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Generic Properties of Module Maps and Characterizing Inverse Limits of C*-Algebras of Compact Operators

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Abstract. We study closedness of the range, adjointability and generalized invertibility of modular operators between Hilbert modules over locally C*-algebras of coefficients. Our investigations and the recent results of M. Frank reveal a number of equivalence properties of the category of Hilbert modules over locally C*-algebras which characterize precisely the inverse limit of C*-algebras of the C*-algebra of compact operators.

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1. Introduction

Locally C*-algebras are generalizations of C*-algebras. A locally C*-algebra is a complete Hausdorff complex topological $*$-algebra $\mathscr{A}$, whose topology is determined by its continuous C*-seminorms in the sense that the net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for every continuous C*-seminorm p on $\mathscr{A}$. Locally C*-algebras were first introduced by Inoue [13] and studied more by Phillips and Fragoulopoulou [8,21]. See also the book of Joita [14] and references therein. Hilbert modules are essentially objects like Hilbert spaces by allowing the inner product to take values in a (locally) C*-algebra rather than the field of complex numbers. They play an important role in the modern theory of operator algebras, in noncommutative geometry and in quantum groups, see [10].

Throughout the present paper we refer to C*-subalgebras of the C*-algebras of compact operators on Hilbert spaces as *C*-algebras of compact operators*. Recall that a C*-algebra of compact operators is a c_0-direct sum of elementary C*-algebras $\mathscr{K}(H_i)$ of all compact operators acting on Hilbert spaces $H_i,\ i\in I$, cf. [2, Theorem 1.4.5].

Magajna and Schweizer, respectively, have shown that C*-algebras of compact operators can be characterized by the property that every closed (and coinciding with its biorthogonal

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complement, respectively) submodule of every Hilbert C*-module over them is automatically an orthogonal summand, cf. [20, 22]. Together with results of Arambašić, Bakić and Guljaš [1, 3, 9], numerous generic properties of the category of Hilbert C*-modules over C*-algebras which characterize precisely the C*-algebras of compact operators have been found by Frank and the author in [5–7]. The later work motivate us to study some properties of modular operators, such as closedness of the range, adjointability, polar decomposition and generalized invertibility of module maps between Hilbert modules over locally C*-algebras of coefficients. These help us to obtain a number of equivalence properties which describe precisely the inverse limit of C*-algebras of compact operators.

In the present paper we recall some definitions and simple facts about Hilbert modules over locally C*-algebras and the module maps between them. Then we study the closedness of the range and adjointability of module maps, in fact we will prove that a bounded module map between Hilbert modules over locally C*-algebras is adjointable if and only if its graph is an orthogonal summand (compare [5]). A bounded adjointable module map possesses a generalized inverse if and only if it has a closed range. Finally, for a given locally C*-algebra $\mathscr{A}$ we demonstrate that any bounded $\mathscr{A}$-module map between arbitrary $\mathscr{A}$-modules possesses an adjoint $\mathscr{A}$-module map, if and only if the images of all bounded $\mathscr{A}$-module maps with closed range between arbitrary Hilbert $\mathscr{A}$-modules are orthogonal summands, if and only if every bounded $\mathscr{A}$-module map between arbitrary Hilbert $\mathscr{A}$-modules has polar decomposition, if and only if every bounded $\mathscr{A}$-module map between arbitrary Hilbert $\mathscr{A}$-modules has generalized inverse, if and only if $\mathscr{A}$ is an inverse limit of C*-algebras of compact operators.

2. Preliminaries

Suppose $\mathscr{A}$ is a locally C*-algebra and $S(\mathscr{A})$ is the set of all continuous C*-seminorms on $\mathscr{A}$. For every $p \in S(\mathscr{A})$, the quotient $*$-algebra $\mathscr{A}/N_p^{\mathscr{A}}$ is denoted by $\mathscr{A}_p$, where $N_p^{\mathscr{A}} = \{a \in \mathscr{A} : p(a) = 0\}$ is a C*-algebra in the C*-norm induced by p. The canonical map from $\mathscr{A}$ to $\mathscr{A}_p$ is denoted by $\pi_p^{\mathscr{A}}$ and a_p is reserved to denote $\pi_p^{\mathscr{A}}(a)$. For $p, q \in S(\mathscr{A})$ with $p \geq q$, the surjective canonical map $\pi_{pq}^{\mathscr{A}} : \mathscr{A}_p \to \mathscr{A}_q$ is defined by $\pi_{pq}^{\mathscr{A}}(\pi_p^{\mathscr{A}}(a)) = \pi_q^{\mathscr{A}}(a)$ for all $a \in \mathscr{A}$. Then $\{\mathscr{A}_p; \pi_{pq}^{\mathscr{A}}\}_{p,q \in S(\mathscr{A}), p \geq q}$ is an inverse system of C*-algebras and $\varprojlim_p \mathscr{A}_p$ is a locally C*-algebra which can be identified with $\mathscr{A}$. We refer to the book [8] and papers [13, 21] for more information and useful examples. A morphism of locally C*-algebras is a continuous $*$-morphism from a locally C*-algebra $\mathscr{A}$ to another locally C*-algebra $\mathscr{B}$. An isomorphism of locally C*-algebras from $\mathscr{A}$ to $\mathscr{B}$ is a bijective map $\Phi : \mathscr{A} \to \mathscr{B}$ such that Φ and Φ^{-1} are morphisms of locally C*-algebras.

A (right) *pre-Hilbert module* over a locally C*-algebra algebra $\mathscr{A}$ is a right $\mathscr{A}$-module E, compatible with the complex algebra structure, equipped with an $\mathscr{A}$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathscr{A}$, $(x, y) \mapsto \langle x, y \rangle$, which is $\mathscr{A}$-linear in the second variable y and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \text{ and } \langle x, x \rangle \geq 0 \text{ with equality if and only if } x = 0.$$

A pre-Hilbert $\mathscr{A}$-module E is a Hilbert $\mathscr{A}$-module if E is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p \in S(\mathscr{A})}$ where $\overline{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)}$, $\xi \in E$. If E, F are two Hilbert $\mathscr{A}$-modules then the set of all ordered pairs of elements

$E \oplus F$ from E and F is a Hilbert $\mathscr{A}$-module with respect to the $\mathscr{A}$-valued inner product $\langle (x_1,y_1),(x_2,y_2)\rangle = \langle x_1,x_2\rangle_E + \langle y_1,y_2\rangle_F$. It is called the direct *orthogonal sum of* E *and* F.

We say that a Hilbert $\mathscr{A}$-submodule X of a Hilbert $\mathscr{A}$-module E is a topological summand if E can be decomposed into the direct sum of the Banach $\mathscr{A}$-submodule X and of another Banach $\mathscr{A}$-submodule Y. The notation is $E = X \dot{+} Y$. If, moreover, the decomposition can be arranged as an orthogonal one (i.e. $X \perp Y$) then the Hilbert $\mathscr{A}$-submodule X is an orthogonal summand of the Hilbert $\mathscr{A}$-module E. In this case, we write $E = X \oplus Y$ and $Y = X^{\perp}$.

Let E be a Hilbert $\mathscr{A}$-module and $p \in S(\mathscr{A})$, then $N_p^E = \{\xi \in E;\ \overline{p}_E(\xi) = 0\}$ is a closed submodule of E and $E_p = E/N_p^E$ is a Hilbert $\mathscr{A}_p$-module with $(\xi + N_p^E)\pi_p^{\mathscr{A}}(a) = \xi a + N_p^E$ and $\langle \xi + N_p^E, \eta + N_p^E\rangle = \pi_p^{\mathscr{A}}(\langle \xi,\eta\rangle)$. The canonical map from E onto E_p is denoted by σ_p^E and ξ_p is reserved to denote $\sigma_p^E(\xi)$. For $p,q \in S(\mathscr{A})$ with $p \geq q$, the surjective canonical map $\sigma_{pq}^E : E_p \to E_q$ is defined by $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$ for all $\xi \in E$. Then $\{E_p;\mathscr{A}_p;\sigma_{pq}^E,\pi_{pq}^{\mathscr{A}}\}_{p,q\in S(\mathscr{A}),p\geq q}$ is an inverse system of Hilbert C*-modules in the following sense:

- $\sigma_{pq}^E(\xi_p a_p) = \sigma_{pq}^E(\xi_p)\pi_{pq}^{\mathscr{A}}(a_p),\ \xi_p \in E_p,\ a_p \in \mathscr{A}_p,\ p,q \in S(\mathscr{A}),\ p \geq q,$
- $\langle \sigma_{pq}^E(\xi_p), \sigma_{pq}^E(\eta_p)\rangle = \pi_{pq}^{\mathscr{A}}(\langle \xi_p,\eta_p\rangle),\ \xi_p,\eta_p \in E_p,\ p,q \in S(\mathscr{A}),\ p \geq q,$
- $\sigma_{qr}^E \circ \sigma_{pq}^E = \sigma_{pr}^E$ if $p,q,r \in S(\mathscr{A})$, $p \geq q \geq r$, and
- $\sigma_{pp}(\xi_p) = \xi_p,\ \xi_p \in E_p,\ p \in S(\mathscr{A})$.

In this case, $\varprojlim_p E_p$ is a Hilbert $\mathscr{A}$-module which can be identified with E.

Let E and F be Hilbert $\mathscr{A}$-modules and $T : E \to F$ be an $\mathscr{A}$-module map. The module map T is called *bounded* if for each $p \in S(\mathscr{A})$, there is $K_p > 0$ such that $\overline{p}_F(Tx) \leq K_p\,\overline{p}_E(x)$ for all $x \in E$. The module map T is called *adjointable* if there exists an $\mathscr{A}$-module map $T^* : F \to E$ with the property $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all $x \in E$, $y \in F$. It is well known that every adjointable $\mathscr{A}$-module map is bounded, cf. [14, Lemma 2.2.3]. The set $\mathscr{L}_{\mathscr{A}}(E,F)$ of all bounded adjointable $\mathscr{A}$-module maps from E into F becomes a locally convex space with topology defined by the family of seminorms $\{\tilde{p}_{\mathscr{L}_{\mathscr{A}}(E,F)}\}_{p\in S(\mathscr{A})}$, in which, $\tilde{p}_{\mathscr{L}_{\mathscr{A}}(E,F)}(T) = \|(\pi_p^{\mathscr{A}})_*(T)\|_{\mathscr{L}_{\mathscr{A}_p}(E_p,F_p)}$ and $(\pi_p^{\mathscr{A}})_* : \mathscr{L}_{\mathscr{A}}(E,F) \to \mathscr{L}_{\mathscr{A}_p}(E_p,F_p)$ is defined by $(\pi_p^{\mathscr{A}})_*(T)(\xi + N_p^E) = T\xi + N_p^F$ for all $T \in \mathscr{L}_{\mathscr{A}}(E,F)$, $\xi \in E$. Suppose $p,q \in S(\mathscr{A})$, $p \geq q$ and $(\pi_{pq}^{\mathscr{A}})_* : \mathscr{L}_{\mathscr{A}_p}(E_p,F_p) \to \mathscr{L}_{\mathscr{A}_q}(E_q,F_q)$ is defined by $(\pi_{pq}^{\mathscr{A}})_*(T_p)(\sigma_q^E(\xi)) = \sigma_{pq}^F(T_p(\sigma_p^E(\xi))$. Then $\{\mathscr{L}_{\mathscr{A}_p}(E_p,F_p);\ (\pi_{pq}^{\mathscr{A}})_*\}_{p,q\in S(\mathscr{A}),p\geq q}$ is an inverse system of Banach spaces and $\varprojlim_p \mathscr{L}_{\mathscr{A}_p}(E_p,F_p)$ can be identified by $\mathscr{L}_{\mathscr{A}}(E,F)$. In particular, topologizing, $\mathscr{L}_{\mathscr{A}}(E,E)$ becomes a locally C*-algebra which is abbreviated by $\mathscr{L}_{\mathscr{A}}(E)$. Proofs of the above facts can be found in Sections 2.1 and 2.2 of the book [14]. Hilbert modules over locally C*-algebras have been studied systematically in the book [14] and the papers [15–18, 21].

We use the notations $\mathrm{Ker}(\cdot)$ and $\mathrm{Ran}(\cdot)$ for kernel and range of module maps, respectively. A bounded $\mathscr{A}$-module map $P : E \to E$ is said to be idempotent if $P^2 = P$. If, in addition, P is adjointable and $P^* = P$ then P is said to be projection. It is known that a Hilbert $\mathscr{A}$-submodule X of a Hilbert $\mathscr{A}$-module E is an orthogonal summand (a topological summand, respectively) if and only if there exists a projection (an idempotent, respectively) on E whose range is X.

Lemma 2.1. *Suppose $P : E \to E$ is a bounded $\mathscr{A}$-module map. Then P is an idempotent if and only if $(\pi_p^{\mathscr{A}})_*(P) : E_p \to E_p$, $(\pi_p^{\mathscr{A}})_*(P)(\xi + N_p^E) = P\xi + N_p^E$ is an idempotent for each $p \in S(\mathscr{A})$. In particular, P is a projection in $\mathscr{L}_{\mathscr{A}}(E)$ if and only if $(\pi_p^{\mathscr{A}})_*(P)$ is a projection in $\mathscr{L}_{\mathscr{A}_p}(E_p)$ for each $p \in S(\mathscr{A})$.*

Proof. Suppose P is an idempotent and $p \in S(\mathscr{A})$. Then $(\pi_p^{\mathscr{A}})_*(P) : E_p \to E_p$ is a bounded $\mathscr{A}_p$-module map and for each $x_p, y_p \in E_p$ we have

$$((\pi_p^{\mathscr{A}})_*(P))^2 x_p = P^2 x + N_p^E = Px + N_p^E = (\pi_p^{\mathscr{A}})_*(P) x_p,$$

that is, $(\pi_p^{\mathscr{A}})_*(P)$ is an idempotent.

Conversely, suppose $(\pi_p^{\mathscr{A}})_*(P) : E_p \to E_p$ is an idempotent. We obtain

$$\begin{aligned}\pi_p^{\mathscr{A}}(\langle P^2 x, y\rangle - \langle Px, y\rangle) &= \langle (\pi_p^{\mathscr{A}})_*(P^2) x_p, y_p\rangle - \langle (\pi_p^{\mathscr{A}})_*(P) x_p, y_p\rangle \\ &= \langle ((\pi_p^{\mathscr{A}})_*(P))^2 x_p, y_p\rangle - \langle (\pi_p^{\mathscr{A}})_*(P) x_p, y_p\rangle = 0\end{aligned}$$

for all $p \in S(\mathscr{A})$ and $x, y \in E$. We therefore have $\langle P^2 x, y\rangle = \langle Px, y\rangle$, i.e., P is an idempotent. A similar argument shows that P is selfadjoint if and only if $(\pi_p^{\mathscr{A}})_*(P)$ is. This proves the second statement. ∎

Corollary 2.1. *Suppose F and E are Hilbert $\mathscr{A}$-modules which are identified with $\varprojlim_p F_p$ and $\varprojlim_p E_p$, respectively. If E is a $\mathscr{A}$-submodule of F, E is topologically (orthogonally) complemented if and only if E_p is topologically (orthogonally) complemented for each $p \in S(\mathscr{A})$.*

Lemma 2.2. *Let T be a module map in $\mathscr{L}_{\mathscr{A}}(E,F)$ which can be identified by $(T_p)_p$ in $\varprojlim_p \mathscr{L}_{\mathscr{A}_p}(E_p, F_p)$. Then T has closed range if and only if $(\pi_p^{\mathscr{A}})_*(T)$ has closed range for each $p \in S(\mathscr{A})$.*

Proof. For $P \in \mathscr{L}_{\mathscr{A}}(F)$, $\mathrm{Ran}(P) = \mathrm{Ran}(T)$ if and only if $\mathrm{Ran}((\pi_p^{\mathscr{A}})_*(P)) = \mathrm{Ran}((\pi_p^{\mathscr{A}})_*(T))$ for each $p \in S(\mathscr{A})$. The result follows the above fact, Lemma 2.1 and [15, Theorem 2.2]. ∎

Closed submodules of Hilbert modules need not be orthogonally complemented at all, but [15, Theorem 2.2], which is an extension of [19, Theorem 3.2], states under which conditions closed submodules may be orthogonally complemented. For the special choice of modular operator $T \in \mathscr{L}_{\mathscr{A}}(E,F)$ with closed range one has:

- $\mathrm{Ker}(T)$ is orthogonally complemented in E, with complement $\mathrm{Ran}(T^*)$,
- $\mathrm{Ran}(T)$ is orthogonally complemented in F, with complement $\mathrm{Ker}(T^*)$,
- the map $T^* \in \mathscr{L}_{\mathscr{A}}(F,E)$ has a closed range, too.

An $\mathscr{A}$-module map $U \in \mathscr{L}_{\mathscr{A}}(E,F)$ is said to be unitary if $U^*U = 1_E$ and $UU^* = 1_F$. If there exists a unitary element of $\mathscr{L}_{\mathscr{A}}(E,F)$ then we say that E and F are unitarily equivalent Hilbert $\mathscr{A}$-modules. Two Hilbert $\mathscr{A}$-modules E and F are isomorphic if and only if there is a unitary operator from E to F, cf. [14, Corollary 2.5.4].

Lemma 2.3. (cf. [14, Remark 2.5.2]) *Suppose $U \in \mathscr{L}_{\mathscr{A}}(E,F)$. Then U is unitary if and only if $(\pi_p^{\mathscr{A}})_*(U) : E_p \to F_p$ is a unitary operator for all $p \in S(\mathscr{A})$.*

Let E,F be $\mathscr{A}$-modules and $T:E\to F$ be an $\mathscr{A}$-module map then $\mathscr{A}$-submodule $G(T)=\{(x,Tx):\ x\in E\}$ is called the *graph* of T. If T is bounded $\mathscr{A}$-module map then $G(T)$ is a closed $\mathscr{A}$-submodule of the Hilbert $\mathscr{A}$-module $E\oplus F$. It is well known that a bounded module map between Hilbert C*-modules is adjointable if and only if its graph is an orthogonal summand, see e.g., [4,5]. The problem are restudied in the case of unbounded module maps between Hilbert C*-modules in [6]. In this section we study adjointability of bounded $\mathscr{A}$-module maps between Hilbert modules over locally C*-algebras. In fact, we show that the Hilbert $\mathscr{A}_p$-modules $G(T)_p$ and $G((\pi_p^{\mathscr{A}})_*(T))$ are isomorphic and then we lift Corollary 2.4 of [5] to the case of Hilbert modules over locally C*-algebras.

Lemma 2.4. *Suppose $T:E\to F$ is bounded $\mathscr{A}$-module map then the Hilbert $\mathscr{A}_p$-modules $G(T)_p$ and $G((\pi_p^{\mathscr{A}})_*(T))$ are isomorphic for every $p\in S(\mathscr{A})$.*

Proof. Suppose $p\in S(\mathscr{A})$ and $T_p=(\pi_p^{\mathscr{A}})_*(T)$. We define the $\mathscr{A}$-module maps

$$U_p:G(T)_p\to G(T_p),\ U_p((x,Tx)+N_p^{G(T)})=(x_p,T_px_p)$$

and

$$W_p:G(T_p)\to G(T)_p,\ W_p(x_p,T_px_p)=((x,Tx)+N_p^{G(T)}),$$

we obtain

$$\langle U_p((x,Tx)+N_p^{G(T)}),(y_p,T_py_p)\rangle=\langle (x,Tx)+N_p^{G(T)},W_p(y_p,T_py_p)\rangle$$

for all $x\in E,\ y_p\in E_p$. That is, U_p is adjointable and $U_p^*=W_p$. We also have $U_pU_p^*=U_p^*U_p=1$ on $G(T)_p$, i.e., $G(T)_p$ and $G(T_p)$ are unitarily equivalent. The result now follows from [14, Corollary 2.5.4]. ∎

Lemma 2.5. *A bounded $\mathscr{A}$-module map $T:E\to F$ is adjointable if and only if $(\pi_p^{\mathscr{A}})_*(T):E_p\to F_p$ is adjointable for each $p\in S(\mathscr{A})$. In this situation, the adjoint of $(\pi_p^{\mathscr{A}})_*(T)$ is $(\pi_p^{\mathscr{A}})_*(T^*)$.*

Proof. Suppose $T:E\to F$ is adjointable then $\pi_p^{\mathscr{A}}(\langle Tx,y\rangle)=\pi_p^{\mathscr{A}}(\langle x,T^*y\rangle)$ for all $x\in E$, $y\in F$ and $p\in S(\mathscr{A})$. We therefore have $\langle(\pi_p^{\mathscr{A}})_*(T)x_p,y_p\rangle=\langle x_p,(\pi_p^{\mathscr{A}})_*(T^*)y_p\rangle$, for all $x_p\in E_p$, $y_p\in F_p$ and $p\in S(\mathscr{A})$, i.e., $(\pi_p^{\mathscr{A}})_*(T)$ is adjointable and its adjoint is $(\pi_p^{\mathscr{A}})_*(T^*)$.

Conversely, suppose $T_p=(\pi_p^{\mathscr{A}})_*(T):E_p\to F_p$ is adjointable for all $p\in S(\mathscr{A})$. Suppose $S:F\to E$ is defined by $Sy=(T_p^*y_p)_p$, $y=(y_p)_p\in F=\varprojlim_p F_p$. Then S is well defined, since

$$\sigma_{pq}^E(T_p^*y_p)=(\pi_{pq}^{\mathscr{A}})_*(T_p^*)(\sigma_{pq}^F(y_p))=T_q^*y_q$$

for all $p,q\in S(\mathscr{A})$ with $p\geq q$. Furthermore, we have

$$\pi_p^{\mathscr{A}}(\langle Tx,y\rangle)=\langle T_px_p,y_p\rangle=\langle x_p,T_p^*y_p\rangle=\pi_p^{\mathscr{A}}(\langle x,Sy\rangle)$$

for all $x=(x_p)_p\in E$, $y=(y_p)_p\in F$ and $p\in S(\mathscr{A})$. Therefore $\langle Tx,y\rangle=\langle x,Sy\rangle$, i.e., T is adjointable and $S=T^*$. ∎

Proposition 2.1. *A bounded $\mathscr{A}$-module map $T:E\to F$ possesses an adjoint map $T^*:F\to E$ if and only if the graph of T is an orthogonal summand of the Hilbert $\mathscr{A}$-module $E\oplus F$.*

Proof. Using Lemmas 2.4, 2.5 and [5, Corollary 2.4], we conclude that T is adjointable if and only if every $(\pi_p^{\mathscr{A}})_*(T)$ is adjointable, if and only if every $G((\pi_p^{\mathscr{A}})_*(T))$ is an orthogonal summand, if and only if $G(T)_p$ is an orthogonal summand.

According Corollary 2.1 and the fact that $G(T) = \varprojlim_p G(T)_p$, the $\mathscr{A}_p$-submodule $G(T)_p$ is an orthogonal summand in $(E \oplus F)_p$ if and only if the $\mathscr{A}$-submodule $G(T)$ is an orthogonal summand in $E \oplus F$, which completes the proof. ∎

3. Polar decomposition and generalized inverses

The polar decomposition is a useful tool that represents an operator as a product of a partial isometry and a positive element. It is well known that every bounded operator on Hilbert spaces has polar decomposition. In general bounded adjointable $\mathscr{A}$-module maps between Hilbert $\mathscr{A}$-modules do not have polar composition, but M. Joita has given a necessary and sufficient condition for bounded adjointable module maps to admit polar decomposition. She has proved that a bounded adjointable operator T has polar decomposition if and only if $\overline{\mathrm{Ran}(T)}$ and $\overline{\mathrm{Ran}(|T|)}$ are orthogonal direct summands. The reader is encouraged to see [15, Theorem 2.8, Proposition 2.10] and [14, Section 3.3] for more information and the proof of this fact. See also [25, Theorem 15.3.7].

Definition 3.1. *An adjointable module map $T: E \to F$ has a polar decomposition if there is a partial isometry $V: E \to F$ such that $T = V|T|$, and $\mathrm{Ker}(V) = \mathrm{Ker}(T)$, $\mathrm{Ran}(V) = \overline{\mathrm{Ran}(T)}$, $\mathrm{Ker}(V^*) = \ker(T^*)$ and $\mathrm{Ran}(V^*) = \overline{\mathrm{Ran}(|T|)}$.*

Proposition 3.1. *A bounded adjointable $\mathscr{A}$-module map $T: E \to F$ has a polar decomposition if and only if $(\pi_p^{\mathscr{A}})_*(T)$ has a polar decomposition for each $p \in S(\mathscr{A})$. In this situation, $T = V|T|$ if and only if $(\pi_p^{\mathscr{A}})_*(T) = (\pi_p^{\mathscr{A}})_*(V)|(\pi_p^{\mathscr{A}})_*(T)|$ for each $p \in S(\mathscr{A})$.*

Definition 3.2. *Let $T \in \mathscr{L}_{\mathscr{A}}(E,F)$, then a bounded adjointable operator $T^\dagger \in \mathscr{L}_{\mathscr{A}}(F,E)$ is called the generalized inverse of T if*

$$TT^\dagger T = T,\ T^\dagger T T^\dagger = T^\dagger,\ (TT^\dagger)^* = TT^\dagger \text{ and } (T^\dagger T)^* = T^\dagger T. \tag{3.1}$$

The notation $T^\dagger$ is reserved to denote the generalized inverse of T. These properties imply that $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections. Moreover, $\mathrm{Ran}(T^\dagger) = \mathrm{Ran}(T^\dagger T)$, $\mathrm{Ran}(T) = \mathrm{Ran}(TT^\dagger)$, $\mathrm{Ker}(T) = \mathrm{Ker}(T^\dagger T)$ and $\mathrm{Ker}(T^\dagger) = \mathrm{Ker}(TT^\dagger)$ which lead us to $E = \mathrm{Ker}(T^\dagger T) \oplus \mathrm{Ran}(T^\dagger T) = \mathrm{Ker}(T) \oplus \mathrm{Ran}(T^\dagger)$ and $F = \mathrm{Ker}(T^\dagger) \oplus \mathrm{Ran}(T)$.

Xu and Sheng in [26] have shown that a bounded adjointable operator between two Hilbert C*-modules admits a bounded generalized inverse if and only if the operator has closed range. The reader should be aware of the fact that a bounded adjointable operator may admit an unbounded operator as its generalized, see [7, 12, 23, 24] for more detailed information.

Lemma 3.1. *Let $T \in \mathscr{L}_{\mathscr{A}}(E,F)$, then T has closed range if and only if $\mathrm{Ker}(T)$ is orthogonally complemented in E and T is bounded below on $\mathrm{Ker}(T)^\perp$, i.e. for each $p \in S(\mathscr{A})$ there exist $c_p > 0$ such that $\overline{p}_F(Tx) \geq c_p\,\overline{p}_E(x)$, for all $x \in \mathrm{Ker}(T)^\perp$. In this case, $(T_{|\mathrm{Ker}(T)^\perp})^{-1}$ is a bounded module map on $\mathrm{Ran}(T)$.*

Proof. Let first $\mathrm{Ran}(T)$ be closed then $\mathrm{Ker}(T)$ is orthogonally complemented in E. Identifying T with $(T_p) \in \varprojlim_p \mathscr{L}_{\mathscr{A}_p}(E_p, F_p)$, then for every $p \in S(\mathscr{A})$ the range of T_p is closed.

According to [6, Proposition 1.3], $\mathrm{Ker}(T_p)$ is orthogonally complemented and there exists $c_p > 0$ such that $\|T_p x_p\| \geq c_p \|x_p\|$ for all $x \in \mathrm{Ker}(T_p)^{\perp}$. The latter inequality implies that

$$\begin{aligned}\overline{p}_F(Tx)^2 &= p(\langle Tx, Tx\rangle) = \|\pi_p^{\mathscr{A}}(\langle Tx, Tx\rangle)\\ &= \|\langle \sigma_p^F(Tx), \sigma_p^F(Tx)\rangle\| = \|\langle T_p(\sigma_p^E(x)), T_p(\sigma_p^E(x))\rangle\|\\ &\geq c_p^2 \|\langle \sigma_p^E(x), \sigma_p^E(x)\rangle\| = c_p^2 \|\pi_p^{\mathscr{A}}(\langle x, x\rangle)\| = c_p^2\, \overline{p}_E(x)^2.\end{aligned}$$

Consequently, for each $p \in S(\mathscr{A})$ there exists $c_p > 0$ such that $\overline{p}_F(Tx) \geq c_p\, \overline{p}_E(x)$, for all $x \in \mathrm{Ker}(T)^{\perp}$.

The converse can be proved by a similar manner as the proof of [6, Proposition 1.3], and so we omitted it. The second assertion follows from the first assertion. ∎

Proposition 3.2. *Suppose $T \in \mathscr{L}_{\mathscr{A}}(E,F)$. The operator T has a generalized inverse if and only if T has a closed range.*

Proof. Suppose T has a generalized inverse, then $TT^{\dagger}$ is an orthogonal projection which implies the closedness of $\mathrm{Ran}(T) = \mathrm{Ran}(TT^{\dagger})$.

Conversely, suppose $\mathrm{Ran}(T)$ is closed then $E = \mathrm{Ker}(T) \oplus \mathrm{Ran}(T^*)$ and $F = \mathrm{Ker}(T^*) \oplus \mathrm{Ran}(T)$ by [15, Theorem 2.2]. According to Lemma 3.1, the module maps $T_{|\mathrm{Ker}(T)^{\perp}}$ and $T^*_{|\mathrm{Ker}(T^*)^{\perp}}$ are invertible on $\mathrm{Ran}(T)$ and $\mathrm{Ran}(T^*)$, respectively, which allow us to define $\mathscr{A}$-module map $T^{\dagger} : F \to E$ and $T^{\dagger *} : E \to F$ by

$$T^{\dagger}x = \begin{cases} (T_{|\mathrm{Ker}(T)^{\perp}})^{-1}x & \text{if } x \in \mathrm{Ran}(T)\\ 0 & \text{if } x \in \mathrm{Ker}(T^*),\end{cases}$$

$$T^{\dagger *}x = \begin{cases} (T^*_{|\mathrm{Ker}(T^*)^{\perp}})^{-1}x & \text{if } x \in \mathrm{Ran}(T^*)\\ 0 & \text{if } x \in \mathrm{Ker}(T).\end{cases}$$

Using the orthogonal direct sum decompositions, the module maps $T^{\dagger}$ and $T^{\dagger *}$ satisfy $\langle T^{\dagger}x, y\rangle = \langle x, T^{\dagger *}y\rangle$ for all $x \in F$ and $y \in E$, which implies that $T^{\dagger} \in \mathscr{L}_{\mathscr{A}}(F,E)$. Moreover, T and $T^{\dagger}$ satisfy (3.1), i.e., $T^{\dagger}$ is the generalized inverse of T. ∎

Corollary 3.1. *Suppose $T \in \mathscr{L}_{\mathscr{A}}(E,F)$. The module map T has generalized inverse if and only if $(\pi_p^{\mathscr{A}})_*(T)$ has generalized inverse for each $p \in S(\mathscr{A})$. In this case, $((\pi_p^{\mathscr{A}})_*(T))^{\dagger} = (\pi_p^{\mathscr{A}})_*(T^{\dagger})$ for each $p \in S(\mathscr{A})$.*

The above result follows from the previous proposition, Lemma 2.2 and [26, Theorem 2.2]. Let $\mathscr{A}$ be a locally C*-algebra and $\mathfrak{a} \in \mathscr{A}$. An element $\mathfrak{a}^{\dagger} \in \mathscr{A}$ is called the generalized inverse of $\mathfrak{a}$ if $\mathfrak{a}$ and $\mathfrak{a}^{\dagger}$ satisfy (3.1). Generalized inverses in C*-algebras have been investigated by Harte and Mbekhta [11]. The main result of their paper now reads as follows:

Corollary 3.2. *Suppose $\mathscr{A}$ is a unital locally C*-algebra and $\mathfrak{a} \in \mathscr{A}$. Then $\mathfrak{a}$ has a generalized inverse if and only if $\mathfrak{a}\mathscr{A}$ is a closed right ideal in $\mathscr{A}$.*

Since every locally C*-algebra is a right $\mathscr{A}$-module on its own, the fact directly follows from Proposition 3.2.

4. Inverse limits of C*-algebras of compact operators

We closed the paper with characterizing the inverse limit of C*-algebras of compact operators via the generic properties of module maps. To deduce the following theorem just one needs to use [5, Theorem 2.6], Corollary 2.1, Lemmas 2.2, 2.5, 3.1 and Proposition 3.1.

Theorem 4.1. *Let $\mathscr{A}$ be a locally C*-algebra. The following conditions are equivalent:*

(i) *$\mathscr{A}$ is an inverse limit of C*-algebras of compact operators.*

(ii) *For every Hilbert $\mathscr{A}$-module E every Hilbert $\mathscr{A}$-submodule $F \subseteq E$ is automatically orthogonally complemented, i.e. F is an orthogonal summand.*

(iii) *For every Hilbert $\mathscr{A}$-module E Hilbert $\mathscr{A}$-submodule $F \subseteq E$ that coincides with its biorthogonal complement $F^{\perp\perp} \subseteq E$ is automatically orthogonally complemented in E.*

(iv) *For every pair of Hilbert $\mathscr{A}$-modules E, F, every bounded $\mathscr{A}$-module map $T : E \to F$ possesses an adjoint bounded $\mathscr{A}$-module map $T^* : F \to E$.*

(v) *The kernels of all bounded $\mathscr{A}$-module maps between arbitrary Hilbert $\mathscr{A}$-modules are orthogonal summands.*

(vi) *The image of all bounded $\mathscr{A}$-module maps with norm closed range between arbitrary Hilbert $\mathscr{A}$-modules are orthogonal summands.*

(vii) *For every pair of Hilbert $\mathscr{A}$-modules E, F, every bounded $\mathscr{A}$-module map $T : E \to F$ has polar decomposition, i.e. there exists a unique partial isometry V with initial set $\overline{\mathrm{Ran}(|T|)}$ and the final set $\overline{\mathrm{Ran}(T)}$ such that $T = V|T|$.*

(viii) *For every pair of Hilbert $\mathscr{A}$-modules E, F, every bounded $\mathscr{A}$-module map $T : E \to F$ has generalized inverse.*

(ix) *For every Hilbert $\mathscr{A}$-module E every Hilbert $\mathscr{A}$-submodule is automatically topologically complemented there, i.e. it is a topological direct summand.*

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