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Real Hypersurfaces in Nearly Kaehler 6-Sphere

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Abstract. In this paper we characterize Hopf hypersurfaces in the nearly Kaehler 6-Sphere S^6 using some restrictions on the characteristic vector field $\xi = -JN$, where J is the almost complex structure on S^6 and N is the unit normal to the hypersurface. It is shown that if the characteristic vector field ξ of a compact and connected real hypersurface M of the nearly Kaehler sphere S^6 is harmonic and the Ricci curvature in the direction of ξ is non-negative, then M is a Hopf hypersurface and therefore congruent to either a totally geodesic hypersphere or a tube over almost complex curve on S^6. It is also observed that similar result holds if ξ is Jacobi-type vector field (a notion similar to Jacobi fields along geodesics). We also show that if a connected real hypersurface M is a Ricci soliton with potential vector field ξ, then M is congruent to an open piece of either a totally geodesic hypersphere or a tube over an almost complex curve in S^6.

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1. Introduction

It is known that the 6-dimensional unit sphere S^6 has a nearly Kaehler structure (J,g), where J is an almost complex structure defined on S^6 using the vector cross product of purely imaginary Cayley numbers R^7 and g is the induced metric on S^6 as a hypersurface of R^7. Regarding the submanifolds of the nearly Kaehler S^6, Gray [17] has proved that it does not have any complex hypersurface. However, there are 4-dimensional CR-submanifolds in S^6 and have been studied in [6, 19, 20]. Moreover, 2- and 3-dimensional totally real submanifolds of S^6 have been quite extensively studied (cf. [4–6, 8, 9, 11, 13–15]). However hypersurfaces of the nearly Kaehler S^6 have not been studied that extensively, as one comes across only [1,10,12]. Almost complex curves (2-dimensional almost complex submanifolds) in S^6 have been studied in [3,18], and recently, Berndt *et al.* [1] have shown that the geometry of almost complex curves in S^6 is related to Hopf hypersurfaces (Real hypersurfaces with the 1-dimensional foliation induced by the distribution which is obtained by applying almost complex structure J to the normal bundle of the hypersurface is totally

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geodesic) of S^6. This relationship between the almost complex curves and Hopf hypersurfaces in S^6 makes the study of Hopf hypersurfaces in S^6 more interesting. In [1], the authors proved that a connected Hopf hypersurface of the nearly Kaehler S^6 is an open part of either a geodesic hypersphere of S^6 or a tube around an almost complex curve in S^6. Therefore it is an interesting question to obtain different characterizations of the Hopf hypersurface in S^6. Let J be the almost complex structure on the nearly Kaehler sphere S^6 and M be an orientable real hypersurface of S^6 with unit normal vector field N. Then the unit vector field ξ defined by $\xi = -JN$ on M is called the characteristic vector field of the real hypersurface M. In this paper, we use different restrictions on the characteristic vector field ξ to obtain characterizations of the Hopf hypersurface in S^6. It is observed that if the characteristic vector field ξ of the compact real hypersurface M is harmonic and the Ricci curvature of M in the direction of ξ is non-negative, then ξ is Killing and in particular the hypersurface M is a Hopf hypersurface (cf. section-3). It is well known that a Killing vector field on a Riemannian manifold is a Jacobi vector field along any geodesic, however a smooth vector field that is a Jacobi vector field along each geodesic need not be a Killing vector field. We define a Jacobi-type vector field on a Riemannian manifold (which in particular implies that a Jacobi-type of vector field is Jacobi field along each geodesic). This leads to the question of finding condition under which a Jacobi-type vector fields are Killing vector fields. We use this notion for the characteristic vector field ξ of the compact real hypersurface M of S^6 and show that if ξ is Jacobi-type vector field on M, then necessarily it is Killing vector field and in particular the hypersurface M is a Hopf hypersurface (cf. section-4). Finally, in the last section of this paper, we show that if the real hypersurface M of the nearly Kaehler S^6 is a Ricci soliton (cf. [7]) with potential vector field ξ, then M is a Hopf hypersurface.

2. Preliminaries

Let S^6 be the nearly Kaehler 6-sphere with nearly Kaehler structure (J,g), where J is the almost complex structure and g is the almost Hermitian metric on S^6. Then we have

$$(\overline{\nabla}_X J)(X) = 0, \quad g(JX,JY) = g(X,Y), \quad X,Y \in \mathfrak{X}(S^6), \tag{2.1}$$

where $\overline{\nabla}$ is the Riemannian connection with respect to the almost Hermitian metric g and $\mathfrak{X}(S^6)$ is the Lie algebra of smooth vector fields on S^6. The tensor field G of type $(2,1)$ defined on S^6 by $G(X,Y) = (\overline{\nabla}_X J)(Y)$, $X,Y \in \mathfrak{X}(S^6)$ has the properties as described in the following:

Lemma 2.1. [15] *(a)* $G(X,JY) = -JG(X,Y)$, *(b)* $G(X,Y) = -G(Y,X)$
(c) $(\overline{\nabla}_X G)(Y,Z) = g(Y,JZ)X + g(X,Z)JY - g(X,Y)JZ$, $X,Y,Z \in \mathfrak{X}(S^6)$.

Let M be an orientable real hypersurface of S^6, ∇ be the Riemannian connection with respect to the induced metric on M which we denote by the same letter g and N be the unit normal vector field. Then we have

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX,Y)N, \quad \overline{\nabla}_X N = -AX, X,Y \in \mathfrak{X}(M), \tag{2.2}$$

where A is the shape operator of the hypersurface M. The Gauss and Codazzi equations for the hypersurface are

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY \tag{2.3}$$

$$(\nabla A)(X,Y) = (\nabla A)(Y,X) \tag{2.4}$$

for $X,Y,Z \in \mathfrak{X}(M)$, where $(\nabla A)(X,Y) = \nabla_X AY - A(\nabla_X Y)$. The Ricci tensor Ric and the scalar curvature S of the hypersurface M are given by

$$Ric(X,Y) = 4g(X,Y) + 5\alpha g(AX,Y) - g(AX,AY), \tag{2.5}$$

$$S = 20 + 25\alpha^2 - \|A\|^2, \tag{2.6}$$

where $\alpha = 1/5trA$ is the mean curvature and $\|A\|^2 = trA^2$ is the square of the length of the shape operator of the hypersurface.

A real hypersurface M of the nearly Kaehler sphere S^6 is said to be a Hopf hypersurface if the characteristic vector field ξ of M is an eigenvector of the shape operator A. In particular if M is a Hopf hypersurface, then the integral curves of the characteristic vector field ξ are geodesics and it is known that a connected Hopf hypersurface in nearly Kaehler sphere S^6 is congruent to open piece of either a totally geodesic hypersphere or a tube over an almost complex curve in S^6 (cf. [1]).

Using the almost complex structure J of S^6, we define a unit vector field $\xi \in \mathfrak{X}(M)$ by $\xi = -JN$, with dual 1-form $\eta(X) = g(X,\xi)$. For a $X \in \mathfrak{X}(M)$, we set $JX = \phi(X) + \eta(X)N$, where $\phi(X)$ is the tangential component of JX. Then it follows that ϕ is a $(1,1)$ tensor field on M. Using $J^2 = -I$, it is easy to see that (ϕ,ξ,η,g) defines an almost contact metric structure on M, that is (cf. [2])

$$\phi^2 = -I + \eta \otimes \xi, \quad 5, \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi(\xi) = 0 \tag{2.7}$$

and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, $X,Y \in \mathfrak{X}(M)$. Using the fact $G(X,X) = 0$, $X \in \mathfrak{X}(M)$, we immediately obtain the following

$$(\nabla_X \phi)(X) = \eta(X)AX - g(AX,X)\xi, \quad g(\nabla_X \xi, X) = g(\phi AX, X), \quad X \in \mathfrak{X}(M). \tag{2.8}$$

Note that as ϕ is skewsymmetric, on a real hypersurface M we can construct a local orthonormal frame $\{e_1, \phi e_1, e_2, \phi e_2, \xi\}$ on M, called an adapted frame. Also using $J\xi = N$ and Lemma 2.1, we immediately arrive at

$$\nabla_X \xi = \phi AX - G(X,N), \quad X \in \mathfrak{X}(M). \tag{2.9}$$

On an orientable hypersurface M of S^6 we let $D = Ker\eta = \{X \in \mathfrak{X}(M) : \eta(X) = 0\}$. Then D is a 4-dimensional smooth distribution on M, and that for each $X \in D$, $JX \in D$, that is D is invariant under the almost complex structure J. We have the following

Lemma 2.2. [10] *Let M be an orientable compact real hypersurface of S^6. Then*

$$\int_M \{Ric(\xi,\xi) - 4 + Tr(\phi A)^2\} dv = 0.$$

3. Real hypersurfaces with harmonic characteristic vector field

Recall that the Laplacian operator Δ acting on smooth vector fields on a Riemannian manifold (M,g) is defined by

$$\Delta X = \sum_{i=1}^{n} \left(\nabla_{e_i}\nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X\right), \quad X \in \mathfrak{X}(M),$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame on M and a vector field X is said to be harmonic if $\Delta X = 0$ (cf. [16]). It is known that the operator Δ is negative semidefinite self adjoint with respect to the inner product $\langle , \rangle$ defined for compactly supported smooth vector fields on M by

$$\langle X,Y\rangle = \int g(X,Y).$$

In this section we study real hypersurfaces of the nearly Kaehler S^6 that has harmonic characteristic vector field. First, we prove the following:

Theorem 3.1. *Let M be an orientable compact and connected real hypersurface of the nearly Kaehler S^6. If the characteristic vector field ξ satisfies*

$$Ric(\xi,\xi) \geq -g(\Delta\xi,\xi)$$

then ξ is Killing and in particular M is a Hopf hypersurface which is therefore congruent to either a totally geodesic hypersphere or a tube over an almost complex curve in S^6.

Proof. Use equations (2.1), (2.2), (2.9) and Lemma 2.1, to compute

$$\begin{aligned} &\nabla_X\nabla_X\xi - \nabla_{\nabla_X X}\xi \\ &= \nabla_X \phi AX - \nabla_X G(X,N) - \phi A(\nabla_X X) + G(\nabla_X X,N) \\ &= (\nabla_X\phi)(AX) + \phi(\nabla_X A)(X) - \left(\overline{\nabla}_X G\right)(X,N) + G(X,AX) + g(AX,G(X,N))N \end{aligned}$$

$$= (\nabla_X\phi)(AX) + \phi(\nabla_X A)(X) + \eta(X)X - \|X\|^2\xi + G(X,AX) - g(G(X,AX),N))N, \tag{3.1}$$

where we also used the fact that $g(G(X,Y),Z) = -g(Y,G(X,Z))$, $X,Y,Z \in \mathfrak{X}(S^6)$. Choosing a local orthonormal frame $\{e_1,...,e_5\}$ on M that diagonalizes A as $A(e_i) = \lambda_i e_i$, and using equation (2.8), we compute

$$\begin{aligned} \sum(\nabla_{e_i}\phi)(Ae_i) &= \sum\lambda_i(\nabla_{e_i}\phi)(e_i) = \sum\lambda_i(\eta(e_i)Ae_i - g(Ae_i,e_i)\xi) \\ &= \sum(\eta(Ae_i)Ae_i - g(Ae_i,Ae_i)\xi) = A^2\xi - \|A\|^2\xi. \end{aligned} \tag{3.2}$$

Note that using Codazzi equation for hypersurface and symmetry of the shape operator A, it can be easily shown that the gradient $\nabla\alpha$ of the mean curvature α satisfies

$$5\nabla\alpha = \sum(\nabla_{e_i}A)(e_i)$$

and consequently, we have

$$\sum\phi(\nabla_{e_i}A)(e_i) = 5\phi(\nabla\alpha). \tag{3.3}$$

It trivially follows that

$$\sum G(e_i,Ae_i) = 0. \tag{3.4}$$

Using equations (3.2)-(3.4) in the equation (3.1), we get the following expression for the Laplacian $\Delta\xi$

$$\Delta\xi = A^2\xi - \|A\|^2\xi + 5\phi(\nabla\alpha) - 4\xi. \tag{3.5}$$

Note that the operator $\phi A - A\phi$ is a symmetric operator and consequently, we have

$$\|\phi A - A\phi\|^2 = 2Tr(\phi A)^2 + 2\|A\|^2 - 2\|A\xi\|^2,$$

which together with equation (3.5) gives

$$\frac{1}{2}\|\phi A - A\phi\|^2 + g(\Delta\xi,\xi) = Tr(\phi A)^2 - 4.$$

Using above equation in Lemma 2.2, we arrive at

$$\int_M \left\{ \frac{1}{2} \|\phi A - A\phi\|^2 + Ric(\xi,\xi) + g(\Delta\xi,\xi) \right\} dv = 0, \tag{3.6}$$

which together with the condition in the hypothesis of the theorem gives $\phi A = A\phi$, that is

$$\begin{aligned}\left(\pounds_\xi g\right)(X,Y) &= g(\nabla_X\xi,Y) + g(\nabla_Y\xi,X) \\ &= g\left((\phi A - A\phi)(X,Y) - g\left(G(X,N),Y\right) - g\left(G(Y,N),X\right) = 0\right.\end{aligned}$$

This proves that ξ is Killing and in particular M is a Hopf hypersurface and then the rest of the result follows from the main theorem in [1] with complete and connected M. ∎

As a particular case of above theorem we have the following:

Corollary 3.1. *Let M be an orientable compact and connected real hypersurface of the nearly Kaehler S^6. If the characteristic vector field ξ is harmonic and the Ricci curvature of M in the direction of ξ is non-negative, then ξ is Killing and in particular M is a Hopf hypersurface which is therefore congruent to either a totally geodesic hypersphere or a tube over an almost complex curve in S^6.*

4. Real hypersurfaces with Jacobi-type characteristic vector field

It is well known that a Killing vector field on a Riemannian manifold (M,g) is a Jacobi field along each geodesic of M. However, the converse is not true as for example the position vector field on the Euclidean space R^n is a Jacobi field along each geodesic of R^n which is not a Killing vector field. Motivated by the definition of a Jacobi field along a geodesic, we define a Jacobi-type vector field u on a Riemannian manifold (M,g) that satisfies

$$\nabla_X\nabla_X u - \nabla_{\nabla_X X} u + R(u,X)X = 0, \quad X \in \mathfrak{X}(M),$$

where ∇ is the Riemannian connection and R is the curvature tensor field of the Riemannian manifold (M,g). Naturally a Jacobi-type vector field is a Jacobi field along each geodesic of M. It is an interesting question to obtain condition under which a Jacobi-type vector field on a Riemannian manifold is Killing. In this section, we study compact real hypersurfaces of the nearly Kaehler sphere S^6 whose characteristic vector field ξ is Jacobi-type vector field and show that it is Killing. We prove the following:

Theorem 4.1. *Let M be an orientable compact and connected real hypersurface of the nearly Kaehler S^6. If the characteristic vector field ξ is a Jacobi-type vector field on M, then ξ is Killing and in particular M is a Hopf hypersurface which is therefore congruent to either a totally geodesic hypersphere or a tube over an almost complex curve in S^6.*

Proof. Let the characteristic vector field ξ of the real hypersurface be Jacobi-type vector field. Then we have

$$\nabla_X\nabla_X\xi - \nabla_{\nabla_X X}\xi + R(\xi,X)X = 0, \quad X \in \mathfrak{X}(M)$$

replacing X by e_i for a local orthonormal frame $\{e_1,\ldots,e_5\}$ on M in the above equation and summing these equations we arrive at

$$\Delta\xi + \sum R(\xi,e_i)e_i = 0.$$

Taking inner product with ξ in the above equation we get

$$Ric(\xi,\xi)+g(\Delta\xi,\xi)=0,$$

which together with equation (3.6) gives $\phi A=A\phi$. Then as in Theorem 3.1, we get the result. ∎

5. Real hypersurfaces as Ricci soliton

A Riemannian manifold (M,g) is said to be a Ricci soliton if there exist a vector field X called potential field and a constant λ satisfying

$$Ric+\frac{1}{2}\pounds_X g=\lambda g \tag{5.1}$$

and the Ricci soliton is said to stable, shrinking or expanding according as the constant $\lambda=0$, $\lambda>0$ or $\lambda<0$ (cf. [7]). In this section we study connected real hypersurface M of the nearly Kaehler S^6 which acquires the status of a Ricci soliton with potential field the characteristic vector field ξ of M and prove that in this case also M is a Hopf hypersurface. We prove the following:

Theorem 5.1. *Let M be an orientable connected real hypersurface of the nearly Kaehler sphere S^6 with characteristic vector field ξ. If M is a Ricci soliton with potential field ξ, then M is a Hopf hypersurface and therefore congruent to open piece of either a totally geodesic hypersphere or a tube over almost complex curve in S^6.*

Proof. Since the real hypersurface M is a Ricci soliton with potential field ξ, by equations (2.5) and (5.1), we have

$$Ric(\xi,\xi)=\lambda=4+5\alpha f-\|A\xi\|^2, \tag{5.2}$$

where f is the smooth function defined by $f=g(A\xi,\xi)$. Moreover using equation (2.9) together with Lemma 2.1, we have

$$\left(\pounds_\xi g\right)(X,Y)=g((\phi A-A\phi)(X),Y),\quad X,Y\in\mathfrak{X}(M).$$

Thus using equations (2.5), (2.9) and the above equation together with Lemma 2.1 in equation (5.1), we arrive at

$$-A^2X+5\alpha AX+(4-\lambda)X+\frac{1}{2}\phi AX-\frac{1}{2}A\phi X=0,\quad X\in\mathfrak{X}(M). \tag{5.3}$$

Define two vector fields $u,v\in D=Ker\eta$ by $u=\nabla_\xi\xi$ and $A\xi=v+f\xi$. Then as $J\xi=N$, it follows by Lemma 2.1 and equation (2.9) that

$$u=\phi(v),\quad v=-\phi(u),\quad \|u\|^2=\|v\|^2. \tag{5.4}$$

Taking $X=\xi$ in equation (5.3) and using $A\xi=v+f\xi$, and equations (5.2), (5.3), we arrive at

$$Av=\|v\|^2\xi+(5\alpha-f)v+\frac{1}{2}u, \tag{5.5}$$

where we used equation (5.2) in the form $\lambda=4+5\alpha f-f^2-\|v\|^2$. Similarly taking $X=v$ in equation (5.3) and using equations (5.2), (5.4), (5.5) , we get

$$Au=-\frac{1}{4}v+\frac{5}{2}\alpha u. \tag{5.6}$$

Now taking inner product with u in equation (5.5) and with v in equation (5.6) and using symmetry of shape operator A, we get

$$\frac{1}{2}\left\|u\right\|^2 = -\frac{1}{4}\left\|v\right\|^2,$$

which together with equation (5.4) gives $u = v = 0$, that is $A\xi = f\xi$, and hence M is a Hopf hypersurface and this with the main result in [1] proves the theorem. ∎

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