BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36**(2) (2013), 435–446

Nonlinear Boundary Value Problems for First Order Integro-Differential Equations with Impulsive Integral Conditions

ZHENHAI LIU, JIANGFENG HAN AND LIJING FANG
College of Sciences, Guangxi University for Nationalities, Nanning, Guangxi 530006, P. R. China
zhhliu100@yahoo.com.cn, hanjiangfengok@163.com, lijingfang100@126.com

Abstract. This paper is concerned with the nonlinear boundary value problems for first order integro-differential equations with impulsive integral conditions. By using of the method of lower and upper solutions coupled with the monotone iterative technique, we give conditions for the existence of extremal solutions.

2010 Mathematics Subject Classification: 34K10, 34K45

Keywords and phrases: Impulsive integro-differential equations, nonlinear boundary valued problems, monotone iterative technique, existence of solutions.

1. Introduction

In this paper, we consider the following nonlinear boundary value problem:

$$\begin{cases} x'(t) = f(t, x(t), (Tx)(t), (Sx)(t)), & t \in J^-, \\ \Delta x(t_k) = I_k(\int_{t_k-\tau_k}^{t_k} x(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} x(s)ds), & k = 1, 2, \cdots, m, \\ g(x(0), x(T)) = 0, \end{cases} \tag{1.1}$$

where $f \in C(J \times R^3, R)$, $g \in (R^2, R)$, $J = [0, T]$, $J^- = J - \{t_1, t_2, \cdots t_m\}$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $I_k \in C(R, R)$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $0 < \sigma_{k-1} \le (t_k - t_{k-1})/2$, $0 \le \tau_k \le (t_k - t_{k-1})/2$, $k = 1, 2 \cdots m$, and

$$(Tx)(t) = \int_0^t k(t, s)x(s)ds, \quad (Sx)(t) = \int_0^T h(t, s)x(s)ds.$$

$k \in C(D, R^+)$, $D = \{(t, s) \in J \times J : \ t \ge s\}$, $h \in C(J \times J, R^+)$.

Recently, the general theory of impulsive differential equations has become an important aspect of differential equations for its extensively application. As an important branch, boundary value problems (BVPS) have drawn much attention (cf. [1–23]).

In the problem (1.1), we deal with the nonlinear boundary value problem $g(x(0), x(T)) = 0$, which includes three typical boundary valued problems:

Communicated by Shangjiang Guo.
Received: April 30, 2011; *Revised:* July 21, 2011.

(i) If $g(x(0),x(T)) = x(0)-x(T)$, (1.1) reduces to the periodic boundary value problem: $x(0) = x(T)$, which have been considered by many authors (cf. [6,8,9,13,17,20,23]).

(ii) If $g(x(0),x(T)) = x(0)+x(T)$, (1.1) reduces to the anti-periodic boundary value problem: $x(0) = -x(T)$, which also have been considered by many authors (cf. [1,2,5,7,14,16,18]).

(iii) If $g(x(0),x(T)) = x(0)-d$, for any $d \in R$, (1.1) reduces to initial value problem: $x(0) = d$ (cf. [3,12] and the references therein).

It is well known that the monotone iterative technique offers an approach for obtaining approximate solutions of nonlinear differential equations. There also exist several works devoted to the applications of this technique to boundary value problems of impulsive differential equations. In [1, 2], the authors discussed the anti-periodic boundary value problem of impulsive differential equations with monotone iterative technique. And in [6, 8, 9, 13, 17], the authors discussed the periodic boundary value problem of impulsive differential equations with the same technique. However, in all papers connected with applications of the monotone iterative technique to impulsive problems, the authors assumed that $\Delta x(t_k) = I_k(x(t_k))$ that is a short-term rapid change of the state at impulse points t_k depends on the left side of their limits of $x(t_k)$ (cf. [8,9,15]).

Just recently, Jessada Tariboon [21] discussed a kind of functional differential equations with the new impulsive integral conditions $\Delta x(t_k) = I_k\left(\int_{t_k-\tau_k}^{t_k} x(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} x(s)ds\right)$. We note that the new jump conditions depend on the functional of path history on $[t_k-\tau_k, t_k]$ before impulse points t_k and functional of path history on $(t_{k-1}, t_{k-1}+\sigma_{k-1}]$ after the past impulse points t_{k-1}. It should be noticed that BVP (1.1) has a memory of the past state and the history of the effects of impulses.

Chen and Sun [4] and Jankowski [10,11] discussed the nonlinear boundary value problem of first order impulsive functional differential equations. Tariboon [21] considered boundary value problems for first order functional differential equations with impulsive integral conditions. Encouraged by the papers [4, 10, 11, 21], we first establish a new comparison principle for nonlinear boundary value problems for first order integro-differential equations with impulsive integral conditions and then obtain the existence of extremal solutions by the upper-lower solution and monotone iterative techniques.

2. Preliminaries and lemmas

Let $PC(J) = \{x : J \to R; x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist, and $x(t_k^-) = x(t_k),\ k = 1,2,\cdots,m\}$. $PC^1(J) = \{x \in PC(J) : x'(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist, and $x'(t_k^-) = x'(t_k),\ k = 1,2,\cdots,m\}$. It is well known that $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\| x \|_{PC} = \sup\{|x(t)| : t \in J\}, \quad \| x \|_{PC^1} = \max\{\| x \|_{PC}, \| x' \|_{PC}\}.$$

Denote $a = \max\{t_{k+1}-t_k,\ k = 0,1,2\cdots m\}$.

A function $x \in PC^1(J)$ is called a solution of problem (1.1) if it satisfies (1.1). In the sequel, we shall need the following lemmas.

Lemma 2.1. *Let $x(t) \in PC^1(J)$ such that*

$$(2.1)\qquad \begin{cases} x'(t) + Mx(t) + N(t)(Tx)(t) + N_1(t)(Sx)(t) \le 0, & t \in J^-, \\ \Delta x(t_k) \le -L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} x(s)ds, & k = 1, 2, \cdots, m, \\ x(0) \le \mu x(T), \end{cases}$$

where $M > 0$, $N(t), N_1(t) \in C(J, R^+), N(t) + N_1(t) \not\equiv 0$ *in* J, $0 \le L_k < 1$, $0 < \sigma_{k-1} \le (t_k - t_{k-1})/2$, $0 \le \tau_k \le (t_k - t_{k-1})/2$, $k = 1, 2 \cdots m$, $0 < \mu \le e^{MT}$. *Suppose in addition that*

$$(2.2)\qquad (\mu e^{-MT})^{-1}\Big[\int_0^T q(s)ds + \frac{1}{M}\sum_{k=1}^m L_k(e^{M(a-\sigma_{k-1})} - e^{M\tau_k})\Big] \le 1,$$

with

$$q(t) = N(t)\int_0^t k(t,s)e^{M(t-s)}ds + N_1(t)\int_0^T h(t,s)e^{M(t-s)}ds.$$

Then $x(t) \le 0$ *on* J.

Proof. Let $u(t) = e^{Mt}x(t)$, then we have

$$(2.3)\qquad \begin{cases} u'(t) \le -N(t)\int_0^t k(t,s)e^{M(t-s)}u(s)ds - N_1(t)\int_0^T h(t,s)e^{M(t-s)}u(s)ds, & t \in J^-, \\ \Delta u(t_k) \le -L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} e^{M(t_k-s)}u(s)ds, & k = 1, 2, \cdots, m, \\ u(0) \le \mu e^{-MT}u(T). \end{cases}$$

Obviously, the function $u(t)$ and $x(t)$ have the same sign.

Suppose, to the contrary, that $u(t) > 0$ for some $t \in J$. It is enough to consider the following two cases:

(i) There exists a $t^- \in J$, such that $u(t^-) > 0$, and $u(t) \ge 0$ for all $t \in J$.

(ii) There exist t_*, $t^* \in J$, such that $u(t_*) < 0$, $u(t^*) > 0$.

Case (i) In view of (2.3), we know that $u'(t) \le 0$ on J^- and $\Delta u(t_k) \le 0$, hence $u(t)$ is non-increasing, which implies $u(0) \ge u(t^-) > 0$,$u(T) \le u(0) \le \mu e^{-MT}u(T)$.

If $0 < \mu < e^{MT}$, we get $u(T) \le 0$, furthermore $u(0) \le 0$, which is a contradiction.

If $\mu = e^{MT}$, then $u(0) \le u(T)$,but $u(t)$ is non-increasing, so $u(t) = \text{constant} = u(t^-) > 0$, in view of (2.3), we have $0 = u'(t) \le -N(t)\int_0^t k(t,s)e^{M(t-s)}u(s)ds - N_1(t)\int_0^T h(t,s)e^{M(t-s)}u(s)ds < 0$, which is a contradiction.

Case (ii) Let $t_* \in (t_i, t_{i+1}]$, $i \in \{0, 1, 2, \cdots m\}$, such that $u(t_*) = \inf\{u(t) : t \in J\} < 0$, and $t^* \in (t_j, t_{j+1}]$, $j \in \{0, 1, 2, \cdots m\}$, such that $u(t^*) > 0$.

If $t_* < t^*$, then $i \le j$. Integrating the differential inequality in (2.3) from t_* to t^*, we obtain

$$\begin{aligned} &u(t^*) - u(t_*) \\ &\le -N(t)\int_{t_*}^{t^*} ds\int_0^s k(s,r)e^{M(s-r)}u(r)dr - N_1(t)\int_{t_*}^{t^*} ds\int_0^T h(s,r)e^{M(s-r)}u(r)dr + \sum_{k=i+1}^{j}\Delta u(t_k) \\ &\le -u(t_*)\int_{t_*}^{t^*} q(s)ds + \sum_{k=i+1}^{j}\Delta u(t_k) \le -u(t_*)\int_{t_*}^{t^*} q(s)ds - u(t_*)\sum_{k=i+1}^{j} L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} e^{M(t_k-s)}ds \\ &\le -u(t_*)\Big[\int_0^T q(s)ds + \sum_{k=1}^{m} L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} e^{M(t_k-s)}ds\Big] \end{aligned}$$

$$= -u(t_*)[\int_0^T q(s)ds + \frac{1}{M}\sum_{k=1}^{m} L_k(e^{M(t_k-t_{k-1}-\sigma_{k-1})} - e^{M\tau_k})]$$
$$\leq -u(t_*)(\mu e^{-MT})^{-1}[\int_0^T q(s)ds + \frac{1}{M}\sum_{k=1}^{m} L_k(e^{M(a-\sigma_{k-1})} - e^{M\tau_k})] \leq -u(t_*),$$

which is a contradiction to $u(t^*) > 0$.

If $t_* > t^*$, then $i \geq j$.

(a) Suppose that $u(T) > 0$. Integrating the differential inequality in (2.3) from t_* to T, we obtain

$$u(T) - u(t_*)$$
$$\leq -N(t)\int_{t_*}^T ds \int_0^s k(s,r)e^{M(s-r)}u(r)dr - N_1(t)\int_{t_*}^T ds\int_0^T h(s,r)e^{M(s-r)}u(r)dr + \sum_{k=i+1}^{m}\Delta u(t_k)$$
$$\leq -u(t_*)\int_{t_*}^T q(s)ds + \sum_{k=i+1}^{m}\Delta u(t_k) \leq -u(t_*)\int_{t_*}^T q(s)ds - u(t_*)\sum_{k=i+1}^{m} L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} e^{M(t_k-s)}ds$$
$$\leq -u(t_*)[\int_0^T q(s)ds + \sum_{k=1}^{m} L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} e^{M(t_k-s)}ds]$$
$$= -u(t_*)[\int_0^T q(s)ds + \frac{1}{M}\sum_{k=1}^{m} L_k(e^{M(t_k-t_{k-1}-\sigma_{k-1})} - e^{M\tau_k})]$$
$$\leq -u(t_*)(\mu e^{-MT})^{-1}[\int_0^T q(s)ds + \frac{1}{M}\sum_{k=1}^{m} L_k(e^{M(a-\sigma_{k-1})} - e^{M\tau_k})] \leq -u(t_*).$$

Then $u(T) \leq 0$,which is a contradiction.

(b) Suppose that $u(T) \leq 0$, then

$$0 < u(t^*) \leq u(0) - u(t_*)\int_0^{t^*} q(s)ds + \sum_{k=1}^{j}\Delta u(t_k).$$

On the other hand

$$u(T) \leq u(t_*) - u(t_*)\int_{t_*}^T q(s)ds + \sum_{k=i+1}^{m}\Delta u(t_k).$$

This implies

$$0 < u(t^*) \leq \mu e^{-MT}u(T) - u(t_*)\int_0^{t^*} q(s)ds + \sum_{k=1}^{j}\Delta u(t_k)$$
$$\leq \mu e^{-MT}u(t_*) - \mu e^{-MT}u(t_*)\int_{t_*}^T q(s)ds + \mu e^{-MT}\sum_{k=i+1}^{m}\Delta u(t_k) - u(t_*)\int_0^{t^*} q(s)ds + \sum_{k=1}^{j}\Delta u(t_k).$$

So we obtain that

$$(\mu e^{-MT})^{-1}[\int_0^T q(s)ds + \frac{1}{M}\sum_{k=1}^{m} L_k(e^{M(a-\sigma_{k-1})} - e^{M\tau_k})] > 1,$$

which is a contradiction.

The proof is complete. ∎

Let us consider the linear boundary value problem of (1.1):

$$
\begin{cases}
x'(t) + Mx(t) + N(t)(Tx)(t) + N_1(t)(Sx)(t) = \sigma(t), & t \in J^-, \\
\Delta x(t_k) = -L_k \displaystyle\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} x(s)ds + I_k\bigg(\int_{t_k-\tau_k}^{t_k} \eta(s)ds \\
\qquad - \displaystyle\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \eta(s)ds\bigg) + L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \eta(s)ds, \\
g(\eta(0),\eta(T)) + M_1(x(0)-\eta(0)) - M_2(x(T)-\eta(T)) = 0,
\end{cases}
\tag{2.4}
$$

where $M > 0$, $N(t), N_1(t) \in C(J, R^+)$, $0 \le L_k < 1$, $0 < \sigma_{k-1} \le (t_k - t_{k-1})/2$, $0 \le \tau_k \le (t_k - t_{k-1})/2$, $k = 1, 2 \cdots m$, and σ, $\eta \in PC(J)$.

Lemma 2.2. *$x \in PC^1(J)$ is a solution of (2.4) if and only if $x \in PC(J)$ is a solution of the impulsive integral equation:*

$$
\begin{aligned}
x(t) = Ce^{-Mt}B\eta + \int_0^T G(t,s)F(s)ds + \sum_{k=1}^m G(t,t_k)\bigg[-L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} x(s)ds \\
+ I_k\bigg(\int_{t_k-\tau_k}^{t_k} \eta(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \eta(s)ds\bigg) + L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \eta(s)ds\bigg],
\end{aligned}
$$

where $F(t) = \sigma(t) - N(t)(Tx)(t) - N_1(t)(Sx)(t)$, $B\eta = -g(\eta(0),\eta(T)) + M_1\eta(0) - M_2\eta(T)$, $C = (M_1 - M_2e^{-MT})^{-1}$, $M_1 \ne M_2e^{-MT}$ and

$$
G(t,s) = \begin{cases} CM_2e^{M(s-t-T)} + e^{M(s-t)}, & 0 \le s < t \le T \\ CM_2e^{M(s-t-T)}, & 0 \le t \le s \le T. \end{cases}
$$

Proof. If $x(t)$ is a solution of (2.4), by directly computation we have the following

$$
\begin{aligned}
x(t) = Ce^{-Mt}B\eta + \int_0^T G(t,s)F(s)ds + \sum_{k=1}^m G(t,t_k)\bigg[-L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} x(s)ds \\
+ I_k\bigg(\int_{t_k-\tau_k}^{t_k} \eta(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \eta(s)ds\bigg) + L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \eta(s)ds\bigg].
\end{aligned}
$$

If $x(t)$ is a solution of the above mentioned integral equation, then for any $t \in J^-$, we have

$$
\begin{aligned}
x'(t) = -M\bigg\{&Ce^{-Mt}B\eta + \int_0^t \Big(CM_2e^{M(s-t-T)} + e^{M(s-t)}\Big)F(s)ds + \int_t^T CM_2e^{M(s-t-T)}F(s)ds \\
&+ \sum_{0<t_k<t} \Big(CM_2e^{M(t_k-t-T)} + e^{M(t_k-t)}\Big)\bigg[-L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} x(s)ds + I_k\bigg(\int_{t_k-\tau_k}^{t_k} \eta(s)ds \\
&- \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \eta(s)ds\bigg) + L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \eta(s)ds\bigg] + \sum_{t\le t_k<T} CM_2e^{M(t_k-t-T)}\bigg[-L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} x(s)ds \\
&+ I_k(\int_{t_k-\tau_k}^{t_k} \eta(s)ds - \int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} \eta(s)ds) + L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} \eta(s)ds\bigg]\bigg\} + F(t) \\
= -M\bigg\{&Ce^{-Mt}B\eta + \int_0^T G(t,s)F(s)ds + \sum_{k=1}^m G(t,t_k)\bigg[-L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} x(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+I_k\left(\int_{t_k-\tau_k}^{t_k}\eta(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\eta(s)ds\right)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\eta(s)ds\Bigg]\Bigg\}+F(t)\\
&=-Mx(t)+F(t).
\end{aligned}
$$

$$
\begin{aligned}
\Delta x(t_k)&=x(t_k^+)-x(t_k^-)\\
&=\left(CM_2e^{-MT}+1\right)\Bigg[-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x(s)ds+I_k\left(\int_{t_k-\tau_k}^{t_k}\eta(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\eta(s)ds\right)\\
&\quad+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\eta(s)ds\Bigg]-CM_2e^{-MT}\Bigg[-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x(s)ds+I_k\Bigg(\int_{t_k-\tau_k}^{t_k}\eta(s)ds\\
&\quad-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\eta(s)ds\Bigg)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\eta(s)ds\Bigg]\\
&=-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x(s)ds+I_k\left(\int_{t_k-\tau_k}^{t_k}\eta(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\eta(s)ds\right)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\eta(s)ds.
\end{aligned}
$$

$$
\begin{aligned}
x(0)=CB\eta+\int_0^T CM_2e^{M(s-T)}F(s)ds+\sum_{k=1}^m CM_2e^{M(t_k-T)}\Bigg[-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x(s)ds\\
+I_k\left(\int_{t_k-\tau_k}^{t_k}\eta(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\eta(s)ds\right)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\eta(s)ds\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
x(T)&=Ce^{-MT}B\eta+\int_0^T CM_2e^{M(s-2T)}+e^{M(s-T)}F(s)ds+\sum_{k=1}^m CM_2e^{M(t_k-2T)}+e^{M(t_k-T)}\\
&\quad\times\Bigg[-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x(s)ds+I_k\left(\int_{t_k-\tau_k}^{t_k}\eta(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\eta(s)ds\right)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\eta(s)ds\Bigg].
\end{aligned}
$$

Then

$$M_1x(0)-M_2x(T)=CM_1B\eta-CM_2e^{-MT}B\eta=B\eta=-g(\eta(0),\eta(T))+M_1\eta(0)-M_2\eta(T).$$

This yields $g(\eta(0),\eta(T))+M_1(x(0)-\eta(0))-M_2(x(T)-\eta(T))=0$. The proof is complete. ∎

Lemma 2.3. *Assume that* $M>0,\ 0\le L_k<1,\ 0<\sigma_{k-1}\le(t_k-t_{k-1})/2,\ 0\le\tau_k\le(t_k-t_{k-1})/2,\ k=1,2\cdots m$ *and the following inequality holds*

$$
\begin{aligned}
&\sup_{t\in J}\int_0^T G(t,s)\left[N(s)\int_0^s k(s,r)dr+N_1(s)\int_0^T h(s,r)dr\right]ds\\
&\qquad\qquad+u\sum_{k=1}^m L_k(a-(\sigma_{k-1}+\tau_k))<1,
\end{aligned}
\tag{2.5}
$$

where $N(t),N_1(t)\in C(J,R^+)$, $u=\max(|CM_1|,|CM_2|)$, $C=(M_1-M_2e^{-MT})^{-1}$, $M_1\neq M_2e^{-MT}$ $G(t,s)$ *is defined as in Lemma 2.2. Then (2.4) has a unique solution.*

Proof. For convenience, we set for any fixed $\eta\in PC(J)$

$$(Ax)(t)=Ce^{-Mt}B\eta+\int_0^T G(t,s)F(s)ds+\sum_{k=1}^m G(t,t_k)\Bigg[-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x(s)ds$$

$$+I_k\left(\int_{t_k-\tau_k}^{t_k}\eta(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\eta(s)ds\right)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\eta(s)ds\Bigg].$$

If $x,y\in PC^1(J)$, are two solutions of (2.4), by Lemma 2.2, they satisfy the following two impulsive integral equation, respectively: $x(t)=(Ax)(t),\ y(t)=(Ay)(t)$. Since $\max_{t\in J}\{G(t,s)\}=\max(|CM_1|,|CM_2|)=u$, we have

$$\begin{aligned}
\| x-y\|_{PC} &=\| (Ax)(t)-(Ay)(t)\|_{PC}\\
&=\Bigg\|\int_0^T G(t,s)\Bigg[N(s)\int_0^s k(s,r)(-x(r)+y(r))dr+N_1(s)\int_0^T h(s,r)(-x(r)+y(r))dr\Bigg]ds\\
&\quad+\sum_{k=1}^{m}-L_kG(t,t_k)\left(\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x(s)ds-\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}y(s)ds\right)\Bigg\|_{PC}\\
&\le\sup_{t\in J}\int_0^T G(t,s)\Bigg[N(s)\int_0^s k(s,r)dr+N_1(s)\int_0^T h(s,r)dr\Bigg]ds\,\| x-y\|_{PC}\\
&\quad+u\sum_{k=1}^{m}L_k\,(a-(\sigma_{k-1}+\tau_k))\,\| x-y\|_{PC}\\
&=\Bigg\{\sup_{t\in J}\int_0^T G(t,s)\Bigg[N(s)\int_0^s k(s,r)dr+N_1(s)\int_0^T h(s,r)dr\Bigg]ds\\
&\quad+u\sum_{k=1}^{m}L_k(a-(\sigma_{k-1}+\tau_k))\Bigg\}\,\| x-y\|_{PC}\,.
\end{aligned}$$

From (2.5) and the Banach fixed point theorem, the impulsive integral equation $x=Ax$ has a unique fixed point $x\in PC^1(J)$. By Lemma 2.2, x is also the unique solution of (2.4). The proof is complete. ∎

3. Main result

In this section, we establish existence criteria for solution of problem (1.1) by the method of lower and upper solutions and the monotone iterative technique,we shall need the following definition.

Definition 3.1. *A function* $\alpha\in PC^1(J)$ *is called a lower solution of* (1.1) *if:*

$$\begin{cases}\alpha'(t)\le f(t,\alpha(t),(T\alpha)(t),(S\alpha)(t)), & t\in J^-,\\ \Delta\alpha(t_k)\le I_k\left(\int_{t_k-\tau_k}^{t_k}\alpha(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\alpha(s)ds\right), & k=1,2,\cdots,m,\\ g(\alpha(0),\alpha(T))\le 0.\end{cases}$$

Analogously, $\beta\in PC^1(J)$ *is called an upper solution of* (1.1) *if:*

$$\begin{cases}\beta'(t)\ge f(t,\beta(t),(T\beta)(t),(S\beta)(t)), & t\in J^-,\\ \Delta\beta(t_k)\ge I_k\left(\int_{t_k-\tau_k}^{t_k}\beta(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\beta(s)ds\right), & k=1,2,\cdots,m,\\ g(\beta(0),\beta(T))\ge 0.\end{cases}$$

For convenience, let us list the following conditions:

(H1) $\alpha(t)$, $\beta(t)$ are lower and upper solutions of (1.1) such that $\alpha(t)\le\beta(t)$.

(H2) There exist constants $M>0$ such that

$$f(t,x,y,z)-f(t,\bar{x},\bar{y},\bar{z})\geq -M(x-\bar{x})-N(t)(y-\bar{y})-N_1(t)(z-\bar{z}),$$

wherever $N(t),N_1(t)\in C(J,R^+),N(t)+N_1(t)\not\equiv 0$ in J, $\alpha(t)\leq\bar{x}(t)\leq x(t)\leq\beta(t),(T\alpha)(t)\leq\bar{y}(t)\leq y(t)\leq(T\beta)(t),(S\alpha)(t)\leq\bar{z}(t)\leq z(t)\leq(S\beta)(t)$.

(H3) There exist constants $0\leq L_k<1$ for $k=1,2,\cdots m$, such that

$$\begin{aligned}I_k&\left(\int_{t_k-\tau_k}^{t_k}x(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}x(s)ds\right)\\&-I_k\left(\int_{t_k-\tau_k}^{t_k}y(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}y(s)ds\right)\geq -L_k\left(\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x(s)-y(s)ds\right),\end{aligned}$$

wherever $\alpha(t_k)\leq y(t_k)\leq x(t_k)\leq\beta(t_k),\ k=1,2,\cdots m$.

Remark 3.1. The assumption (H3) was also used by Tariboon in [21].

(H4) There exist constants $M_1,\ M_2$ with $0\leq M_2e^{-MT}<M_1,\ M_1>0$ such that

$$g(x,y)-g(x^-,y^-)\leq M_1(x-x^-)-M_2(y-y^-),$$

wherever $\alpha(0)\leq x^-\leq x\leq\beta(0)$, and $\alpha(T)\leq y^-\leq y\leq\beta(T)$

(H5) The inequalities (2.2) and (2.5) hold.

Let $[\alpha(t),\beta(t)]=\left\{x\in PC^1(J):\alpha(t)\leq x(t)\leq\beta(t)\ \forall t\in J\right\}$.

Now we are in the position to establish the main results of this paper.

Theorem 3.1. *Let (H1)–(H5) hold. Then there exist monotone sequences* $\{\alpha_n(t)\},\{\beta_n(t)\}\subset PC^1(J)$ *with* $\alpha=\alpha_0\leq\alpha_1\leq\cdots\alpha_n\leq\cdots\leq\beta_n\leq\cdots\leq\beta_1\leq\beta_0=\beta$ *such that* $\lim_{n\to\infty}\alpha_n=x_*(t)$, $\lim_{n\to\infty}\beta_n=x^*(t)$, *uniformly on J. Moreover,* $x_*(t),x^*(t)$ *are minimal and maximal solution of* (1.1) *in* $[\alpha(t),\beta(t)]$, *respectively.*

Proof. For each $\eta\in[\alpha(t),\beta(t)]$, we consider (2.4) with

$$\sigma(t)=f(t,\eta(t),(T\eta)(t),(S\eta)(t))+M\eta(t)+N(t)(T\eta)(t)+N_1(t)(S\eta)(t).$$

By Lemma 2.3, we know that for any $\eta\in PC(J)$, (2.4) has a unique solution $x\in PC^1(J)$.

Now we define an operator B as: $x=A\eta$. then the operator B has the following properties:

(a). $\alpha_0\leq B\alpha_0,\ B\beta_0\leq\beta_0$.

(b). $B\eta_1\leq B\eta_2$, if $\alpha_0\leq\eta_1\leq\eta_2\leq\beta_0$.

To prove (a), let $\alpha_1=B\alpha_0$, and $m(t)=\alpha_0(t)-\alpha_1(t)$.

$$\begin{aligned}m'(t)&=\alpha_0'(t)-\alpha_1'(t)\\&\leq f(t,\alpha_0(t),(T\alpha_0)(t),(S\alpha_0)(t))-\big[f(t,\alpha_0(t),(T\alpha_0)(t),(S\alpha_0)(t))+M\alpha_0(t)\\&\quad+N(t)(T\alpha_0)(t)+N_1(t)(S\alpha_0)-M\alpha_1(t)-N(t)(T\alpha_1)(t)-N_1(t)(S\alpha_1)(t)\big]\\&=-M(\alpha_0(t)-\alpha_1(t))-N(t)(T(\alpha_0-\alpha_1))(t)-N_1(t)(S(\alpha_0-\alpha_1))(t)\\&=-Mm(t)-N(t)(Tm)(t)-N_1(t)(Sm)(t).\end{aligned}$$

$$\begin{aligned}\Delta m(t_k)&=\Delta\alpha_0(t_k)-\Delta\alpha_1(t_k)\\&\leq I_k\left(\int_{t_k-\tau_k}^{t_k}\alpha_0(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\alpha_0(s)ds\right)-\Big[-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\alpha_1(s)ds\end{aligned}$$

$$+I_k\left(\int_{t_k-\tau_k}^{t_k}\alpha_0(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\alpha_0(s)ds\right)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\alpha_0(s)ds\Bigg]$$
$$=-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}(\alpha_0(s)-\alpha_1(s))ds=-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}m(s)ds.$$

$$\begin{aligned}m(0)&=\alpha_0(0)-\alpha_1(0)\\&\le\alpha_0(T)-\left[-\frac{1}{M_1}g(\alpha_0(0),\alpha_0(T))+\alpha_0(0)+\frac{M_2}{M_1}(\alpha_1(T)-\alpha_0(T))\right]\\&=\frac{M_2}{M_1}(\alpha_0(T)-\alpha_1(T))=\frac{M_2}{M_1}m(T).\end{aligned}$$

By Lemma 2.1, we get $m(t)\le 0$ for $t\in J$, that is, $\alpha_0\le B\alpha_0$. Similarly, we can prove that $B\beta_0\le\beta_0$.

To prove (b), let $m(t)=x_1(t)-x_2(t)$, where $x_1=B\eta_1$, $x_2=B\eta_2$.

$$\begin{aligned}m'(t)&=x_1'(t)-x_2'(t)\\&=[f(t,\eta_1(t),(T\eta_1)(t),(S\eta_1)(t))+M\eta_1(t)+N(t)(T\eta_1)(t)+N_1(t)(S\eta_1)(t)-Mx_1(t)\\&\quad-N(t)(Tx_1)(t)-N_1(t)(Sx_1)(t)]-[f(t,\eta_2(t),(T\eta_2)(t),(S\eta_2)(t))+M\eta_2(t)\\&\quad+N(t)(T\eta_2)(t)+N_1(t)(S\eta_2)(t)-Mx_2(t)-N(t)(Tx_2)(t)-N_1(t)(Sx_2)(t)]\\&\le-M(x_1(t)-x_2(t))-N(t)(T(x_1-x_2))(t)-N_1(t)(S(x_1-x_2))(t)\\&=-Mm(t)-N(t)(Tm)(t)-N_1(t)(Sm)(t).\end{aligned}$$

$$\begin{aligned}&\Delta m(t_k)=\Delta x_1(t_k)-\Delta x_2(t_k)\\&\le\left[-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x_1(s)ds+I_k\left(\int_{t_k-\tau_k}^{t_k}\eta_1(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\eta_1(s)ds\right)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\eta_1(s)ds\right]\\&\quad-\left[-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}x_2(s)ds+I_k\left(\int_{t_k-\tau_k}^{t_k}\eta_2(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\eta_2(s)ds\right)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\eta_2(s)ds\right]\\&\le-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}(x_1(s)-x_2(s))ds=-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}m(s)ds.\end{aligned}$$

$$\begin{aligned}m(0)&=x_1(0)-x_2(0)\\&=-\frac{1}{M_1}g(\eta_1(0),\eta_1(T))+\eta_1(0)+\frac{M_2}{M_1}(x_1(T)-\eta_1(T))\\&\quad\left[-\frac{1}{M_1}g(\eta_2(0),\eta_2(T))+\eta_2(0)+\frac{M_2}{M_1}(x_2(T)-\eta_2(T))\right]\\&\le\frac{M_2}{M_1}m(T).\end{aligned}$$

By Lemma 2.1, we get $m(t)\le 0$ for $t\in J$, that is $B\eta_1\le B\eta_2$. Then (b) is proved.

Let $\alpha_n=B\alpha_{n-1}$ and $\beta_n=B\beta_{n-1}$ for $k=1,2,\cdots$ we get

$$\alpha=\alpha_0\le\alpha_1\le\cdots\alpha_n\le\cdots\le\beta_n\le\cdots\le\beta_1\le\beta_0=\beta$$

Obviously, each α_i, β_i $(i=1,2\cdots)$ satisfies:

$$\begin{cases} \alpha_i'(t)+M\alpha_i(t)+N(t)(T\alpha_i)(t)+N_1(t)(S\alpha_i)(t)=f(t,\alpha_{i-1}(t),(T\alpha_{i-1})(t),(S\alpha_{i-1})(t)) \\ \qquad +M\alpha_{i-1}(t)+N(t)(T\alpha_{i-1})(t)+N_1(t)(S\alpha_{i-1})(t), \quad t\in J^-, \\ \Delta\alpha_i(t_k)=-L_k\displaystyle\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\alpha_i(s)ds+I_k\biggl(\int_{t_k-\tau_k}^{t_k}\alpha_{i-1}(s)ds \\ \qquad -\displaystyle\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\alpha_{i-1}(s)ds\biggr)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\alpha_{i-1}(s)ds, \quad k=1,2,\cdots,m, \\ g(\alpha_{i-1}(0),\alpha_{i-1}(T))+M_1(\alpha_i(0)-\alpha_{i-1}(0))-M_2(\alpha_i(T)-\alpha_{i-1}(T))=0, \end{cases}$$

and

$$\begin{cases} \beta_i'(t)+M\beta_i(t)+N(t)(T\beta_i)(t)+N_1(t)(S\beta_i)(t)=f(t,\beta_{i-1}(t),(T\beta_{i-1})(t),(S\beta_{i-1})(t)) \\ \qquad +M\beta_{i-1}(t)+N(t)(T\beta_{i-1})(t)+N_1(t)(S\beta_{i-1})(t), \quad t\in J^-, \\ \Delta\beta_i(t_k)=-L_k\displaystyle\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\beta_i(s)ds+I_k\biggl(\int_{t_k-\tau_k}^{t_k}\beta_{i-1}(s)ds \\ \qquad -\displaystyle\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\beta_{i-1}(s)ds\biggr)+L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\beta_{i-1}(s)ds, \quad k=1,2,\cdots,m, \\ g(\beta_{i-1}(0),\beta_{i-1}(T))+M_1(\beta_i(0)-\beta_{i-1}(0))-M_2(\beta_i(T)-\beta_{i-1}(T))=0. \end{cases}$$

Therefore there exist x_* and x^* such that

$$\lim_{n\to\infty}\alpha_n=x_*(t), \quad \lim_{n\to\infty}\beta_n=x^*(t)$$

uniformly on J. Moreover, $x_*(t)$, $x^*(t)$ are solutions of (1.1) in $[\alpha(t),\beta(t)]$.

To prove that $x_*(t)$, $x^*(t)$ are extremal solutions of (1.1), let $x(t)\in[\alpha(t),\beta(t)]$ be any solution of (1.1), that is:

$$\begin{cases} x'(t)=f(t,x(t),(Tx)(t),(Sx)(t)), & t\in J^-, \\ \Delta x(t_k)=I_k\left(\int_{t_k-\tau_k}^{t_k}x(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}x(s)ds\right), & k=1,2,\cdots,m, \\ g(x(0),x(T))=0. \end{cases}$$

Suppose that there exists a positive integer n such that $\alpha_n(t)\le x\le\beta_n(t)$ on J. Then, let $m(t)=\alpha_{n+1}(t)-x(t)$, we have:

$$\begin{aligned} m'(t)&=\alpha_{n+1}'(t)-x'(t) \\ &=[f(t,\alpha_n(t),(T\alpha_n)(t),(S\alpha_n)(t))+M\alpha_n(t)+N(t)(T\alpha_n)(t)+N_1(t)(S\alpha_n)(t) \\ &\quad -M\alpha_{n+1}(t)-N(t)(T\alpha_{n+1})(t)-N_1(t)(S\alpha_{n+1})(t)]-f(t,x(t),(Tx)(t),(Sx)(t)) \\ &\le -M(\alpha_{n+1}(t)-x(t))-N(t)(T(\alpha_{n+1}-x))(t)-N_1(t)(S(\alpha_{n+1}-x))(t) \\ &=-Mm(t)-N(t)(Tm)(t)-N_1(t)(Sm)(t). \end{aligned}$$

$$\begin{aligned} \Delta m(t_k)&=\Delta\alpha_{n+1}(t_k)-\Delta x(t_k) \\ &=\biggl[-L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\alpha_{n+1}(s)ds+I_k\biggl(\int_{t_k-\tau_k}^{t_k}\alpha_n(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}\alpha_n(s)ds\biggr) \\ &\quad +L_k\int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k}\alpha_n(s)ds\biggr]-I_k\biggl(\int_{t_k-\tau_k}^{t_k}x(s)ds-\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}}x(s)ds\biggr) \end{aligned}$$

$$\leq -L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} (\alpha_{n+1}(s) - x(s))ds = -L_k \int_{t_{k-1}+\sigma_{k-1}}^{t_k-\tau_k} m(s)ds.$$

$$\begin{aligned} m(0) = \alpha_{n+1}(0) - x(0) &= -\frac{1}{M_1} g(\alpha_n(0), \alpha_n(T)) + \alpha_n(0) + \frac{M_2}{M_1}(\alpha_{n+1}(T) - \alpha_n(T)) - x(0) \\ &\leq \frac{1}{M_1}[-g(x(0), x(T)) + M_1 x(0) - M_2 x(T)] + \frac{M_2}{M_1}\alpha_{n+1}(T) - x(0) = \frac{M_2}{M_1} m(T). \end{aligned}$$

By Lemma 2.1, $m(t) \leq 0$ on J, i.e, $\alpha_{n+1}(t) \leq x$ on J. Similarly we obtain $x \leq \beta_{n+1}(t)$ on J. Since $\alpha_0 \leq x(t) \leq \beta_0$ on J, by induction we get $\alpha_n(t) \leq x \leq \beta_n(t)$ on J for every n. Therefore, $x_*(t) \leq x(t) \leq x^*(t)$ on J by taking $n \to \infty$. The proof is complete. ∎

Example 3.1. Consider the following boundary value problem:

$$\begin{cases} x'(t) = -2x(t) + \frac{1}{12}\sin^2 x(t) \int_0^t x(s)ds - \frac{1}{10}\int_0^1 x(s)ds, & t \in [0,1], t \neq \frac{1}{2}, \\ \Delta x(\frac{1}{2}) = \frac{1}{2}\int_0^{\frac{1}{4}} x(s)ds, \\ x(0) - 2x(1) - x^2(1) + 1 = 0. \end{cases} \tag{3.1}$$

Let $L_1 = 1/2, M = 2, N(t) = N_1(t) = 1/10, k(t,s) = h(t,s) = 1, J = [0,1], \mu = e^M, M_1 = M_2 = 1$. Then for $x_i, y_i, z_i, i = 1,2, x_1 \geq x_2, y_1 \geq y_2, z_1 \geq z_2$,

$$\begin{aligned} f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2) &= -2(x_1 - x_2) + \frac{1}{12}\left(\sin x_1^2 - \sin x_2^2\right)(y_1 - y_2) - \frac{1}{10}(z_1 - z_2) \\ &\geq -2(x_1 - x_2) - \frac{1}{10}(y_1 - y_2) - \frac{1}{10}(z_1 - z_2), \end{aligned}$$

$$\frac{1}{2}\int_0^{\frac{1}{4}} x(s)ds - \frac{1}{2}\int_0^{\frac{1}{4}} y(s)ds = \frac{1}{2}\int_0^{\frac{1}{4}} x(s) - y(s)ds \geq -\frac{1}{2}\int_{\frac{1}{4}}^{\frac{1}{2}} x(s) - y(s)ds,$$

where $x \geq y$. And

$$(x(0) - 2x(1) - x^2(1) + 1) - (y(0) - 2y(1) - y^2(1) + 1) \leq (x(0) - y(0)) - (x(1) - y(1)),$$

where $y(0) \leq x(0)$ and $y(1) \leq x(1)$. Thus the conditions (H2),(H3) and (H4) hold. Direct computation shows that

$$\int_0^1 q(s)ds + \frac{1}{4}(e^{\frac{1}{2}} - 1) = \frac{1}{20}e^2 + \frac{1}{4}e^{\frac{1}{2}} + \frac{1}{40}e^{-2} - \frac{7}{8} < \frac{7}{20} < 1.$$

$$\begin{aligned} \sup_{t\in J} \int_0^T G(t,s)\left[N(s)\int_0^s k(s,r)dr + N_1(s)\int_0^T h(s,r)dr\right]ds \\ + u\sum_{k=1}^m L_k(a - (\sigma_{k-1} + \tau_k)) = \frac{11}{40}(1 - e^{-2})^{-1} < 1. \end{aligned}$$

Therefore, the condition (H5) holds. It is easy to verify that (3.1) admits lower solution $\alpha(t)$ and upper solution $\beta(t)$ given by

$$\alpha(t) = \begin{cases} -1, & t \in \left[0, \frac{1}{2}\right], \\ -2, & t \in \left(\frac{1}{2}, 1\right], \end{cases} \qquad \beta(t) = \begin{cases} \frac{1}{10}, & t \in \left[0, \frac{1}{2}\right], \\ \frac{1}{5}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Obviously, $\alpha(t) \leq \beta(t)$. And thus the conclusion of Theorem 3.1 holds for (3.1).

Acknowledgement. Project supported by NNSF of China Grant No. 11271087, No. 612 63006 and Guangxi Scientific Experimental (China-ASEAN Research) Centre No. 20120116.

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