s}]}(xy) \geq 1/2 \Rightarrow$ either $\nu_A(xy) \leq 1-s < 1$ or $\nu_A(xy) < s < 1 \Rightarrow \nu_A(xy) < 1 \Rightarrow A_{\underline{0}}(xy) \geq 1/2$.

Also, if $A_{\underline{0}}(x) \vee A_{\underline{0}}(y) = 0$, then obviously $A_{\underline{0}}(xy) \geq 0$. Thus, in all cases we have $A_{\underline{0}}(xy) \geq A_{\underline{0}}(x) \vee A_{\underline{0}}(y)$. ∎

Theorem 3.2. *Let $A = (\mu_A, \nu_A)$ be a non-zero (α, β)-intuitionistic fuzzy ideal of R. If $\alpha \neq \in \wedge q$, then the support A^* is an ideal of R.*

Proof. Let $x, y \in A^*$ and $r \in R$. Then, $A_{\underline{0}}(x) > 0$ and $A_{\underline{0}}(y) > 0$. From Theorem 3.1, we have $A_{\underline{0}}(x+y) \geq A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) > 0$. Thus, $x+y \in A^*$. Similarly, $-x \in A^*$. Also, $A_{\underline{0}}(xr) \geq A_{\underline{0}}(x) \vee A_{\underline{0}}(r) > 0$, because $A_{\underline{0}}(x) > 0$ and so $xr \in A^*$. Similarly, $ry \in A^*$. Hence, A^* is an ideal of R. ∎

Theorem 3.3. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy ideal with thresholds (s,t) of R. Then, for any $p \in (s,t]$, $A_{\overline{p}}$ is an ideal of R.*

Proof. Let $x, y \in A_{\overline{p}} = \{x \in R \mid [x_p \in A] > 0\}$. Then, $[x_p \in A] > 0$ and $[y_p \in A] > 0$, which implies that $p \leq 1 - \nu_A(x)$ and $p \leq 1 - \nu_A(y)$. Now, $\nu_A(x+y) \wedge (1-s) \leq (\nu_A(x) \vee \nu_A(y)) \vee (1-t)$, implies $(1 - \nu_A(x+y)) \vee s \geq (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \wedge t \geq p \wedge p \wedge t = p$. Thus, $1 - \nu_A(x+y) \geq p$, and so $[(x+y)_p \in A] \geq 1/2 > 0$. Therefore, $x+y \in A_{\overline{p}}$. Similarly, $-x \in A_{\overline{p}}$. Let $r \in R$. Now, $\nu_A(xr) \wedge (1-s) \leq (\nu_A(x) \wedge \nu_A(r)) \vee (1-t)$, implies $(1 - \nu_A(xr)) \vee s \geq ((1 - \nu_A(x)) \vee (1 - \nu_A(r))) \wedge t \geq (p \vee (1 - \nu_A(r))) \wedge t \geq p \wedge t = p$. Thus, $1 - \nu_A(xr) \geq p$, and so $[(xr)_p \in A] \geq 1/2 > 0$. Therefore, $xr \in A_{\overline{p}}$. Similarly, we have $rx \in A_{\overline{p}}$. Hence, $A_{\overline{p}}$ is an ideal of R. ∎

Theorem 3.4. *An IFS $A = (\mu_A, \nu_A)$ of R is a $(\in, \in)$-intuitionistic fuzzy ideal of R if and only if A is an intuitionistic fuzzy ideal of R with thresholds $(0,1)$.*

Proof. Suppose that $A = (\mu_A, \nu_A)$ is a $(\in, \in)$-intuitionistic fuzzy ideal of R. To show A is an intuitionistic fuzzy ideal of R with thresholds $(0,1)$ i.e. to show

(1) $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$;
(2) $\mu_A(-x) \geq \mu_A(x)$;
(3) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$;
(4) $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$;
(5) $\nu_A(-x) \leq \nu_A(x)$;
(6) $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$, for all $x, y \in R$.

For (1), let $t = \mu_A(x) \wedge \mu_A(y)$. Then, $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$, which implies that $A_t(x) = 1$ and $A_t(y) = 1$, and so $[x_t \in A] = 1$ and $[y_t \in A] = 1$. Now $1 \geq [(x_t + y_t) \in A] \geq [x_t \in A] \wedge [y_t \in A] = 1 \Rightarrow [(x_t + y_t) \in A] = 1 \Rightarrow \mu_A(x+y) \geq t = \mu_A(x) \wedge \mu_A(y)$.

In a similar manner we can prove (2).

(3) Let $t = \mu_A(x) \vee \mu_A(y)$, then either $\mu_A(x) = t$ or $\mu_A(y) = t$, which implies either $A_t(x) = 1$ or $A_t(y) = 1$, and so either $[x_t \in A] = 1$ or $[y_t \in A] = 1$. Now $1 \geq [(x_t y_t) \in A] \geq [x_t \in A] \vee [y_t \in A] = 1 \Rightarrow [(xy)_t \in A] = 1 \Rightarrow \mu_A(xy) \geq t = \mu_A(x) \vee \mu_A(y)$.

(4) If $\nu_A(x+y) = 0$, then it is obvious. Let $s = \nu_A(x+y) > 0$ and let $t \in [0,1]$ be such that $t > 1 - s = 1 - \nu_A(x+y)$, then we have $0 = [(x_t + y_t) \in A] \geq [x_t \in A] \wedge [y_t \in A] \Rightarrow [x_t \in A] \wedge [y_t \in A] = 0 \Rightarrow [x_t \in A] = 0$ or $[y_t \in A] = 0$ i.e., either $t > 1 - \nu_A(x)$ or $t > 1 - \nu_A(y) \Rightarrow$ either $\nu_A(x) > 1 - t$ or $\nu_A(y) > 1 - t \Rightarrow \nu_A(x) \vee \nu_A(y) > 1 - t$. Therefore, $\nu_A(x) \vee \nu_A(y) \geq \vee\{1 - t \mid t > 1 - s\} = \vee\{1 - t \mid s > 1 - t\} = s = \nu_A(x+y)$. Thus, $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$.

Similarly, we have (5).

Lastly, if $\nu_A(xy) = 0$, then it is obvious. Let $s = \nu_A(xy) > 0$ and let $t \in [0,1]$ be such that $t > 1 - s = 1 - \nu_A(xy)$, then we have $0 = [(x_t y_t) \in A] \geq [x_t \in A] \vee [y_t \in A] \Rightarrow [x_t \in A] \vee [y_t \in A] = 0 \Rightarrow [x_t \in A] = 0$ and $[y_t \in A] = 0$ i.e., $t > 1 - \nu_A(x)$ and $t > 1 - \nu_A(y) \Rightarrow \nu_A(x) > 1 - t$ and $\nu_A(y) > 1 - t \Rightarrow \nu_A(x) \wedge \nu_A(y) > 1 - t$. Therefore, $\nu_A(x) \wedge \nu_A(y) \geq \vee\{1 - t \mid t > 1 - s\} = \vee\{1 - t \mid s > 1 - t\} = s = \nu_A(xy)$. Thus, $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Conversely, we assume A is an intuitionistic fuzzy ideal of R with thresholds $(0,1)$. We need to show $A = (\mu_A, \nu_A)$ is a $(\in, \in)$-intuitionistic fuzzy ideal of R. Let $x, y \in R$ and $s, t \in (0,1]$.

Let $a = [x_s \in A] \wedge [y_t \in A]$.

Case I. $a = 1$. Then, $[x_s \in A] = 1$ and $[y_t \in A] = 1 \Rightarrow \mu_A(x) \geq s$ and $\mu_A(y) \geq t \Rightarrow \mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t \Rightarrow [(x_s + y_t) \in A] = 1 \geq 1 = [x_s \in A] \wedge [y_t \in A]$.

Case II. $a = 1/2$. Then, $[x_s \in A] \geq 1/2$ and $[y_t \in A] \geq 1/2 \Rightarrow 1 - \nu_A(x) \geq s$ and $1 - \nu_A(y) \geq t \Rightarrow 1 - \nu_A(x+y) \geq 1 - \nu_A(x) \vee \nu_A(y) = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t \Rightarrow [(x_s + y_t) \in A] \geq 1/2 = [x_s \in A] \wedge [y_t \in A]$.

Case III. $a = 0$. Then, the result is obvious. Thus, in all cases we have $[(x_s + y_t) \in A] \geq [x_s \in A] \wedge [y_t \in A]$. In a similar manner we can prove that $[-x_s \in A] \geq [x_s \in A]$.

Let $b = [x_s \in A] \vee [y_t \in A]$.

Case I. $b = 1$. Then, either $[x_s \in A] = 1$ or $[y_t \in A] = 1 \Rightarrow$ either $\mu_A(x) \geq s$ or $\mu_A(y) \geq t \Rightarrow \mu_A(xy) \geq \mu_A(x) \vee \mu_A(y) \geq s \vee t \Rightarrow [(x_s y_t) \in A] = 1 \geq 1 = [x_s \in A] \vee [y_t \in A]$.

Case II. $b = 1/2$. Then, either $[x_s \in A] = 1/2$ or $[y_t \in A] = 1/2 \Rightarrow$ either $1 - \nu_A(x) \geq s$ or $1 - \nu_A(y) \geq t \Rightarrow 1 - \nu_A(xy) \geq 1 - \nu_A(x) \wedge \nu_A(y) = (1 - \nu_A(x)) \vee (1 - \nu_A(y)) \geq s \vee t \Rightarrow [(x_s y_t) \in A] \geq 1/2 = [x_s \in A] \vee [y_t \in A]$. Hence, A is a $(\in, \in)$-intuitionistic fuzzy ideal of R. ∎

As a consequence of Theorem 3.3 and Theorem 3.4, we have the following:

Theorem 3.5. *If an IFS $A = (\mu_A, \nu_A)$ of R is a $(\in, \in)$-intuitionistic fuzzy ideal of R, then for any $p \in (0,1]$, $A_{\overline{p}}$ is an ideal of R.*

Theorem 3.6. *An IFS $A = (\mu_A, \nu_A)$ of R is a $(\in, \in \vee q)$-intuitionistic fuzzy ideal of R if and only if A is an intuitionistic fuzzy ideal of R with thresholds $(0, 0.5)$.*

Proof. Suppose that $A = (\mu_A, \nu_A)$ is a $(\in, \in \vee q)$-intuitionistic fuzzy ideal of R. To show A is an intuitionistic fuzzy ideal of R with thresholds $(0, 0.5)$ i.e. to show

(1) $\mu_A(x+y) \geq (\mu_A(x) \wedge \mu_A(y)) \wedge 0.5$;
(2) $\mu_A(-x) \geq \mu_A(x) \wedge 0.5$;
(3) $\mu_A(xy) \geq (\mu_A(x) \vee \mu_A(y)) \wedge 0.5$;
(4) $\nu_A(x+y) \leq (\nu_A(x) \vee \nu_A(y)) \vee 0.5$;
(5) $\nu_A(-x) \leq \nu_A(x) \vee 0.5$;
(6) $\nu_A(xy) \leq (\nu_A(x) \wedge \nu_A(y)) \vee 0.5$, for all $x, y \in R$.

For (1), let $t = (\mu_A(x) \wedge \mu_A(y)) \wedge 0.5$, then $\mu_A(x) \geq t$, $\mu_A(y) \geq t \Rightarrow [x_t \in A] = 1$, $[y_t \in A] = 1$. Therefore, from (1) of Definition 3.4 we have $1 \geq [(x_t + y_t) \in \vee qA] \geq [x_t \in A] \wedge [y_t \in A] = 1$. Thus, $[(x_t + y_t) \in \vee qA] = 1$,

$\Rightarrow [(x_t + y_t) \in A] \vee [(x_t + y_t)qA] = 1$,
$\Rightarrow [(x_t + y_t) \in A] = 1$ or $[(x_t + y_t)qA] = 1$,
$\Rightarrow \mu_A(x+y) \geq t$ or $\mu_A(x+y) + t > 1$,
$\Rightarrow \mu_A(x+y) \geq t$ or $\mu_A(x+y) > 1 - t \geq 0.5 \geq t$,
$\Rightarrow \mu_A(x+y) \geq t = (\mu_A(x) \wedge \mu_A(y)) \wedge 0.5$.

Similarly, we can prove (2).

(3)Let $t = (\mu_A(x) \vee \mu_A(y)) \wedge 0.5 = (\mu_A(x) \wedge 0.5) \vee (\mu_A(y) \wedge 0.5)$. This implies that $(\mu_A(x) \wedge 0.5) = t$ or $(\mu_A(y) \wedge 0.5) = t \Rightarrow \mu_A(x) \geq t$ or $\mu_A(y) \geq t \Rightarrow [x_t \in A] = 1$ or $[y_t \in A] = 1$. Therefore, from (3) of Definition 3.4 we have
$1 \geq [(x_t y_t) \in \vee qA] \geq [x_t \in A] \vee [y_t \in A] = 1$. Thus, $[(x_t y_t) \in \vee qA] = 1$,

$\Rightarrow [(x_t y_t) \in A] = 1$ or $[(x_t y_t)qA] = 1$,
$\Rightarrow \mu_A(xy) \geq t$ or $\mu_A(xy) + t > 1$,
$\Rightarrow \mu_A(xy) \geq t$ or $\mu_A(xy) > 1 - t \geq 0.5 \geq t$,
$\Rightarrow \mu_A(xy) \geq t = (\mu_A(x) \vee \mu_A(y)) \wedge 0.5$.

(4) Let $\nu_A(x) \vee \nu_A(y) \vee 0.5 = 1 - s$. Then, $\nu_A(x) \leq 1 - s$ and $\nu_A(y) \leq 1 - s \Rightarrow s \leq 1 - \nu_A(x)$ and $s \leq 1 - \nu_A(y) \Rightarrow [x_s \in A] \geq 1/2$ and $[y_s \in A] \geq 1/2$. Therefore, from (1) of definition 3.4 we have, $1 \geq [(x_t + y_t) \in \vee qA] \geq [x_t \in A] \wedge [y_t \in A] \geq 1/2$. This implies that $[(x_t + y_t) \in A] \vee [(x_t + y_t)qA] \geq 1/2$,

$\Rightarrow [(x_t + y_t) \in A] \geq 1/2$ or $[(x_t + y_t)qA] \geq 1/2$,
$\Rightarrow$ either $s \leq 1 - \nu_A(x+y)$ or $\nu_A(x+y) < s \leq 1 - s$, [since $1 - s \geq 0.5$ so, $s \leq 0.5$]
$\Rightarrow \nu_A(x+y) \leq 1 - s = \nu_A(x) \vee \nu_A(y) \vee 0.5$.

Similarly, we can prove (5).

(6) Let $(\nu_A(x) \wedge \nu_A(y)) \vee 0.5 = 1 - s$. Then
$1 - (\nu_A(x) \vee 0.5) \wedge (\nu_A(y) \vee 0.5) = s$,
$\Rightarrow (1 - \nu_A(x) \vee 0.5) \vee (1 - \nu_A(y) \vee 0.5) = s$,
$\Rightarrow ((1 - \nu_A(x)) \wedge 0.5) \vee ((1 - \nu_A(y)) \wedge 0.5) = s$,
$\Rightarrow (1 - \nu_A(x)) \wedge 0.5 = s$ or $(1 - \nu_A(y)) \wedge 0.5 = s$,
$\Rightarrow (1 - \nu_A(x)) \geq s$ or $(1 - \nu_A(y)) \geq s$,
$\Rightarrow [x_s \in A] \geq 1/2$ or $[y_s \in A] \geq 1/2$,
$\Rightarrow [x_s y_s \in \vee qA] \geq [x_s \in A] \vee [y_s \in A] \geq 1/2$, [By (3) of Definition 3.4]
$\Rightarrow [x_s y_s \in \vee qA] \geq 1/2$,
$\Rightarrow [x_s y_s \in A] \geq 1/2$ or $[x_s y_s qA] \geq 1/2$,

$\Rightarrow s \leq 1 - \nu_A(xy)$ or $\nu_A(xy) < s \leq 1 - s$, [Since $1 - s \geq 0.5$, so $s \leq 0.5$]
$\Rightarrow \nu_A(xy) \leq 1 - s$ or $\nu_A(xy) \leq 1 - s$,
$\Rightarrow \nu_A(xy) \leq 1 - s = (\nu_A(x) \wedge \nu_A(y)) \vee 0.5$

Conversely, we assume A is an intuitionistic fuzzy ideal of R with thresholds $(0, 0.5)$. We claim A is a $(\in, \in \vee q)$-intuitionistic fuzzy ideal of R. Let $x, y \in R$ and for $s, t \in [0, 1]$, let $a = [x_s \in A] \wedge [y_t \in A]$.

Case I. $a = 1$. Then, $[x_s \in A] = 1$ and $[y_t \in A] = 1$, which implies that $\mu_A(x) \geq s$ and $\mu_A(y) \geq t$.
If $[(x_s + y_t) \in \vee qA] \leq 1/2$, then $\mu_A(x+y) < s \wedge t$ and $\mu_A(x+y) \leq 1 - s \wedge t$. Thus, $0.5 > \mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5$. So, $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t$, a contradiction to $\mu_A(x+y) < s \wedge t$. Thus, we must have $[(x_s + y_t) \in \vee qA] = 1$.

Case II. $a = 1/2$. Then, $[x_s \in A] \geq 1/2$ and $[y_t \in A] \geq 1/2$ which implies that $1 - \nu_A(x) \geq s$ and $1 - \nu_A(y) \geq t$. Now

$$1 - \nu_A(x) \vee \nu_A(y) = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$$

If $[(x_s + y_t) \in \vee qA] = 0$, then $(1 - \nu_A(x+y)) < s \wedge t$ and $\nu_A(x+y) \geq s \wedge t$. Now, from $0.5 < \nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y) \vee 0.5$, we get $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$ and $1 - \nu_A(x+y) \geq 1 - \nu_A(x) \vee \nu_A(y) = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$, which contradicts $(1 - \nu_A(x+y)) < s \wedge t$. Therefore, we must have $[(x_s + y_t) \in \vee qA] \geq 1/2 = [x_s \in A] \wedge [y_t \in A]$.

Case III. $a = 0$. Then, the result is obvious. Thus, in all cases, $[(x_s + y_t) \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A]$.

Similarly, we can prove that $[-x_s \in \vee qA] \geq [x_s \in A]$.

Next, we claim that $[(x_s y_t) \in \vee qA] \geq [x_s \in A] \vee [y_t \in A]$. Let $b = [x_s \in A] \vee [y_t \in A]$.

Case I. $b = 1$. Then, either $[x_s \in A] = 1$ or $[y_t \in A] = 1$, which implies either $\mu_A(x) \geq s$ or $\mu_A(y) \geq t$. If $[x_s y_t \in \vee qA] \leq 1/2$, then $[x_s y_t \in A] \leq 1/2$ and $[x_s y_t qA] \leq 1/2 \Rightarrow \mu_A(xy) < s \vee t$ and $s \vee t \leq 1 - \mu_A(xy) \Rightarrow \mu_A(xy) < s \vee t$ and $\mu_A(xy) \leq 1 - s \vee t$. Now, $0.5 > \mu_A(xy) \geq (\mu_A(x) \vee \mu_A(y)) \wedge 0.5$ implies $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y) \geq s \vee t$, a contradiction to $\mu_A(xy) < s \vee t$. Therefore, we must have $[x_s y_t \in \vee qA] = 1$.

Case II. $b = 1/2$. Then, either $[x_s \in A] = 1/2$ or $[y_t \in A] = 1/2$, which implies either $s \leq 1 - \nu_A(x)$ or $t \leq 1 - \nu_A(y)$. If $[x_s y_t \in \vee qA] = 0$, then $[x_s y_t \in A] = 0$ and $[x_s y_t qA] = 0 \Rightarrow s \vee t > 1 - \nu_A(xy)$ and $s \vee t \leq \nu_A(xy) \Rightarrow \nu_A(xy) > 1 - s \vee t$ and $s \vee t \leq \nu_A(xy) \Rightarrow 0.5 < \nu_A(xy) \leq (\nu_A(x) \wedge \nu_A(y)) \vee 0.5 \Rightarrow \nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$. Now, $1 - \nu_A(xy) \geq 1 - \nu_A(x) \wedge \nu_A(y) = (1 - \nu_A(x)) \vee (1 - \nu_A(y)) \geq s \vee t$, a contradiction to $s \vee t > 1 - \nu_A(xy)$. Therefore, we have $[x_s y_t \in \vee qA] \geq 1/2 = [x_s \in A] \vee [y_t \in A]$. Hence, $[x_s y_t \in \vee qA] \geq [x_s \in A] \vee [y_t \in A]$. ∎

As a consequence of Theorem 3.3 and Theorem 3.6, we have the following:

Theorem 3.7. *If an IFS $A = (\mu_A, \nu_A)$ of R is a $(\in, \in \vee q)$-intuitionistic fuzzy ideal of R, then for any $p \in (0, 0.5]$, $A_{\overline{p}}$ is an ideal of R.*

Theorem 3.8. *An IFS $A = (\mu_A, \nu_A)$ of R is a $(\in \wedge q, \in)$-intuitionistic fuzzy ideal of R if and only if A is an intuitionistic fuzzy ideal of R with thresholds $(0.5, 1)$.*

Proof. Suppose that $A = (\mu_A, \nu_A)$ is a $(\in \wedge q, \in)$-intuitionistic fuzzy ideal of R. To show

(1) $\mu_A(x+y) \vee 0.5 \geq \mu_A(x) \wedge \mu_A(y)$;
(2) $\mu_A(-x) \vee 0.5 \geq \mu_A(x)$;
(3) $\mu_A(xy) \vee 0.5 \geq \mu_A(x) \vee \mu_A(y)$;
(4) $\nu_A(x+y) \wedge 0.5 \leq \nu_A(x) \vee \nu_A(y)$;
(5) $\nu_A(-x) \wedge 0.5 \leq \nu_A(x)$;

(6) $\nu_A(xy) \wedge 0.5 \leq \nu_A(x) \wedge \nu_A(y)$, for all $x, y \in R$.

Let $x, y \in R$ and $t = \mu_A(x) \wedge \mu_A(y)$. If $\mu_A(x+y) \vee 0.5 < t = \mu_A(x) \wedge \mu_A(y)$, then $\mu_A(x) \geq t > 0.5$ and $\mu_A(y) \geq t > 0.5$,

$\Rightarrow [x_t \in A] = 1$, $[x_t qA] = 1$, $[y_t \in A] = 1$, $[y_t qA] = 1$,

$\Rightarrow [x_t \in \wedge qA] = 1$, $[y_t \in \wedge qA] = 1$,

$\Rightarrow [x_t \in \wedge qA] \wedge [y_t \in \wedge qA] = 1$.

Therefore, $[(x_t + y_t) \in A] \geq [x_t \in \wedge qA] \wedge [y_t \in \wedge qA] = 1$, which gives $[(x_t + y_t) \in A] = 1 \Rightarrow \mu_A(x+y) \geq t$, a contradiction to our assumption $\mu_A(x+y) \leq \mu_A(x+y) \vee 0.5 < t$. Therefore, we have $\mu_A(x+y) \vee 0.5 \geq t = \mu_A(x) \wedge \mu_A(y)$.

Similarly, we can prove that $\mu_A(-x) \vee 0.5 \geq \mu_A(x)$.

Next, let $t = \mu_A(x) \vee \mu_A(y)$, then $\mu_A(x) = t$ or $\mu_A(y) = t$. If $\mu_A(xy) \vee 0.5 < t$, then either $\mu_A(x) = t > 0.5$ or $\mu_A(y) = t > 0.5$, which implies that $[x_t \in \wedge qA] = 1$, or $[y_t \in \wedge qA] = 1$. Now

$$[(x_t y_t) \in A] \geq [x_t \in \wedge qA] \vee [y_t \in \wedge qA] = 1$$

From which we get $[(x_t y_t) \in A] = 1 \Rightarrow \mu_A(xy) \geq t$, which contradicts to our assumption $\mu_A(xy) < t$. Therefore, we must have $\mu_A(xy) \vee 0.5 \geq t = \mu_A(x) \vee \mu_A(y)$.

(4) let $t = 1 - s = \nu_A(x) \vee \nu_A(y)$, then $1 - s \geq \nu_A(x), 1 - s \geq \nu_A(y)$. If $\nu_A(x+y) \wedge 0.5 > t$, then we have $s \leq 1 - \nu_A(x)$, $s \leq 1 - \nu_A(y)$, $\nu_A(x+y) > t$ and $s > 0.5 > t$, and so $[x_s \in A] \geq 1/2$, $[y_s \in A] \geq 1/2$, $\nu_A(x+y) > t$ and $s > 0.5 > t$. Also, $\nu_A(x) \leq t < s$ and $\nu_A(y) \leq t < s$ imply $[x_s qA] \geq 1/2$, $[y_s qA] \geq 1/2$. Therefore, from $[(x_s + y_s) \in A] \geq [x_s \in \wedge qA] \wedge [y_s \in \wedge qA] \geq 1/2$ we have $[(x_s + y_s) \in A] \geq 1/2$. This implies that $s \leq 1 - \nu_A(x+y)$, which is a contradiction to $\nu_A(x+y) > t = 1 - s$. Hence, $\nu_A(x+y) \wedge 0.5 \leq t = \nu_A(x) \vee \nu_A(y)$.

Similarly, we can prove that $\nu_A(-x) \wedge 0.5 \leq \nu_A(x)$.

(6) Let $t = 1 - s = \nu_A(x) \wedge \nu_A(y)$. Then

$s = (1 - \nu_A(x)) \vee (1 - \nu_A(y))$,

$\Rightarrow s = 1 - \nu_A(x)$ or $s = 1 - \nu_A(y)$,

$\Rightarrow [x_s \in A] \geq 1/2$ or $[y_s \in A] \geq 1/2$.

If $\nu_A(xy) \wedge 0.5 > t$, then $\nu_A(xy) > t$ and $t < 0.5 < s$. Therefore, $s = 1 - \nu_A(x)$ or $s = 1 - \nu_A(y)$ which implies that $\nu_A(x) = 1 - s = t < s$ or $\nu_A(y) = 1 - s = t < s \Rightarrow [x_s qA] \geq 1/2$ or $[y_s qA] \geq 1/2$. Thus, we have

$[x_s \in A] \geq 1/2$ or $[y_s \in A] \geq 1/2$ and $[x_s qA] \geq 1/2$ or $[y_s qA] \geq 1/2$.

Now if $[x_s \in A] \geq 1/2$ and $[x_s qA] = 0$, then we get $s \leq 1 - \nu_A(x)$ and $s \leq \nu_A(x) \Rightarrow \nu_A(x) \leq 1 - s = t < 0.5 < s$, (since $t < 0.5 < s$), which contradicts to $\nu_A(x) \geq s$.

Therefore, $[x_s \in A] \geq 1/2$ and $[x_s qA] = 0$ can't hold simultaneously.Thus, if $[x_s \in A] \geq 1/2$, then $[x_s qA] \geq 1/2$.

Similarly, if $[y_s \in A] \geq 1/2$, then $[y_s qA] \geq 1/2$.

Again, if $[x_s qA] \geq 1/2$ and $[x_s \in A] = 0$, then we get $\nu_A(x) < s$, $s > 1 - \nu_A(x)$. Therefore, $s > \nu_A(x) > 1 - s$, which is true for all $s > 0.5 > t$. Hence, we must have, $\nu_A(x) = 0.5$. Similarly, if $[y_s qA] \geq 1/2$ and $[y_s \in A] = 0$, then $\nu_A(y) = 0.5$. Now, $t = \nu_A(x) \wedge \nu_A(y) = 0.5$, which contradicts to $t < 0.5$. Therefore, we must have

$[x_s qA] \geq 1/2$ and $[x_s \in A] \geq 1/2$ or $[y_s qA] \geq 1/2$ and $[y_s \in A] \geq 1/2$.

Thus, if $[x_s \in A] \geq 1/2$, then $[x_s qA] \geq 1/2$ and vice versa.

or, if $[y_s \in A] \geq 1/2$, then $[y_s qA] \geq 1/2$ and vice versa.

Thus, in all cases, we have

$[(x_s y_s) \in A] \geq [x_s \in \wedge qA] \vee [y_s \in \wedge qA]$,

$\Rightarrow [(x_s y_s) \in A] \geq (([x_s \in A] \vee [y_s \in A]) \wedge ([x_s qA] \vee [y_s \in A])) \wedge (([x_s \in A] \vee [y_s qA]) \wedge$

$([x_s qA] \vee [y_s qA])) \geq 1/2$,

$\Rightarrow [(x_s y_s) \in A] \geq 1/2 \Rightarrow s \leq 1 - \nu_A(xy)$. Therefore, $\nu_A(xy) \leq 1 - s = t$, which contradicts to $\nu_A(xy) > t$. Hence, $\nu_A(xy) \wedge 0.5 \leq t = \nu_A(x) \wedge \nu_A(y)$.

Conversely, we assume A is an intuitionistic fuzzy ideal with thresholds $(0.5, 1)$. Let $x, y \in R$, $s, t \in [0, 1]$ and $a = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$. Then

Case I. $a = 1$. Then, $\mu_A(x) \geq s, \mu_A(x) + s > 1, \mu_A(y) \geq t, \mu_A(y) + t > 1$. This implies that $\mu_A(x) \geq 0.5$ and $\mu_A(y) \geq 0.5$. Now, we have $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y) \geq s \wedge t$, from which we get $[(x_s + y_t) \in A] = 1$.

Case II. $a = 1/2$. Then, $s \leq 1 - \nu_A(x)$, $\nu_A(x) < s, t \leq 1 - \nu_A(y)$, $\nu_A(y) < t$,

$\Rightarrow 1 - \nu_A(x) \geq s > \nu_A(x)$, $1 - \nu_A(y) \geq t > \nu_A(y)$,

$\Rightarrow \nu_A(x) < 0.5$, $\nu_A(x) < 0.5$.

Therefore, $\nu_A(x+y) \wedge 0.5 \leq \nu_A(x) \vee \nu_A(y) \Rightarrow \nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$ which implies that $1 - \nu_A(x+y) \geq (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$. Thus, $[(x_s + y_t) \in A] \geq 1/2$. Hence, $[(x_s + y_t) \in A] \geq [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Similarly, we can prove that $[-x_s \in A] \geq [x_s \in \wedge qA]$. Next, let $b = [x_s \in \wedge qA] \vee [y_t \in \wedge qA]$.

Case I. $b = 1$. Then, either $\mu_A(x) \geq s, \mu_A(x) + s > 1$ or $\mu_A(y) \geq t, \mu_A(y) + t > 1$. This implies, either $\mu_A(x) \geq 0.5$ or $\mu_A(y) \geq 0.5$. Now,

$\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y) \geq s \vee t$, from which we get $[(x_s y_t) \in A] = 1$.

Case II. $a = 1/2$. Then, either $s \leq 1 - \nu_A(x)$, $\nu_A(x) < s$ or $t \leq 1 - \nu_A(y)$, $\nu_A(y) < t$,

$\Rightarrow 1 - \nu_A(x) \geq s > \nu_A(x)$ or $1 - \nu_A(y) \geq t > \nu_A(y)$,

$\Rightarrow \nu_A(x) < 0.5$ or $\nu_A(x) < 0.5$.

Therefore, $\nu_A(xy) \wedge 0.5 \leq \nu_A(x) \wedge \nu_A(y) \Rightarrow \nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$ which implies that $1 - \nu_A(xy) \geq (1 - \nu_A(x)) \vee (1 - \nu_A(y)) \geq s \vee t$. Thus, $[(x_s y_t) \in A] \geq 1/2$. Hence, $[(x_s y_t) \in A] \geq [x_s \in \wedge qA] \vee [y_t \in \wedge qA]$. Therefore, A is a $(\in \wedge q, \in)$-intuitionistic fuzzy ideal of R. ∎

As a consequence of Theorem 3.3 and Theorem 3.8, we have the following:

Theorem 3.9. *If an IFS $A = (\mu_A, \nu_A)$ of R is a $(\in \wedge q, \in)$-intuitionistic fuzzy ideal of R, then for any $p \in (0.5, 1]$, $A_{\overline{p}}$ is an ideal of R.*

Theorem 3.10. *An intuitionistic fuzzy set, $A = (\mu_A, \nu_A)$ of R is a $(\in, \in)$-intuitionistic fuzzy ideal of R if and only if for any $a \in [0, 1]$, A_a is a fuzzy ideal of R.*

Proof. Suppose that A is a $(\in, \in)$-intuitionistic fuzzy ideal of R. Let $x, y \in R$ and $a \in [0, 1]$. Then

$A_a(x+y) = [(x+y)_a \in A] = [(x_a + y_a) \in A] \geq [x_a \in A] \wedge [y_a \in A] = A_a(x) \wedge A_a(y)$,

$A_a(-x) = [-x_a \in A] \geq [x_a \in A] = A_a(x)$,

$A_a(xy) = [(xy)_a \in A] = [(x_a y_a) \in A] \geq [x_a \in A] \vee [y_a \in A] = A_a(x) \vee A_a(y)$,

Hence, A_a is a fuzzy ideal of R.

Conversely, we assume for any $a \in [0, 1]$, A_a is a fuzzy ideal of R. Let $x, y \in R$ and $s, t \in [0, 1]$. We will prove $[(x_s y_t) \in A] \geq [x_s \in A] \vee [y_t \in A]$ and proofs of the other two conditions $[(x_s + y_t) \in A] \geq [x_s \in A] \wedge [y_t \in A]$ and $[-x_s \in A] \geq [x_s \in A]$ are straightforward and can be obtained in the similar manner. Let $a = [x_s \in A] \vee [y_t \in A]$.

Case I. $a = 1$. Then, either $[x_s \in A] = 1$ or $[x_t \in A] = 1$, which gives either $A_s(x) = 1$ or $A_t(y) = 1$. Now, if $A_s(x) = 1$, then $A_s(xy) \geq A_s(x) \vee A_s(y) = 1$. Therefore, $A_s(xy) = 1$, and so $\mu_A(xy) \geq s$. Similarly, if $A_t(y) = 1$, then $\mu_A(xy) \geq t$. Thus, $\mu_A(xy) \geq s \vee t$ which implies that $A_{s \vee t}(xy) = 1$. Hence, $[x_s y_t \in A] = 1$.

Case II. $a = 1/2$. Then, either $[x_s \in A] \geq 1/2$ or $[x_t \in A] \geq 1/2$, which gives either $A_s(x) \geq 1/2$ or $A_t(y) \geq 1/2$. Now, if $A_s(x) \geq 1/2$, then $A_s(xy) \geq A_s(x) \vee A_s(y) \geq 1/2$. Therefore,

$A_s(xy) \geq 1/2$, and so $s \leq 1 - \nu_A(xy)$. Similarly, if $A_t(y) \geq 1/2$, then $t \leq 1 - \nu_A(xy)$. Thus, $s \vee t \leq 1 - \nu_A(xy)$, which implies that $A_{s\vee t}(xy) \geq 1/2$. Hence, $[x_s y_t \in A] \geq 1/2$. Thus, in all cases, we get $[x_s y_t \in A] \geq [x_s \in A] \vee [y_t \in A]$. ∎

Theorem 3.11. *An intuitionistic fuzzy set, $A = (\mu_A, \nu_A)$ of R is a $(\in, \in \vee q)$-intuitionistic fuzzy ideal of R if and only if for any $a \in [0, 0.5]$, A_a is a fuzzy ideal of R.*

Proof. Suppose that A is a $(\in, \in \vee q)$-intuitionistic fuzzy ideal of R. Then, for any $a \in (0, 0.5]$ and $x, y \in R$, we have

$$[x_a y_a \in \vee q] \geq [x_a \in A] \vee [y_a \in A] \Rightarrow A_a(xy) \vee A_{[\underline{a}]}(xy) \geq A_a(x) \vee A_a(y).$$

Since $0 < a \leq 0.5$, therefore we have $a \leq 0.5 \leq 1 - a$. Then

$$A_{[\underline{a}]}(xy) = A_{\underline{1-a}}(xy) \leq A_{\underline{a}}(xy) \leq A_a(xy).$$

Therefore, $A_a(x) \vee A_a(y) \leq A_a(xy) \vee A_{[\underline{a}]}(xy) \leq A_a(xy) \vee A_a(xy) = A_a(xy)$, and so $A_a(xy) \geq A_a(x) \vee A_a(y)$. Similarly, we can prove that $A_a(x+y) \geq A_a(x) \wedge A_a(y)$ and $A_a(-x) \geq A_a(x)$. Therefore, for any $a \in [0, 0.5]$, A_a is a fuzzy ideal of R.

Conversely, we assume for any $a \in [0, 0.5]$, A_a is a fuzzy ideal of R. Let $s, t \in [0, 1]$ and $x, y \in R$.

(1) If $s \wedge t \leq 0.5$, then let $a = [x_s \in A] \wedge [y_t \in A]$.

Case I. $a = 1$. Then, $A_s(x) = 1$ and $A_t(y) = 1$, and so $A_{s\wedge t}(x+y) \geq A_{s\wedge t}(x) \wedge A_{s\wedge t}(y) \geq A_s(x) \wedge A_t(y) = 1$. Therefore, we have $A_{s\wedge t}(x+y) = 1 \Rightarrow [(x_s + y_t) \in A] = 1$. Now, $[(x_s + y_t) \in \vee qA] = [(x_s + y_t) \in A] \vee [(x_s + y_t) qA] = 1$.

Case II. $a = 1/2$. Then, $A_s(x) \geq 1/2$ and $A_t(y) \geq 1/2$, and so $A_{s\wedge t}(x+y) \geq A_{s\wedge t}(x) \wedge A_{s\wedge t}(y) \geq A_s(x) \wedge A_t(y) \geq 1/2$. Therefore, we have $A_{s\wedge t}(x+y) \geq 1/2 \Rightarrow [(x_s + y_t) \in A] \geq 1/2$. Now, $[(x_s + y_t) \in \vee qA] = [(x_s + y_t) \in A] \vee [(x_s + y_t) qA] \geq 1/2$. Therefore, $[(x_s + y_t) \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A]$.

If $s \wedge t > 0.5$, then let $a \in (0, 1)$ such that $1 - s \wedge t < a < 0.5 < s \wedge t$. Now, $A_{[\underline{s\wedge t}]}(x+y) = A_{\underline{1-s\wedge t}}(x+y) \geq A_{s\wedge t}(x+y)$ and $A_{[\underline{s\wedge t}]}(x+y) = A_{\underline{1-s\wedge t}}(x+y) \geq A_a(x+y)$.

Therefore, $[(x_s + y_t) \in \vee qA] = [(x_s + y_t) \in A] \vee [(x_s + y_t) qA] = A_{s\wedge t}(x+y) \vee A_{[\underline{s\wedge t}]}(x+y) = A_{[\underline{s\wedge t}]}(x+y) \geq A_a(x+y) \geq A_a(x) \wedge A_a(y) \geq A_s(x) \wedge A_t(y) = [x_s \in A] \wedge [y_t \in A]$, and hence $[(x_s + y_t) \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A]$.

Similarly, we can prove that $[-x_s \in \vee qA] \geq [x_s \in A]$.

(3) If $s \vee t \leq 0.5$, then let $b = [x_s \in A] \vee [y_t \in A]$.

Case I. $b = 1$. Then, either $A_s(x) = 1$ or $A_t(y) = 1$. If $A_s(x) = 1$, then $A_s(xy) \geq A_s(x) \vee A_s(y) = 1$, and so $A_s(xy) = 1$. This implies that $\mu_A(xy) \geq s$. Similarly, if $A_t(y) = 1$, then $\mu_A(xy) \geq t$. Therefore, we obtain $\mu_A(xy) \geq s \vee t$, from which we get $[(x_s y_t) \in A] = 1$. Thus, $[(x_s y_t) \in \vee qA] = [(x_s y_t) \in A] \vee [(x_s y_t) qA] = 1$.

Case II. $b = 1/2$. Then, either $A_s(x) = 1/2$ or $A_t(y) = 1/2$. If $A_s(x) = 1/2$, then $A_s(xy) \geq A_s(x) \vee A_s(y) \geq 1/2$, and so $s \leq 1 - \nu_A(xy)$. Similarly, if $A_t(y) = 1/2$, then $t \leq 1 - \nu_A(xy)$. Therefore, we have $s \vee t \leq 1 - \nu_A(xy)$ which implies that $A_{s\vee t}(xy) \geq 1/2$. Thus, $[x_s y_t \in A] \geq 1/2$, and so $[(x_s y_t) \in \vee qA] = [(x_s y_t) \in A] \vee [(x_s y_t) qA] \geq 1/2$. Therefore, $[(x_s y_t) \in \vee qA] \geq [x_s \in A] \vee [y_t \in A]$.

If $s \vee t > 0.5$, then let $a \in (0, 1)$ be such that $1 - s \vee t < a < 0.5 < s \vee t$. Now,

$A_{[\underline{s\vee t}]}(xy) = A_{\underline{1-s\vee t}}(xy) \geq A_{s\vee t}(xy)$, and $A_{[\underline{s\vee t}]}(xy) = A_{\underline{1-s\vee t}}(xy) \geq A_a(xy)$.

Therefore, $[(x_s y_t) \in \vee qA] = [(x_s y_t) \in A] \vee [(x_s y_t) qA] = A_{s\vee t}(xy) \vee A_{[\underline{s\vee t}]}(xy) = A_{[\underline{s\vee t}]}(xy) \geq A_a(xy) \geq A_a(x) \vee A_a(y) \geq A_s(x) \vee A_t(y) = [x_s \in A] \vee [y_t \in A]$, and hence $[(x_s y_t) \in \vee qA] \geq [x_s \in A] \vee [y_t \in A]$. ∎

Theorem 3.12. *An intuitionistic fuzzy set, $A = (\mu_A, \nu_A)$ of R is a $(\in \wedge q, \in)$-intuitionistic fuzzy ideal of R if and only if for any $a \in (0.5, 1]$, A_a is a fuzzy ideal of R.*

Proof. Suppose that A is a $(\in \wedge q, \in)$-intuitionistic fuzzy ideal of R. Let $a \in (0.5, 1]$ and $x, y \in R$, then $A_{[\underline{a}]}(x) \geq A_a(x)$. Thus,

$$\begin{aligned} A_a(x+y) &= [(x_a + y_a) \in A] \geq [x_a \in \wedge qA] \wedge [y_a \in \wedge qA] \\ &= A_a(x) \wedge A_{[\underline{a}]}(x) \wedge A_a(y) \wedge A_{[\underline{a}]}(y) = A_a(x) \wedge A_a(y). \end{aligned}$$

Therefore, $A_a(x+y) \geq A_a(x) \wedge A_a(y)$. Similarly, we have $A_a(-x) \geq A_a(x)$.

$$\begin{aligned} A_a(xy) &= [x_a y_a \in A] \geq [x_a \in \wedge qA] \vee [y_a \in \wedge qA] \\ &= (A_a(x) \wedge A_{[\underline{a}]}(x)) \vee (A_a(y) \wedge A_{[\underline{a}]}(y)) = A_a(x) \vee A_a(y). \end{aligned}$$

Therefore, $A_a(xy) \geq A_a(x) \vee A_a(y)$.

Conversely, we assume for any $a \in (0.5, 1]$, A_a is a fuzzy ideal of R. Let $x, y \in R, s, t \in (0, 1]$.

(1) Let $b = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Case I. $b = 1$. Then, $\mu_A(x) \geq s$, $\mu_A(x) > 1 - s$, $\mu_A(y) \geq t$, $\mu_A(y) > 1 - t$. Therefore, $\mu_A(x) > 0.5$, $\mu_A(y) > 0.5$. Let $a = \mu_A(x) \wedge \mu_A(y)$. Then, $a > 0.5$ and $\mu_A(x) \geq a$, $\mu_A(y) \geq a$, and so $A_a(x) = 1, A_a(y) = 1$. Thus, $A_a(x+y) \geq A_a(x) \wedge A_a(y) = 1$ implies $A_a(x+y) = 1$, and so $\mu_A(x+y) \geq a = \mu_A(x) \wedge \mu_A(y) \geq s \wedge t$. Therefore, $[(x_s + y_t) \in A] = 1$.

Case II. $b = 1/2$. Then, $1 - \nu_A(x) \geq s, s > \nu_A(x)$ and $1 - \nu_A(y) \geq t, t > \nu_A(y)$ which implies that $\nu_A(x) < 0.5, \nu_A(y) < 0.5$. Thus, $1 - \nu_A(x) > 0.5$, $1 - \nu_A(y) > 0.5$. Let $a = (1 - \nu_A(x)) \wedge (1 - \nu_A(y))$, then $a > 0.5$. Therefore, $A_a(x+y) \geq A_a(x) \wedge A_a(y) \geq 1/2 \wedge 1/2 = 1/2$, [Since $1 - \nu_A(x) \geq a$, $1 - \nu_A(y) \geq a$]. This implies that $1 - \nu_A(x+y) \geq a = (1 - \nu_A(x)) \wedge (1 - \nu_A(y)) \geq s \wedge t$. Therefore, $[(x_s + y_t) \in A] \geq 1/2 = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

(2) Similarly, we can prove that $[-x_s \in A] \geq [x_s \in \wedge qA]$.

(3) Let $b = [x_s \in \wedge qA] \vee [y_t \in \wedge qA]$.

Case I. $b = 1$. Then, either $\mu_A(x) \geq s$, $\mu_A(x) > 1 - s$ or $\mu_A(y) \geq t$, $\mu_A(y) > 1 - t$. Therefore, $\mu_A(x) > 0.5$ or $\mu_A(y) > 0.5$. Let $a = \mu_A(x) \vee \mu_A(y)$, then $a > 0.5$. Also, $\mu_A(x) = a$ or $\mu_A(y) = a$, and so $A_a(x) = 1$ or $A_a(y) = 1$. Thus, $A_a(xy) \geq A_a(x) \vee A_a(y) = 1$ which implies that $A_a(xy) = 1$, and so $\mu_A(xy) \geq a = \mu_A(x) \vee \mu_A(y) \geq s \vee t$. Therefore, $[(x_s y_t) \in A] = 1$.

Case II. $b = 1/2$. Then, either $1 - \nu_A(x) \geq s, s > \nu_A(x)$ or $1 - \nu_A(y) \geq t, t > \nu_A(y)$, which implies either $\nu_A(x) < 0.5$ or $\nu_A(y) < 0.5$. Thus, $1 - \nu_A(x) > 0.5$ or $1 - \nu_A(y) > 0.5$. Let $a = (1 - \nu_A(x)) \vee (1 - \nu_A(y))$, then $a > 0.5$. Therefore, $A_a(xy) \geq A_a(x) \vee A_a(y) \geq 1/2 \vee 1/2 = 1/2$, [Since $1 - \nu_A(x) = a$ or $1 - \nu_A(y) = a$]. This implies that $1 - \nu_A(xy) \geq a = (1 - \nu_A(x)) \vee (1 - \nu_A(y)) \geq s \vee t$. Therefore, $[(x_s y_t) \in A] \geq 1/2 = [x_s \in \wedge qA] \vee [y_t \in \wedge qA]$. ∎

Theorem 3.13. *An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of R is an intuitionistic fuzzy ideal with thresholds (s, t) of R if and only if for any $a \in (s, t]$, A_a is a fuzzy ideal of R.*

Proof. Suppose that A is an intuitionistic fuzzy ideal with thresholds (s, t) of R. Let $a \in (s, t], x, y \in R$ and $b = A_a(x) \vee A_a(y)$.

Case I. $b = 1$. Then, $A_a(x) = 1$ or $A_a(y) = 1$. This implies that $\mu_A(x) \geq a > s$ or $\mu_A(y) \geq a > s$. Now, $\mu_A(xy) \vee s \geq (\mu_A(x) \vee \mu_A(y)) \wedge t \geq (a \vee a) \wedge t = a$. Therefore $\mu_A(xy) \geq a$ which implies that $A_a(xy) = 1$.

Case II. $b = 1/2$. Then, $A_a(x) = 1/2$ or $A_a(y) = 1/2$, which implies that $1 - \nu_A(x) \geq a$ or $1 - \nu_A(y) \geq a$. Thus, $\nu_A(x) \wedge \nu_A(y) \leq 1 - a < 1 - s$. Now, $\nu_A(xy) \wedge (1 - s) \leq (\nu_A(x) \wedge \nu_A(y)) \vee (1 - t) \leq (1 - a) \vee (1 - t) = 1 - a$, [Since $t \geq a$ and $1 - s > 1 - a$]. Therefore,

$1-\nu_A(xy) \geq a$, and so $A_a(xy) \geq 1/2 = A_a(x) \vee A_a(y)$. Hence, $A_a(xy) \geq A_a(x) \vee A_a(y)$.

Similarly, we have $A_a(x+y) \geq A_a(x) \wedge A_a(y)$ and $A_a(-x) \geq A_a(x)$.

Conversely, we assume for any $a \in (s,t]$, A_a is a fuzzy ideal of R.

(1) To show $\mu_A(x+y) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t$. If $\mu_A(x+y) \vee s < a = \mu_A(x) \wedge \mu_A(y) \wedge t$, then $a \in (s,t]$ and $\mu_A(x) \geq a, \mu_A(y) \geq a$. Thus, from $A_a(x+y) \geq A_a(x) \wedge A_a(y) = 1$, we have $A_a(x+y) = 1$, and so $\mu_A(x+y) \geq a$, which contradicts to $\mu_A(x+y) < a$. Therefore, $\mu_A(x+y) \vee s \geq \mu_A(x) \wedge \mu_A(y) \wedge t$.

(2) Similarly, we have $\mu_A(-x) \vee s \geq \mu_A(x) \wedge t$.

(3) To show $\mu_A(xy) \vee s \geq (\mu_A(x) \vee \mu_A(y)) \wedge t$. If $\mu_A(xy) \vee s < a = (\mu_A(x) \vee \mu_A(y)) \wedge t$, then $a \in (s,t]$ and $\mu_A(x) \geq a$ or $\mu_A(y) \geq a$. Thus, from $A_a(xy) \geq A_a(x) \vee A_a(y) = 1$, we have $A_a(xy) = 1$, and so $\mu_A(xy) \geq a$, which contradicts to $\mu_A(xy) < a$. Therefore, $\mu_A(xy) \vee s \geq (\mu_A(x) \vee \mu_A(y)) \wedge t$.

(4) To show $\nu_A(x+y) \wedge (1-s) \leq (\nu_A(x) \vee \nu_A(y)) \vee (1-t)$. If $\nu_A(x+y) \wedge (1-s) > a = (\nu_A(x) \vee \nu_A(y)) \vee (1-t)$, then $(1-\nu_A(x+y)) \vee s < b = 1-a = (1-\nu_A(x)) \wedge (1-\nu_A(y)) \wedge t$, and so $b \in (s,t]$ and $(1-\nu_A(x)) \geq b$, $(1-\nu_A(y)) \geq b$. Thus, from $A_a(x+y) \geq A_a(x) \wedge A_a(y) \geq 1/2$, we have $A_a(x+y) \geq 1/2$, and so $1-\nu_A(x+y) \geq b = 1-a$. Therefore, $\nu_A(x+y) \leq a$, which contradicts to $\nu_A(x+y) > a$. Hence, $\nu_A(x+y) \wedge (1-s) \leq (\nu_A(x) \vee \nu_A(y)) \vee (1-t)$.

(5) Similarly, we have $\nu_A(-x) \wedge (1-s) \leq \nu_A(x) \vee (1-t)$.

(6) To show $\nu_A(xy) \wedge (1-s) \leq (\nu_A(x) \wedge \nu_A(y)) \vee (1-t)$. If $\nu_A(xy) \wedge (1-s) > a = (\nu_A(x) \wedge \nu_A(y)) \vee (1-t)$, then $(1-\nu_A(xy)) \vee s < b = 1-a = (1-\nu_A(x)) \vee (1-\nu_A(y)) \wedge t$, and so $b \in (s,t]$ and $(1-\nu_A(x)) \geq b$ or $(1-\nu_A(y)) \geq b$. Thus, from $A_a(xy) \geq A_a(x) \vee A_a(y) \geq 1/2$, we have $A_a(xy) \geq 1/2$, and so $1-\nu_A(xy) \geq b = 1-a$. Therefore, $\nu_A(xy) \leq a$, which contradicts to $\nu_A(xy) > a$. Hence, $\nu_A(xy) \wedge (1-s) \leq (\nu_A(x) \wedge \nu_A(y)) \vee (1-t)$.

Hence, $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy ideal with thresholds (s,t) of R. ∎

4. Conclusion

In this article, we have defined a new kind of fuzzy subring and ideal namely, (α,β)-intuitionistic fuzzy subrings and ideals, where $\alpha,\beta \in \{\in, q, \in \wedge q, \in \vee q\}$. Among the 16 such intuitionistic fuzzy ideals, $(\in,\in)$, $(\in,\in \vee q)$ and $(\in \wedge q,\in)$ are significant. We have investigated various properties of (α,β)-intuitionistic fuzzy ideals and attempted to connect intuitionistic fuzzy ideal with thresholds (s,t). In our opinion this is an opening for investigations of different types of (α,β)-intuitionistic fuzzy ideals.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20** (1986), no. 1, 87–96.

[2] K. T. Atanassov, New operations defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **61** (1994), no. 2, 137–142.

[3] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Studies in Fuzziness and Soft Computing, 35, Physica, Heidelberg, 1999.

[4] I. Beg and S. Ashraf, Fuzzy relational calculus, *Bull. Malays. Math. Sci. Soc. (2)*, accepted.

[5] S. K. Bhakat and P. Das, On the definition of a fuzzy subgroup, *Fuzzy Sets and Systems* **51** (1992), no. 2, 235–241.

[6] S. K. Bhakat and P. Das, $(\in,\in \vee q)$-fuzzy subgroup, *Fuzzy Sets and Systems* **80** (1996), no. 3, 359–368.

[7] S. K. Bhakat and P. Das, Fuzzy subrings and ideals redefined, *Fuzzy Sets and Systems* **81** (1996), no. 3, 383–393.

[8] R. Biswas, Intuitionistic fuzzy subgroups, *Mathematical Fortum*, **10** (1989), 37–46.

[9] B. Davvaz, J. Zhan and K. P. Shum, Generalized fuzzy H_v-submodules endowed with interval valued membership functions, *Inform. Sci.* **178** (2008), no. 15, 3147–3159.

[10] B. Davvaz and P. Corsini, Redefined fuzzy H_v-submodules and many valued implications, *Inform. Sci.* **177** (2007), no. 3, 865–875.

[11] B. Davvaz, Intuitionistic hyperideals of semihypergroups, *Bull. Malays. Math. Sci. Soc. (2)* **29** (2006), no. 2, 203–207.

[12] B. Davvaz, P. Corsini and V. Leoreanu-Fotea, Atanassov's intuitionistic (S,T)-fuzzy n-ary sub-hypergroups and their properties, *Inform. Sci.* **179** (2009), no. 5, 654–666.

[13] B. Davvaz and V. Leoreanu-Fotea, Intuitionistic fuzzy n-ary hypergroups, *J. Mult.-Valued Logic Soft Comput.* **16** (2010), no. 1-2, 87–103.

[14] H. Hedayati, Generalized fuzzy k-ideals of semirings with interval-valued membership functions, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 3, 409–424.

[15] C. Gunduz and B. Davvaz, The universal coefficient theorem in the category of intuitionistic fuzzy modules, *Util. Math.* **81** (2010), 131–156.

[16] Y. B. Jun, A. Khan and M. Shabir, Ordered semigroups characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 3, 391–408.

[17] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall PTR, Upper Saddle River, NJ, 1995.

[18] J. N. Mordeson and D. S. Malik, *Fuzzy Commutative Algebra*, World Sci. Publishing, River Edge, NJ, 1998.

[19] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.* **35** (1971), 512–517.

[20] M. Shabir and A. Khan, Fuzzy quasi-ideals of ordered semigroups, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 1, 87–102.

[21] M. Shabir, M. S. Arif, A. Khan and M. Aslam, On Intuitionistic Fuzzy Prime Bi-Ideals of Semigroups, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 4, 983–996.

[22] J. Wang, X. Lin and Y. Yin, Intuitionistic fuzzy ideals with thresholds (α, β) of rings, *Int. Math. Forum* **4** (2009), no. 21-24, 1119–1127.

[23] X. Yuan, H. Li and E. S. Lee, On the definition of the intuitionistic fuzzy subgroups, *Comput. Math. Appl.* **59** (2010), no. 9, 3117–3129.

[24] S. Yamak, O. Kazanc and B. Davvaz, Divisible and pure intuitionistic fuzzy subgroups and their properties, *Int. J. Fuzzy Syst.* **10** (2008), no. 4, 298–307.

[25] X. H. Yuan, H. X. Li and K. B. Sun, The cut sets, decomposition theorems and representation theorems on intuitionistic fuzzy sets and interval-valued fuzzy sets, *Science in China Series F: Information Sciences*, **39** (2009), no. 9, 933–945.

[26] L. A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.