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The Structure of Some Classes of 3-Dimensional Normal Almost Contact Metric Manifolds

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Abstract. The object of the present paper is to study ξ-projectively flat and ϕ-projectively flat 3-dimensional normal almost contact metric manifolds. An illustrative example is given.

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1. Introduction

Let M be an almost contact manifold and (ϕ,ξ,η) its almost contact structure. This means, M is an odd-dimensional differentiable manifold and ϕ, ξ, η are tensor fields on M of types $(1,1)$, $(1,0)$, $(0,1)$ respectively, such that

$$\phi^2 = -I+\eta\otimes\xi, \quad \eta(\xi)=1, \quad \phi\xi=0, \quad \eta\circ\phi=0. \tag{1.1}$$

Let $\mathbb{R}$ be the real line and t a coordinate on $\mathbb{R}$. Define an almost complex structure J on $M\times\mathbb{R}$ by

$$J\left(X,\frac{\lambda d}{dt}\right)=\left(\phi X-\lambda\xi,\eta(X)\frac{d}{dt}\right),$$

where the pair $(X,\lambda d/dt)$ denotes a tangent vector to $M\times\mathbb{R}$, f is a smooth function on $M\times R$, X and $\lambda d/dt$ being tangent to M and $\mathbb{R}$ respectively. M with the structure (ϕ,ξ,η) is said to be normal if the structure J is integrable [1], [2]. The necessary and sufficient condition for (ϕ,ξ,η) to be normal is

$$[\phi,\phi]+2d\eta\otimes\xi=0,$$

where the pair $[\phi,\phi]$ is the Nijenhuis tensor of ϕ defined by

$$[\phi,\phi](X,Y)=[\phi X,\phi Y]+\phi^2[X,Y]-\phi[\phi X,Y]-\phi[X,\phi Y],$$

for any $X, Y\in T(M)$;

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We say that the form η has rank $r = 2s$ if $(d\eta)^s \neq 0$, and $\eta \wedge (d\eta)^s = 0$, and has rank $r = 2s+1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. We also say that r is the rank of the structure (ϕ, ξ, η).

A Riemannian metric g on M satisfying the condition

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.2}$$

for any $X,Y \in T(M)$, is said to be compatible with the structure (ϕ, ξ, η). If g is such a metric, then the quadruple (ϕ, ξ, η, g) is called an almost contact metric (shortly a.c.m.) structure on M and M is an (a.c.m.) manifold. On such a manifold we also have $\eta(X) = g(X, \xi)$, for any $X \in T(M)$ and we can always define the 2-form Φ by

$$\Phi(X,Y) = g(X, \phi Y),$$

where $X, Y \in T(M)$.

It is no hard to see that if $\dim M = 3$, then two Riemannian metrics g and $\acute{g}$ are compatible with the same almost contact structure (ϕ, ξ, η) on M if and only if $\acute{g} = \sigma g + (1-\sigma)\eta \otimes \eta$, for a certain positive function σ on M.

A normal (a.c.m.) structure (ϕ, ξ, η, g) satisfying additionally the condition $d\eta = \Phi$ is called Sasakian. Of course, any such structure on M has rank 3. Also a normal almost contact metric structure satisfying the condition $d\Phi = 0$ is said to be quasi-Sasakian [3]. Contact metric manifolds have been studied by several authors [5,7,16]. Also if we consider $\tilde{M}^n$ be a complex n-dimensional Kaehler manifold and M a real hypersurface of $\tilde{M}^n$. We denote by $\tilde{g}$ and $\tilde{J}$ a Kaehler metric tensor and its Hermitian Structure tensor, respectively. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where ϕ is a (1,1)-type tensor field, η is a 1-form and ξ is a unit vector field on M. The induced Riemannian metric on M is denoted by g. Then by the properties of $(\tilde{g}, \tilde{J})$, we see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M. Real hypersurfaces of a complex manifold have been studied by [10, 19] and many others.

In a recent paper [14], Olszak studied the curvature properties of normal almost contact manifold of dimension three with several examples. De, Yildiz and Funda [9] studied locally ϕ-symmetric normal (a.c.m.) manifolds of dimension 3. Also De and Kalam [8] recently characterized certain curvature conditions on 3-dimensional normal almost contact manifolds. Since at each point $p \in M$ the tangent space $T_p(M)$ can be decomposed into the direct sum $T_p(M) = \phi(T_p(M)) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of $T_p(M)$ generated by ξ_p, the conformal curvature tensor C is a map

$$C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M)) \oplus \{\xi_p\}, \quad p \in M.$$

One has the following well known particular cases: (1) the projection of the image of C in $\phi(T_p(M))$ is zero; (2) the projection of the image of C in $\{\xi_p\}$ is zero; and (3) the projection of the image of $C \mid_{\phi(T_p(M)) \times \phi(T_p(M)) \times \phi(T_p(M))}$ in $\phi(T_p(M))$ is zero. An (a.c.m.) manifold satisfying the cases (1), (2) and (3) is said to be conformally symmetric [11], ξ-conformally flat [20] and ϕ-conformally flat [4] respectively.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let M be a n-dimensional Riemannian manifold. If there exist an one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidian space such that any geodesic of the

Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [13]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.3}$$

for $X,Y,Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact, M is projectively flat (that is $P = 0$) if and only if the manifold is of constant curvature [17, pp. 84–85]. Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

The present paper is devoted to study ξ-projectively flat and ϕ-projectively flat normal(a.c.m.) metric manifold of dimension 3. After preliminaries in section 3, we prove that a compact 3-dimensional normal (a.c.m.) manifold is ξ-projectively flat if and only if the manifold is β-Sasakian. In the next section, it is proved that a 3-dimensional normal (a.c.m.) manifold is ϕ-projectively flat if and only if it is an Einstein manifold provided $\alpha,\beta = constant$. Finally we cited of a normal almost contact metric manifold.

2. Preliminaries

For a normal (a.c.m.) structure (ϕ,ξ,η,g) on M , we have [14]

$$\nabla_X\xi = \alpha\{X - \eta(X)\xi\} - \beta\phi X, \tag{2.1}$$

where $2\alpha = \operatorname{div}\xi$ and $2\beta = \operatorname{tr}(\phi\nabla\xi)$, $\operatorname{div}\xi$ is the divergence of ξ defined by $\operatorname{div}\xi = \operatorname{trace}\{X \longrightarrow \nabla_X\xi\}$ and $\operatorname{tr}(\phi\nabla\xi) = \operatorname{trace}\{X \longrightarrow \phi\nabla_X\xi\}$. As a consequence of (2.1) we have

$$\begin{aligned}(\nabla_X\phi)(Y) &= g(\phi\nabla_X\xi,Y)\xi - \eta(Y)\phi\nabla_X\xi \\ &= \alpha\{g(\phi X,Y)\xi - \eta(Y)\phi X\} + \beta\{g(X,Y)\xi - \eta(Y)X\},\end{aligned} \tag{2.2}$$

$$\begin{aligned}R(X,Y)\xi = {} & \{Y\alpha + (\alpha^2-\beta^2)\eta(Y)\}\phi^2X - \{X\alpha + (\alpha^2-\beta^2)\eta(X)\}\phi^2Y \\ & + \{Y\beta + 2\alpha\beta\eta(Y)\}\phi X - \{X\beta + 2\alpha\beta\eta(X)\}\phi Y,\end{aligned} \tag{2.3}$$

$$\begin{aligned}S(X,Y) = {} & \left(\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2\right)g(X,Y) - \left\{\frac{r}{2} + \xi\alpha + 3(\alpha^2-\beta^2)\right\}\eta(X)\eta(Y) \\ & - (\eta(Y)X\alpha + \eta(X)Y\alpha) - \{\eta(Y)(\phi X)\beta + \eta(X)(\phi Y)\beta\}\end{aligned} \tag{2.4}$$

$$S(Y,\xi) = -Y\alpha - (\phi Y)\beta - \{\xi\alpha + 2(\alpha^2-\beta^2)\}\eta(Y), \tag{2.5}$$

$$\xi\beta + 2\alpha\beta = 0, \tag{2.6}$$

where R denotes the curvature tensor and S is the Ricci tensor.

On the other hand, the curvature tensor in a 3-dimensional Riemannian manifold always satisfies

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = {} & g(X,W)S(Y,Z) - g(X,Z)S(Y,W) + g(Y,Z)S(X,W) \\ & - g(Y,W)S(X,Z) - \frac{r}{2}\left[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\right],\end{aligned} \tag{2.7}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$ and r is the scalar curvature.

From (2.3) we can derive that

$$\tilde{R}(\xi,Y,Z,\xi) = -(\xi\alpha+\alpha^2-\beta^2)g(\phi Y,\phi Z)-(\xi\beta+2\alpha\beta)g(Y,\phi Z). \tag{2.8}$$

By (2.5), (2.7) and (2.8) we obtain for $\alpha,\beta = constant$,

$$S(Y,Z) = \left(\frac{r}{2}+\alpha^2-\beta^2\right)g(\phi Y,\phi Z)-2(\alpha^2-\beta^2)\eta(Y)\eta(Z). \tag{2.9}$$

Applying (2.9) in (2.7) we get

$$\begin{aligned} R(X,Y)Z = {} & \left(\frac{r}{2}+2(\alpha^2-\beta^2)\right)\{g(Y,Z)X-g(X,Z)Y\}+g(X,Z)\Big\{\Big(\frac{r}{2}+3(\alpha^2 \\ & -\beta^2)\Big)\eta(Y)\xi\Big\}-\Big\{\frac{r}{2}+3(\alpha^2-\beta^2)\Big\}\eta(Y)\eta(Z)X-g(Y,Z)\Big\{\Big(\frac{r}{2} \\ & +3(\alpha^2-\beta^2)\Big)\eta(X)\xi\Big\}+\left(\frac{r}{2}+3(\alpha^2-\beta^2)\right)\eta(X)\eta(Z)Y. \end{aligned}$$

From (2.6) it follows that if $\alpha,\beta =$ constant, then the manifold is either β-Sasakian, or α-Kenmotsu [12] or cosymplectic [1].

Proposition 2.1. *A 3-dimensional normal almost contact metric manifold with $\alpha,\beta =$ constant is either β-Sasakian, or α-Kenmotsu or cosymplectic.*

Definition 2.1. *An almost $C(\lambda)$-manifold M is an almost co-Hermitian manifold such that the Riemannian curvature tensor satisfies the following property:*
there exist $\lambda \in R$ such that for all $X,Y,Z,W \in T(M)$:

$$\begin{aligned} R(X,Y,Z,W) = {} & R(X,Y,\phi Z,\phi W)+\lambda\{-g(X,Z)g(Y,W)+g(X,W)g(Y,Z) \\ & +g(X,\phi Z)g(Y,\phi W)-g(X,\phi W)g(Y,\phi Z)\}. \end{aligned}$$

A normal almost $C(\lambda)$-manifold is a $C(\lambda)$-manifold. If we take $\lambda = -\alpha^2$ for $\alpha > 0$, then we get $C(-\alpha^2)$-manifold.

We note that β-Sasakian manifold are quasi-Sasakian [3]. They provide examples of $C(\lambda)$-manifolds with $\lambda \geq 0$.

An α-Kenmotsu manifold is a $C(-\alpha^2)$-manifold [12].

Cosymplectic manifolds provide a natural setting for time dependent mechanical systems as they are locally product of a Kaehler manifold and a real line or a circle [6].

3. 3-dimensional ξ-projectively flat normal almost contact metric manifolds

ξ-conformally flat K-contact manifolds have been studied by Zhen, Cabrerizo and Fernandez [20]. In this section we study ξ-projectively flat normal (a.c.m.) manifold. Analogous to the definition of ξ-conformally flat (a.c.m.) manifold we define ξ-projectively flat (a.c.m.) manifolds.

Definition 3.1. *A normal almost contact metric manifold M is called ξ-projectively flat if the condition $P(X,Y)\xi = 0$ holds on M, where projective curvature tensor P is defined by (1.3).*

Putting $Z=\xi$ in (1.3) and using (2.3) and (2.5), we get

$$
\begin{aligned}
P(X,Y)\xi = &-\frac{1}{2}\{(Y\alpha)X-(X\alpha)Y\}+\{(Y\alpha)\eta(X)-(X\alpha)\eta(Y)\}\xi\\
&+(Y\beta)\phi X-(X\beta)\phi Y+2\alpha\beta\{\eta(Y)\phi X-\eta(X)\phi Y\}\\
&+\frac{1}{2}\big[(\phi Y)\beta X-(\phi X)\beta Y+(\xi\alpha)\{\eta(Y)X-\eta(X)Y\}\big].
\end{aligned}
\tag{3.1}
$$

Now assume that M is a compact 3-dimensional ξ-projectively flat normal (a.c.m.) manifold. Then from (3.1) we can write

$$
\begin{aligned}
&-\frac{1}{2}\{(Y\alpha)X-(X\alpha)Y\}+\{(Y\alpha)\eta(X)-(X\alpha)\eta(Y)\}\xi\\
&+(Y\beta)\phi X-(X\beta)\phi Y+2\alpha\beta\{\eta(Y)\phi X-\eta(X)\phi Y\}\\
&+\frac{1}{2}\{(\phi Y)\beta X-(\phi X)\beta Y+(\xi\alpha)(\eta(Y)X-\eta(X)Y)\}=0.
\end{aligned}
\tag{3.2}
$$

Putting $Y=\xi$ in (3.2) and using (2.6), we obtain

$$(X\alpha)\xi+(\phi X)\beta\xi-(\xi\alpha)\eta(X)\xi=0$$

which implies

$$(X\alpha)+(\phi X)\beta-(\xi\alpha)\eta(X)=0. \tag{3.3}$$

Now (3.3) can be written as

$$(X\alpha)+g(\operatorname{grad}\beta,\phi X)-(\xi\alpha)\eta(X)=0. \tag{3.4}$$

Differentiating (3.4) covariantly along Y, we get

$$
\begin{aligned}
&\nabla_Y(X\alpha)+g(\nabla_Y\operatorname{grad}\beta,\phi X)+g(\operatorname{grad}\beta,(\nabla_Y\phi)X)\\
&\qquad -Y(\xi\alpha)\eta(X)-(\xi\alpha)(\nabla_Y\eta)(X)=0.
\end{aligned}
\tag{3.5}
$$

Hence, by antisymmetrization with respect to X and Y, we have from (3.5)

$$
\begin{aligned}
&g(\nabla_Y\operatorname{grad}\beta,\phi X)-g(\nabla_X\operatorname{grad}\beta,\phi Y)+g(\operatorname{grad}\beta,(\nabla_Y\phi)X)-g(\operatorname{grad}\beta,(\nabla_X\phi)Y)\\
&\qquad -Y(\xi\alpha)\eta(X)+X(\xi\alpha)\eta(Y)-(\xi\alpha)\{(\nabla_Y\eta)(X)-(\nabla_X\eta)(Y)\}=0.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&g(\nabla_Y\operatorname{grad}\beta,\phi X)-g(\nabla_X\operatorname{grad}\beta,\phi Y)+\{(\nabla_Y\phi)X\beta-(\nabla_X\phi)Y\beta\}\\
&\qquad -Y(\xi\alpha)\eta(X)+X(\xi\alpha)\eta(Y)+2(\xi\alpha)d\eta(X,Y)=0.
\end{aligned}
\tag{3.6}
$$

Using (2.2) and $d\eta=\beta\Phi$ [14], (3.6) yields

$$
\begin{aligned}
&g(\nabla_Y\operatorname{grad}\beta,\phi X)-g(\nabla_X\operatorname{grad}\beta,\phi Y)+\{2\alpha g(\phi Y,X)\xi-\alpha(\eta(X)\phi Y-\eta(Y)\phi X)\\
&-\beta(\eta(X)Y-\eta(Y)X)\}\beta-\{Y(\xi\alpha)\eta(X)-X(\xi\alpha)\eta(Y)\}+2\beta(\xi\alpha)\Phi(X,Y)=0.
\end{aligned}
\tag{3.7}
$$

Let $\{e_1,e_2,\xi\}$ be an orthonormal ϕ-basis where $\phi e_1=-e_2$ and $\phi e_2=e_1$. Taking $Y=e_1$ and $X=e_2$ in (3.7), we find that

$$g(\nabla_{e_1}\operatorname{grad}\beta,e_1)+g(\nabla_{e_2}\operatorname{grad}\beta,e_2)=2\alpha(\xi\beta)+2\beta(\xi\alpha). \tag{3.8}$$

On the other hand (2.6) yields $g(\operatorname{grad}\beta,\xi)=-2\alpha\beta$, whence by covariant differentiation we get, on account of (2.1)

$$g(\nabla_\xi\operatorname{grad}\beta,\xi)=-2\alpha(\xi\beta)-2\beta(\xi\alpha). \tag{3.9}$$

Denoting by $\triangle$ the Laplacian defined by $\triangle = \operatorname{div}\operatorname{grad}$, in view of (3.8) and (3.9) we have $\triangle\beta = 0$. Since M is compact, β is a constant. Now if $\beta \neq 0$, (2.6) implies $\alpha = 0$. This implies M is a β-Sasakian manifold. Conversely, if M is a β-Sasakian manifold, then from (3.1) it is easy to see that $P(X,Y)\xi = 0$. Hence we can state the following:

Theorem 3.1. *A compact 3-dimensional normal almost contact metric manifold is ξ-projectively flat if and only if it is a β-Sasakian manifold.*

4. 3-dimensional ϕ-projectively flat normal almost contact metric manifolds

Analogous to the definition of ϕ-conformally flat contact metric manifold [4], we define ϕ-projectively flat normal almost contact metric manifold. In this connection we can mention the work of Ozgur [15] who has studied ϕ-projectively flat Lorentzian Para-Sasakian manifolds.

Definition 4.1. *A 3-dimensional normal almost contact metric manifold satisfying the condition*

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0$$

is called ϕ-Projectively flat.

Let us assume that M is a 3-dimensional ϕ-projectively flat normal (a.c.m.) manifold. It can be easily seen that $\phi^2 P(\phi X, \phi Y)\phi Z = 0$ holds if and only if

$$g(P(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

for $X,Y,Z,W \in T(M)$.

Using (1.3) and (1.1), ϕ-projectively flat means

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2}\big\{S(\phi Y, \phi Z)g(\phi X, \phi W) - S(\phi X, \phi Z)g(\phi Y, \phi W)\big\}. \tag{4.1}$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in M and using the fact that $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.1) and summing up with respect to i, then we have

$$\sum_{i=1}^{2} g\big(R(\phi e_i, \phi Y)\phi Z, \phi e_i\big) = \frac{1}{2}\sum_{i=1}^{2}\big\{S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - S(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\big\}. \tag{4.2}$$

It can be easily verified that

$$\sum_{i=1}^{2} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + (\xi\alpha + \alpha^2 - \beta^2)g(\phi Y, \phi Z),$$

$$\sum_{i=1}^{2} g(\phi e_i, \phi e_i) = 2, \qquad \sum_{i=1}^{2} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = S(\phi Y, \phi Z).$$

So using (1.2) and (2.4), the equation (4.2) becomes

$$\left(\frac{r}{2} + 3(\xi\alpha + \alpha^2 - \beta^2)\right)\big\{g(Y,Z) - \eta(Y)\eta(Z)\big\} = 0,$$

which gives $r = -6(\xi\alpha + \alpha^2 - \beta^2)$. So we state the following:

Proposition 4.1. *The scalar curvature r of a 3-dimensional ϕ-projectively flat normal almost contact metric manifold is $-6(\xi\alpha + \alpha^2 - \beta^2)$.*

Also if $r = -6(\xi\alpha + \alpha^2 - \beta^2)$, it follows from (2.4) that the manifold is an Einstein manifold provided $\alpha, \beta = constant$. Hence we can state the following:

Proposition 4.2. *A 3-dimensional ϕ-projectively flat normal almost contact metric manifold is an Einstein manifold, provided $\alpha, \beta = constant$.*

It is known [18] that a 3-dimensional Einstein manifold is a manifold of constant curvature. Also M is projectively flat if and only if it is of constant curvature [17]. Now trivially, projectively flatness implies ϕ-projectively flat. Hence using Proposition 4.2 we can state the following:

Theorem 4.1. *A 3-dimensional normal almost contact metric manifold is ϕ-projectively flat if and only if it is an Einstein manifold, provided $\alpha, \beta = constant$.*

5. Example of a 3-dimensional normal almost contact metric manifold

We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are standard coordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_1,e_2) = g(e_2,e_3) = 0,$$
$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1,$$

that is, the form of the metric becomes

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in T(M)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W),$$

for any $Z, W \in T(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1,e_3] = e_1e_3 - e_3e_1 = z\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial z}\right) - z\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}\right) = z^2\frac{\partial^2}{\partial x\partial z} - z^2\frac{\partial^2}{\partial z\partial x} - z\frac{\partial}{\partial x} = -e_1.$$

Similarly

$$[e_1,e_2] = 0 \quad \text{and} \quad [e_2,e_3] = -e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) \\ & - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned} \tag{5.1}$$

which is known as Koszul's formula.

Using (5.1) we have

$$2g(\nabla_{e_1}e_3,e_1) = -2g(e_1,e_1) = 2g(-e_1,e_1). \tag{5.2}$$

Again by (5.1)

$$2g(\nabla_{e_1}e_3,e_2) = 0 = 2g(-e_1,e_2) \tag{5.3}$$

and

$$2g(\nabla_{e_1}e_3,e_3) = 0 = 2g(-e_1,e_3). \tag{5.4}$$

From (5.2), (5.3) and (5.4) we obtain

$$2g(\nabla_{e_1}e_3,X) = 2g(-e_1,X),$$

for all $X \in T(M)$. Thus

$$\nabla_{e_1}e_3 = -e_1.$$

Therefore, (5.1) further yields

$$\begin{aligned} \nabla_{e_1}e_3 &= -e_1, & \nabla_{e_1}e_2 &= 0, & \nabla_{e_1}e_1 &= e_3, \\ \nabla_{e_2}e_3 &= -e_2, & \nabla_{e_2}e_2 &= e_3, & \nabla_{e_2}e_1 &= 0, \\ \nabla_{e_3}e_3 &= 0, & \nabla_{e_3}e_2 &= 0, & \nabla_{e_3}e_1 &= 0. \end{aligned} \tag{5.5}$$

(5.5) tells us that the manifold satisfies (2.1) for $\alpha = -1$ and $\beta = 0$ and $\xi = e_3$. Hence the manifold is a normal almost contact metric manifold with α, $\beta =$ constants.

It is known that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{5.6}$$

With the help of the above results and using (5.6) it can be easily verified that

$$R(e_1,e_2)e_3 = 0, \quad R(e_2,e_3)e_3 = -e_2, \quad R(e_1,e_3)e_3 = -e_1,$$

$$R(e_1,e_2)e_2 = -e_1, \quad R(e_2,e_3)e_2 = e_3, \quad R(e_1,e_3)e_2 = 0,$$

$$R(e_1,e_2)e_1 = e_2, \quad R(e_2,e_3)e_1 = 0, \quad R(e_1,e_3)e_1 = e_3.$$

From the above expressions of the curvature tensor we obtain

$$S(e_1,e_1) = g\big(R(e_1,e_2)e_2,e_1\big) + g\big(R(e_1,e_3)e_3,e_1\big) = -2.$$

Similarly, we have

$$S(e_2,e_2) = S(e_3,e_3) = -2.$$

Therefore,

$$r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) = -6.$$

We note that here α, β and r are all constants. It is sufficient to check

$$S(e_i,e_i) = -2 = -2(\alpha^2 - \beta^2)g(e_i,e_i),$$

for all $i = 1,2,3$ and $\alpha = -1$, $\beta = 0$. Hence M is an Einstein manifold. Therefore M is ϕ-projectively flat. Thus Theorem 4.1 is verified.

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