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Robustness of Correlation Coefficient and Variance Ratio under Elliptical Symmetry

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Abstract. Parametric robustness of a statistic in a class of distributions implies that the distribution of the statistic is the same for any member of the class of distributions. The bivariate Wishart distribution, based on the class of bivariate elliptical distributions, involves three essential statistics, namely, two sample variances and the product moment correlation coefficient. The distribution of the product moment correlation coefficient is known to be robust in the class of bivariate elliptical distributions. In this paper, we prove that the distribution of the variance ratio is also robust in the class of bivariate elliptical distributions.

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1. Introduction

Consider a two component column random variable $\underset{\sim}{X}$, where $\underset{\sim}{X}' = (X_1, X_2)$, follows the class of bivariate elliptical distributions with mean $\underset{\sim}{\theta}$ (column vector order 2 components) and scale matrix Σ (a 2×2 matrix) where $\underset{\sim}{\theta}' = (\theta_1, \theta_2)$ and $\Sigma = (\sigma_{ik}), i = 1,2; k = 1,2$. Let $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2, \quad \sigma_{12} = \rho\sigma_1\sigma_2$ with $\sigma_1 > 0, \sigma_2 > 0$ and the quantity $\rho(-1 < \rho < 1)$ is the product moment correlation coefficient between X_1 and X_2. Let each of the sample observation $\underset{\sim}{X}_j, (j = 1,2,\cdots,N)$, where $\underset{\sim}{X}'_j, = (X_{1j}, X_{2j}), j = 1,2,\cdots,N$, follow the class of bivariate elliptical distributions. The sample mean vector is $\underset{\sim}{\overline{X}}$ where $\underset{\sim}{\overline{X'}} = (\overline{X_1}, \overline{X_2}), \underset{\sim}{\overline{X_i}} = N^{-1}\sum_{j=1}^N X_{ij}, i = 1,2$ so that the sums of squares and cross product matrix is given by $A = (a_{ik})$, where $a_{ik} = \sum_{j=1}^N (X_{ij} - \bar{X}_{i.})(X_{kj} - \bar{X}_{k.}), i = 1,2; k = 1,2$. Obviously, $a_{ii} = \sum_{j=1}^N (X_{ij} - \bar{X}_i)^2, \quad (i = 1,2)$, and $a_{12} = \sum_{j=1}^N (X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2)$. The sample correlation coefficient is then given by $r = a_{12}/(ms_1 s_2)$, where $m = N-1$ and $s_i^2 = a_{ii}/m(i = 1,2)$.

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Fisher [10] derived the distribution of the matrix A to study the distribution of correlation coefficient for a bivariate normal sample. The distribution of A is also known for the class of elliptical distributions (see, for example, Sutradhar and Ali [28]).

A recent interest among the applied scientists is the use of fat tailed distribution for modeling business data such as stock returns. Since the bivariate t-distribution has fatter tails, it has been increasingly applied for modeling business data. Interested readers may go through Sutradhar and Ali [27], Lange *et al.* [21], Billah and Saleh [4] and Kibria and Saleh [20] among others. A wider class of distributions accommodating bivariate t-distribution or bivariate normal distribution is the class of Compound Normal Distributions. Bivariate elliptical distributions accommodates all these distributions. We assume that in the case of bivariate elliptical distribution, the observations in the sample are uncorrelated but not necessarily independent. There have been a number of publications in the recent years but for interested readers, we refer to Fang and Anderson [7] and Fang and Zhang [8].

It is proved by Ali and Joarder [1] that the distribution of R is robust in the class of bivariate elliptical distributions. Thus the null $(H_0 : \rho = 0)$ or non-null $(H_1 : \rho \neq 0)$ distribution of the test statistic R is robust. However, if the parent population is bivariate normal, the test statistic $(m-1)^{1/2}(R-\rho)(1-R^2)^{-1/2}$ is known to have a t-distribution with $m-1$ degrees of freedom under the null hypothesis. In view of Fang and Anderson [7], Fang and Zhang [8] or Ali and Joarder [1], the test statistic is also robust in the class of bivariate elliptical distributions.

Consider the scaled variances $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$. Assuming that the observations are from a bivariate normal population, Bose [5] and Finney [9] proved that the distribution of variance ratio $H = U/V$ has a correlated $F(m,m;\rho)$ distribution (see equation 5.1) which specializes to usual F-distribution $F(m,m)$ for $\rho = 0$. The random variables U and V have a bivariate chi-square distribution [12] with correlation coefficient ρ^2 and found application in signal processing [11].

Can we relax the assumption of bivariate normality to a broader class of distributions and study the behaviour of the variance ratio? In this paper, we prove that if the sample observations are governed by the class of bivariate elliptical distributions, the distribution of the variance ratio remains the same for any member of the class.

The organization of the paper is as follows: Some mathematical preliminaries are given in Section 2. Section 3 introduces the class of bivariate elliptical distributions. Robustness of the distribution of correlation coefficient and tests on correlation coefficient is presented in Section 4. In Section 5, we prove that the distribution of variance ratio $(H = U/V)$ has a correlated $F(m,m;\rho)$distribution if the observations are governed by the class of bivariate elliptical distributions. Thus we say that the distribution of the variance ratio is robust or the distribution of the variance ratio is invariant in the class of bivariate elliptical distributions.

2. Some mathematical preliminaries

The duplication formula of gamma function is given by

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+(1/2)). \tag{2.1}$$

Multiplying both sides of (2.1) by $2z$ we have

$$(2z)!\sqrt{\pi} = 2^{2z}z!\Gamma\left(z+\tfrac{1}{2}\right). \tag{2.2}$$

The beta function is defined by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \tag{2.3}$$

which can be expressed as $\Gamma(a)\Gamma(b)/\Gamma(a+b)$.

Lemma 2.1. *Let the non-negative random variables X and Y have the joint density function $f(x,y)$. Further let $H = X/Y$. Then the density function of H is given by*

$$g_H(h) = \int_0^\infty f(yh,y)ydy. \tag{2.4}$$

3. The class of bivariate elliptical distributions

In this section, we will present the class of bivariate elliptical distributions which includes the class of compound bivariate normal distribution. The latter includes bivariate t and bivariate normal distributions as special cases.

The probability density function of a bivariate elliptical random variable $\underset{\sim}{X}$, where $\underset{\sim}{X} = (X_1, X_2)$, is given by

$$f_1(\underset{\sim}{x}) \infty \mid \Sigma \mid^{\frac{-1}{2}} g_{N,2}((\underset{\sim}{X} - \underset{\sim}{\theta})'\Sigma^{-1}(\underset{\sim}{X} - \underset{\sim}{\theta})), \tag{3.1}$$

where $\Sigma > 0$ and the normalizing constant is determined by the form of g [28]. Johnson [17] describes the generation of random samples from the class of bivariate elliptical distributions. The density function (3.1) is also called bivariate elliptically contoured distributions [7]. A bivariate random variable is said to have elliptical symmetry if its density function is given by (3.1).

Now consider a sample $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \cdots \underset{\sim}{X}_N (N > 2)$ having the joint probability density function

$$f_2(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots \underset{\sim}{X}_N) \infty \mid \Sigma \mid^{\frac{-N}{2}} g_{N,2}\left(\Sigma_{j=1}^{N}(\underset{\sim j}{x} - \underset{\sim}{\theta})'\Sigma^{-1}(\underset{\frown}{x}_j - \underset{\sim}{\theta})\right), \tag{3.2}$$

where $\Sigma > 0$ and the normalizing constant is determined by the form of g. Each observation $\underset{\sim}{X}_j (j = 1, 2, \cdots, N)$ in (3.2) follows (3.1). Since the observations are uncorrelated but not necessarily independent, (3.2) is called Uncorrelated and Identical Bivariate Elliptical (UIBE) model for the sample. It can be checked that the coefficient of correlation between components X_{1j} and X_{2j} of $X_j (j = 1, 2, \cdots, N)$ is ρ [1]. Note that if $g_{N,2}(u) = e^{-u/2}$ in (3.2), then it defines the joint density function of N independent observations from a bivariate normal distribution.

The model in (3.2) is a bivariate version of the multivariate sampling model considered by Anderson, Fang and Hsu [3], Fang and Anderson [7], Fang and Zhang [8], Kibria and Haq [19] and Kibria [18] among others.

It is worth mentioning that if the sample observations are independent and identical bivariate elliptical distributions, then $f_2(\underset{\sim 1}{x}, \underset{\sim 2}{x}, \cdots, \underset{\sim N}{x}) = \prod_{j=1}^{N} f_1(\underset{\sim j}{x})$, which will be called an Independent and Identical Bivariate.

Theorem 3.1. [28, p. 158] *Let A be the mean centered sum of squares and product matrix based on UIBE model (3.2). Then the density function of the A is given by*

$$f(A) \propto |\Sigma|^{-m/2}|A|^{(m-3)/2} g_{m,2}(tr\Sigma^{-1}A),\ A > 0 \tag{3.3}$$

where $m > 2$ and $\Sigma > 0$. The above can be written in terms of the elements of the matrix A as

(3.4)
$$f(a_{11}, a_{22}, a_{12}) \propto \frac{1}{(\sigma_1\sigma_2)^m} \left(a_{11}a_{22} - a_{12}^2\right)^{(m-3)/2} g_{m,2}\left(\frac{a_{11}}{(1-\rho^2)\sigma_1^2} + \frac{a_{22}}{(1-\rho^2)\sigma_2^2} - \frac{2\rho a_{12}}{(1-\rho^2)\sigma_1\sigma_2}\right),$$

where $a_{11} > 0, a_{22} > 0, -\sqrt{a_{11}a_{22}} < a_{12} < \sqrt{a_{11}a_{22}}, m > 2, -1 < \rho < 1$.

Some other examples of bivariate elliptical distributions are Pearson Type II Distribution, [17, p. 111], Laplace and generalized Laplace Distributions [22] and [16], Pearson Type VII Distribution, Symmetric Kotz Type Distribution, Uniform distribution, Logistic Distribution, Stable Distribution etc. [6, p. 69].

4. Robustness of some tests on correlation coefficient

In the following theorem, we derive the joint density function of scaled variances and correlation coefficient.

Theorem 4.1. *Let S_1^2, S_2^2 and R be variances and correlation coefficient based on UIBE model (3.2). Then the joint density function of $U = mS_1^2/\sigma_1^2, V = mS_2^2/\sigma_2^2$ and R is given by*

$$f_{U,V,R}(u,v,r) \propto (uv)^{(m/2)-1}(1-r^2)^{(m-3)/2} g_{m,2}\left(\frac{u+v}{1-\rho^2} - \frac{2\rho r\sqrt{uv}}{1-\rho^2}\right), \tag{4.1}$$

where $m > 2$ and $-1 < \rho < 1$.

Proof. Consider the transformation $a_{11} = ms_1^2, a_{22} = ms_2^2, a_{12} = mrs_1s_2$ in (3.4) with Jacobian $J((a_{11}, a_{22}, a_{12}) \to (s_1^2, s_2^2, r)) = m^3 s_1 s_2$. Then the density function of S_1^2, S_2^2 and R is given by

$$f_{S_1^2,S_2^2,R} \propto (s_1s_2)^{m-2}(1-r^2)^{(m-3)/2} g_{m,2}\left(\frac{ms_1^2}{(1-\rho^2)\sigma_1^2} + \frac{ms_2^2}{(1-\rho^2)\sigma_2^2} - \frac{2\rho r m s_1 s_2}{\sigma_1\sigma_2(1-\rho^2)}\right).$$

By making the transformation $ms_1^2 = \sigma_1^2 u$, $ms_2^2 = \sigma_2^2 v$, keeping r intact, with Jacobian $J(s_1^2, s_2^2 \to uv) = (\sigma_1\sigma_2/m)^2$, the density of U, V and R is given by (4.1). ∎

The density function in (4.1) depends on the particular form of $g_{m,2}(.)$ implying that the joint distribution of scaled variances and correlation coefficient is not robust. Note that in case of sampling from a bivariate normal population, the joint density function of scaled variances and correlation coefficient is given by

$$f_{U,V,R}(u,v,r) = \frac{(1-\rho^2)^{-m/2}(uv)^{(m-2)/2}(1-r^2)^{(m-3)/2}}{2^m\sqrt{\pi}\,\Gamma(m/2)\Gamma((m-1)/2)} \exp\left(-\frac{u+v}{2(1-\rho^2)} + \frac{\rho r\sqrt{uv}}{1-\rho^2}\right), \tag{4.2}$$

where $m > 2$ and $-1 < \rho < 1$, [15].

The density function of R was derived originally by Fisher [10] assuming that the sample is from a bivariate normal distribution. In case $\rho = 0$, the distribution of R based on a sample governed by UIBE model (3.2) is given by Theorem 4.2.4 of Fang and Zhang [8, p. 137]). For any $\rho(-1 < \rho < 1)$, the distribution of R based on a sample governed by UIBE model (3.2) follows from Theorem 4 of Fang and Anderson [7, p. 10]. The following theorem is due to Ali and Joarder [1] who independently proved that if the sample observations follow UIBE model (3.2), the distribution of R remains the same, for any value of ρ $(-1 < \rho < 1)$, as that obtained by Fisher [10] for the bivariate normal case. We sketch an outline of the proof just for the self containment of the paper.

Theorem 4.2. *The density function of sample correlation coefficient R based on a sample following Uncorrelated and Identical Bivariate Elliptical model (3.2) is given by*

$$f_R(r) = \frac{2^{m-2}\left(1-\rho^2\right)^{m/2}}{\pi\Gamma(m-1)}\left(1-r^2\right)^{(m-3)/2}\sum_{k=0}^{\infty}\frac{(2\rho r)^k}{k!}\Gamma^2\left(\frac{m+k}{2}\right), -1<r<1, \tag{4.3}$$

where $m>2$ *and* $-1<\rho<1$.

Proof. Integrating out u and v from the density function (4.1) of U,V and R, the density function of R is given by

$$f_R(r) \propto (1-r^2)^{(m-3)/2} I(r;\rho,m), \tag{4.4}$$

where

$$I(r;\rho,m) = 4\int_0^\infty\int_0^\infty (y_1y_2)^{m/2-1} g_{m,2}\left(y_1^2+y_2^2-2\rho r\sqrt{y_1y_2}\right)dy_1dy_2.$$

Then the transformation $y_1 = w\cos\theta,\ y_2 = w\sin\theta$ yields

$$I(r;\rho,m) = 2^{-m+3}\int_{\theta=0}^{\pi/2}\int_{w=0}^{\infty}(\sin 2\theta)^{m-1}w^{2m-1}g_{m,2}\left(w^2-\rho r w^2\sin 2\theta\right)^{-\nu-m}dwd\theta.$$

By substituting $w^2=u,\ 2\theta=\alpha$, the above integral can be evaluated to be

$$I(r;\rho,m) = \sum_{k=0}^{\infty}\frac{(2\rho r)^k}{k!}\frac{\Gamma^2((m+k)/2)}{\Gamma(m)}\int_{w=0}^{\infty}w^{m-1}g_{m,2}(w)dw.$$

Since the last integral does not involve r, from (4.4), we have (4.3). ∎

Theorem 4.2 indicates robustness of the correlation coefficient in the class of bivariate elliptical populations. Thus the assumption of bivariate normality under which tests on correlation coefficient are developed can be relaxed to a broader class of bivariate elliptical distributions.

If $\rho=0$, $R^2 \sim \text{Beta}(1/2,(m-1)/2)$, and $\sqrt{m-1}R\left(1-R^2\right)^{-1/2}$ has a student t- distribution with $(m-1)$ degrees of freedom. The likelihood ratio test of the hypothesis $H_0:\rho=0$ against the alternative $H_1:\rho\neq 0$ is accomplished by the above statistic. Acceptance of the null hypothesis does not mean independence unless the sample is from bivariate normal distribution. In view of Ali and Joarder [1], the test is robust in the class of bivariate elliptical distributions though in this case the acceptance of $H_0:\rho=0$ implies uncorrelation but not necessarily independence.

The most popular test is based on $Z=\tanh^{-1}R=\ln\sqrt{(1+R)/(1-R)}$ has an approximate normal distribution with mean $\ln\sqrt{(1+\rho)/(1-\rho)}$ and variance $1/(m-2)$. In view of Ali and Joarder [1], tests developed for correlation coefficient based on bivariate normal distribution by Muddapur [24], Samiuddin [26] and Anderson [2] are all robust. See e.g. Joarder [14].

We warn that the distribution of R is not necessarily robust for independent observations from elliptical population. The model for samples considered in (3.2) implies that the observations $\underset{\sim}{X}_j$ $(j=1,2,\cdots,N)$ are uncorrelated but not necessarily independent. The asymptotic distribution of R for independent observations from bivariate elliptical population was obtained by Muirhead [25, p. 157].

5. The distribution of variance ratio

In this section we will prove that the distribution of $H = U/V$ is robust in the class of bivariate elliptical distributions.

Theorem 5.1. *Let S_1^2, S_2^2 and R be variances and correlation coefficient based on Uncorrelated and Identical Bivariate Elliptical model (3.2). Also let $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$ be scaled sample variances. Then the density function of $H = U/V$ is given by*

$$f_H(h) = \frac{(1-\rho^2)^{m/2}}{B(m/2,m/2)} \frac{h^{(m-2)/2}}{(1+h)^m} \left(1 - \frac{4\rho^2 h}{(1+h)^2}\right)^{-(m+1)/2}, h > 0, \quad (5.1)$$

where $m > 2$ and $-1 < \rho < 1$.

Proof. It follows from (4.1) that the joint density function of U and V is given by

$$f_{U,V}(u,v) \propto (uv)^{(m-2)/2} \int_{-1}^{1} \left(1-r^2\right)^{(m-3)/2} g_{m,2}\left(\tfrac{u+v}{1-\rho^2} - \tfrac{2\rho r\sqrt{uv}}{1-\rho^2}\right) dr. \quad (5.2)$$

Applying Lemma 2.1 in (5.2), the density function of $H = U/V$ is given by

$$f_H(h) \propto h^{(m-2)/2} \int_{r=-1}^{1} \left(1-r^2\right)^{(m-3)/2} \int_{v=0}^{\infty} v^{m-1} g_{m,2}\left(\frac{v(1+h)}{1-\rho^2} - \frac{2\rho r v\sqrt{h}}{1-\rho^2}\right) dv dr.$$

Substituting $\frac{v(1+h)}{1-\rho^2} - \frac{2\rho r v\sqrt{h}}{1-\rho^2} = y$, with the Jacobian $J(v \to y) = \frac{1-\rho^2}{(1+h) - 2\rho r\sqrt{h}}$,
we have

$$f_H(h) \propto \frac{h^{(m-2)/2}}{(1+h)^m} \int_{r=-1}^{1} \left(1 - \frac{2\rho r\sqrt{h}}{1+h}\right)^{-m} \left(1-r^2\right)^{(m-3)/2} dr \int_{y=0}^{\infty} y^{m-1} g_{m,2}(y)\, dy. \quad (5.3)$$

Since the integral in y gets absorbed into the normalizing constant [14] and $2|\rho r|\sqrt{h} \le 1+h$, expanding the binomial term

$$\left(1 - \frac{2\rho r\sqrt{h}}{1+h}\right)^{-m}$$

in (5.3), we have

$$f_H(h) \propto \frac{h^{(m-2)/2}}{(1+h)^m} \sum_{k=0}^{\infty} \frac{\Gamma(m+k)}{\Gamma(m)k!} \left(\frac{2\rho\sqrt{h}}{1+h}\right)^k I(k), \quad (5.4)$$

where $I(k) = \int_{r=-1}^{1} r^k \left(1-r^2\right)^{(m-3)/2} dr$. The integral $I(k)$can be simplified to be

$$I(k) = \frac{[(-1)^k + 1]}{2} \int_{r=0}^{1} u^{(k-1)/2} (1-u)^{(m-3)/2} du,$$

which , by beta integral (2.3), simplifies to

$$I(k) = \frac{[(-1)^k + 1]}{2} \frac{\Gamma((k+1)/2)\Gamma((m-1)/2)}{\Gamma((k+m)/2)}. \quad (5.5)$$

By using (5.5) in (5.4), and simplifying, we have

$$(5.6) \quad f_H(h) \propto \frac{h^{(m-2)/2}}{(1+h)^m} \sum_{k=0}^{\infty} \frac{\Gamma((m+k+1)/2)}{k!} \left(\frac{4\rho\sqrt{h}}{1+h}\right)^k \frac{[(-1)^k+1]}{2} \Gamma((k+1)/2).$$

Since each odd k provide zero measure for the summand in the right hand side of (5.6) , we replace k in the summand by $2k$ and use (2.2) so that (5.6) simplifies to

$$f_H(h) \propto \frac{h^{(m-2)/2}}{(1+h)^m} \sum_{k=0}^{\infty} \frac{\Gamma((m+2k+1)/2)}{k!} \left(\frac{4\rho^2 h}{(1+h)^2}\right)^k,$$

which simplifies to (5.1).

Equation (5.1) is well known for bivariate normal distribution [5, 9]. This proves the robustness of the distribution of the variance ratio in the class of bivariate elliptical distributions. The distribution of test statistic $H = U/V$ given by (5.1) will be denoted by $F(m,m;\rho)$. ∎

6. Conclusion

The testing of equality of variances in presence of correlation with a bivariate normal population has a long history. Under the null hypothesis, the test statistic $H = U/V$ has a $F(m,m;\rho)$ distribution [5, 9] and can be used for testing $H_0 : \sigma_1^2 = \sigma_2^2,\ \rho \neq 0$ against $H_0 : \sigma_1^2 \neq \sigma_2^2,\ \rho \neq 0$ if ρ is known. A test would be to reject the null hypothesis if $H < c$ or $H > k$ such that $P(H > c) = 1 - (\alpha/2)$ and $P(H > k) = \alpha/2$ where the density function of H is given by (5.1).

Finney [9] compared the variability of the measurements of standing height and stem length for different age group of school boys by his method with the help of Hirschfeld [13]. Wilks [29] developed the likelihood ratio test for testing the equality of variances in presence of correlation if the parent population is bivariate or multivariate normal. An excellent review is available in Modarres [23] who also performed Monte Carlo simulation to determine the behaviour of the likelihood ratio test.

In this paper, we proved that the assumption of bivariate normality can be relaxed to the class of bivariate elliptical distributions for testing equality of variances in presence of correlation. However, the acceptance of the null hypothesis would generally mean uncorrelation; it would mean independence in the special case of bivariate normality. The robustness of the distribution of the variance ratio or of the test of equivariance will stimulate statisticians, econometricians and business experts to embark on further investigations in the area, let alone the use of classical results with confidence.

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