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Weyl's Type Theorem and a Local Growth Condition

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Abstract. A bounded linear operator $T \in \mathbf{L}(\mathbb{X})$ acting on a Banach space satisfies a local growth condition of order m for some positive integer m, $T \in \mathrm{loc}(G_m)$, if for every closed subset F of the set of complex numbers and every x in the glocal spectral subspace $\mathbb{X}_T(F)$ there exists an analytic function $f : \mathbb{C} \setminus F \to \mathbb{X}$ such that $(T - \lambda I)f(\lambda) \equiv x$ and $\|f(\lambda)\| \leq M[\mathrm{dist}(\lambda, F)]^{-m}\|x\|$ for some $M > 0$ (independent of F and x). In this paper, we study the stability of generalized Browder-Weyl theorems under perturbations by finite rank operators, by nilpotent operators and, more generally, by algebraic and Riesz operators commuting with T.

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1. Introduction

Throughout this paper, $\mathbf{L}(\mathbb{X})$ denote the algebra of all bounded linear operators acting on a Banach space $\mathbb{X}$. For $T \in \mathbf{L}(\mathbb{X})$, let T^*, $\ker(T)$, $\Re(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote respectively the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by

$$\alpha(T) := \dim\ker(T) \quad \text{and} \quad \beta(T) := \mathrm{codim}\,\Re(T).$$

If the range $\Re(T)$ of T is closed and $\alpha(T) < \infty$ (respectively $\beta(T) < \infty$), then T is called an upper semi-Fredholm (respectively a lower semi-Fredholm) operator. In the sequel $SF_+(\mathbb{X})$ (respectively $SF_-(\mathbb{X})$) will denote the set of all upper (respectively lower) semi-Fredholm operators. If $T \in \mathbf{L}(\mathbb{X})$ is either upper or lower semi-Fredholm, then T is called a *semi-Fredholm operator*, and the index of T is defined by

$$\mathrm{ind}(T) = \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is a *Fredholm operator*. An operator T is called *Weyl* if it is Fredholm of index zero.

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Let $a := a(T)$ be the *ascent* of an operator T; i.e., the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, let $d := d(T)$ be *descent* of an operator T; i.e., the smallest nonnegative integer s such that $\Re(T^s) = \Re(T^{s+1})$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite then $a(T) = d(T)$ [26, Proposition 38.3]. Moreover, $0 < a(T-\lambda I) = d(T-\lambda I) < \infty$ precisely when λ is a pole of the resolvent of T, see Heuser [26, Proposition 50.2].

An operator $T \in \mathbf{L}(\mathbb{X})$ is called *Browder* if it is Fredholm "of finite ascent and descent". The Weyl spectrum of T is defined by $\sigma_W(T) := \{\lambda \in \mathbb{C} : T-\lambda I \text{ is not Weyl}\}$. For $T \in \mathbf{L}(\mathbb{X})$, let $SF_+^-(\mathbb{X}) := \{T \in SF_+(\mathbb{X}) : \operatorname{ind}(T) \leq 0\}$. Then the *upper Weyl spectrum* of T is defined by $\sigma_{SF_+^-}(T) := \left\{\lambda \in \mathbb{C} : T-\lambda I \notin SF_+^-(\mathbb{X})\right\}$. Let $\Delta(T) = \sigma(T) \setminus \sigma_W(T)$ and $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Following Coburn [18], we say that *Weyls theorem* holds for $T \in \mathbf{L}(\mathbb{X})$ (in symbols, $T \in \mathscr{W}$) if $\Delta(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T-\lambda I) < \infty\}$ and that Browder's theorem holds for T (in symbols, $T \in \mathscr{B}$) if $\sigma_b(T) = \sigma_W(T)$, where

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T-\lambda I \text{ is not Browder}\}.$$

Here and elsewhere in this paper, for $K \subset \mathbb{C}$, $\operatorname{iso} K$ is the set of isolated points of K.

According to Rakočević [29], an operator $T \in \mathbf{L}(\mathbb{X})$ is said to satisfy *a-Weyl's theorem* (in symbols, $T \in a\mathscr{W}$) if $\Delta_a(T) = E_a^0(T)$, where

$$E_a^0(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(T-\lambda I) < \infty\}.$$

It is known [29] that an operator satisfying a- Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in \mathbf{L}(\mathbb{X})$ and a nonnegative integer n define T_n to be the restriction of T to $\Re(T^n)$ viewed as a map from $\Re(T^n)$ into $\Re(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $\Re(T^n)$ is closed and T_n is an *upper (respectively a lower) semi-Fredholm operator*, then T is called an upper (respectively a lower) semi-B-Fredholm operator. In this case the index of T is defined as the index of the semi-B-Fredholm operator T_n, see [10]. Moreover, if T_n is a Fredholm operator, then T is called a B-Fredholm operator. A *semi-B-Fredholm operator* is an upper or a lower semi-B-Fredholm operator. An operator $T \in \mathbf{L}(\mathbb{X})$ is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T-\lambda I \text{ is not a B-Weyl operator}\}.$$

Given $T \in \mathbf{L}(\mathbb{X})$, we say that the *generalized Weyl's theorem* holds for T (and we write $T \in g\mathscr{W}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T)$ is the set of all isolated eigenvalues of T, and that the *generalized Browder's theorem* holds for T (in symbols, $T \in g\mathscr{B}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T),$$

where $\pi(T)$ is the set of all poles of T, see [13, Definition 2.13]. It is known [13,25] that

$$g\mathscr{W} \subseteq g\mathscr{B} \cap \mathscr{W} \quad \text{and that} \quad g\mathscr{B} \cup \mathscr{W} \subseteq \mathscr{B}.$$

Moreover, given $T \in g\mathscr{B}$, it is clear that $T \in g\mathscr{W}$ if and only if $E(T) = \pi(T)$. Generalized Weyl's theorem has been studied in [7, 11–14, 22, 30, 31] and the references therein.

Let $SBF_+(\mathbb{X})$ be the class of all *upper semi-B-Fredholm operators*,

$$SBF_+^-(\mathbb{X}) = \{T \in SBF_+(\mathbb{X}) : \mathrm{ind}(T) \le 0\}.$$

The upper B-Weyl spectrum of T

$$\sigma_{SBF_+^-}(T) := \left\{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+^-(\mathbb{X})\right\}.$$

We say that T obeys *generalized a-Weyls theorem* (in symbols, $T \in \mathrm{ga}\mathscr{W}$), if

$$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T);$$

where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ [13, Definition 2.13]. Generalized a-Weyls theorem has been studied in [13, 15, 17, 30, 31].

Definition 1.1. [12] *Let $T \in \mathbf{L}(\mathbb{X})$ and let $s \in \mathbb{N}$. Then T has a uniform descent for $n \ge s$ if $\mathfrak{R}(T) + \ker(T^n) = \mathfrak{R}(T) + \ker(T^s)$ for all $n \ge s$. If, in addition, $\mathfrak{R}(T) + \ker(T^s)$ is closed then T is said to have a topological uniform descent for $n \ge s$.*

Recall from [12] that an operator T is *Drazin invertible* if it has a finite ascent and descent. The Drazin spectrum

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \,\text{is not Drazin invertible}\}.$$

We observe that $\sigma_D(T) = \sigma(T) \setminus \pi(T)$.

Define the set $LD(\mathbb{X})$ by

$$LD(\mathbb{X}) = \left\{T \in \mathbf{L}(\mathbb{X}) : a(T) < \infty \text{ and } \mathfrak{R}(T^{a(T)+1}) \,\text{is closed}\right\}.$$

Definition 1.2. [13] *Let $\mathbb{X}$ be a Banach space. Then $T \in \mathbf{L}(\mathbb{X})$ is called left Drazin invertible if $T \in LD(\mathbb{X})$. The left Drazin spectrum is defined by*

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(\mathbb{X})\}.$$

Definition 1.3. [13] *We will say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I$ is left Drazin invertible and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. We will denote by $\pi^a(T)$ the set of all left poles of T, and by $\pi_0^a(T)$ the set of all left poles of T of finite rank.*

It follows from the preceding description that

$$\sigma_{LD}(T) = \sigma_a(T) \setminus \pi^a(T).$$

Remark 1.1. If $\lambda \in \pi^a(T)$, then it's easily seen that $T - \lambda I$ is an operator of topological uniform descent. Therefore, from [15, Remark 2.6] that λ is isolated in $\sigma_a(T)$.

Following [8], we say that T obeys *generalized a-Browder's theorem* (in symbol, $T \in \mathrm{ga}\mathscr{B}$) if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T).$$

This article also deals with the *single-valued extension property*. This property has a basic role in the local spectral theory, see the recent monograph of Laursen and Neumann [28] or Aiena [3]. In this article consider a localized version of this property, recently studied by several authors [1, 4, 6, 9], and previously by Finch [24].

Let $\mathrm{Hol}(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [24] we say that $T \in \mathbf{L}(\mathbb{X})$ has the *single-valued extension property* (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function

$f: U_\lambda \longrightarrow \mathscr{H}$ which satisfies the equation $(T-\mu)f(\mu)=0$ is the constant function $f \equiv 0$. An operator $T \in \mathbf{B}(\mathscr{H})$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

An operator $T \in \mathbf{L}(\mathbb{X})$ has the SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. The identity theorem for analytic functions ensures that for every $T \in \mathbf{L}(\mathbb{X})$, both T and T^* have the SVEP at the points of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, that both T and T^* have the SVEP at every isolated point of $\sigma(T)=\sigma(T^*)$. The SVEP is inherited by the restrictions to closed invariant subspaces, i.e., if $T \in \mathbf{L}(\mathbb{X})$ has the SVEP at λ_0 and M is closed T-invariant subspace then $T|_M$ has SVEP at λ_0.

The *quasinilpotent part* $H_0(T-\lambda I)$ and the *analytic core* $K(T-\lambda I)$ of $T-\lambda I$ are defined by

$$H_0(T-\lambda I) := \left\{x \in \mathbb{X} : \lim_{n\longrightarrow\infty} \|(T-\lambda I)^n x\|^{\frac{1}{n}} = 0\right\}.$$

and

$$\begin{aligned} K(T-\lambda I) = \big\{x \in \mathbb{X} : &\text{ there exists a sequence } \{x_n\} \subset \mathbb{X} \text{ and } \delta > 0 \text{ for which} \\ & x = x_0, (T-\lambda I)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{for all } n = 1,2,\cdots\big\}. \end{aligned}$$

We note that $H_0(T-\lambda I)$ and $K(T-\lambda I)$ are generally non-closed hyper-invariant subspaces of $T-\lambda I$ such that $(T-\lambda I)^{-p}(0) \subseteq H_0(T-\lambda I)$ for all $p = 0,1,\cdots$ and $(T-\lambda I)K(T-\lambda I) = K(T-\lambda I)$. Recall that if $\lambda \in \operatorname{iso}(\sigma(T))$, then $H_0(T-\lambda I) = \chi_T(\{\lambda\})$, where $\chi_T(\{\lambda\})$ is the glocal spectral subspace consisting of all $x \in \mathscr{H}$ for which there exists an analytic function $f : \mathbb{C}\setminus\{\lambda\} \longrightarrow \mathbb{X}$ that satisfies $(T-\mu I)f(\mu) = x$ for all $\mu \in \mathbb{C}\setminus\{\lambda\}$(see [22]). From [2], the following implication holds for every $T \in \mathbf{L}(\mathbb{X})$,

$$H_0(T-\lambda I) \text{ is closed} \Longrightarrow T \text{ has SVEP at } \lambda.$$

2. Local growth condition

An operator T is said to satisfy a *growth condition of order m*, or to be a (G_m)-operator, if there exists a constant $M > 0$ such that

$$\left\|(T-\lambda I)^{-1}\right\| \leq \frac{M}{[\operatorname{dist}(\lambda, \sigma(T))]^m}$$

for all $\lambda \notin \sigma(T)$. Hyponormal operators are (G_1)-operators [33] and spectral operators of type $m-1$ are (G_m)-operators [23, Theorem XV.6.7]. Not every $T \in (G_m)$ has SVEP. To see this, start by observing that $T \in (G_m) \Rightarrow T^* \in (G_m)$. Hence, if every $T \in (G_m)$ has SVEP, then both T and T^* have SVEP. But this is false, as follows from a consideration of the forward and backward unilateral shifts on a Hilbert space.

Let m be a positive integer. Following [21] we say that $T \in \operatorname{loc}(G_m)$ (or, T satisfies a local growth condition of order m) if for every closed set $F \subset \mathbb{C}$ and every $x \in \mathbb{X}_T(F)$ there exists an analytic function $f : \mathbb{C}\setminus F \to \mathbb{X}$ such that $(T-\lambda I)f(\lambda) \equiv x$ and

$$\|f(\lambda)\| \leq M[\operatorname{dist}(\lambda, F)]^{-m}\|x\| \qquad \text{for some } M > 0$$

(independent of F and x). Hyponormal operators are $\operatorname{loc}(G_1)$ [33] and spectral operators of type $m-1$ are $\operatorname{loc}(G_m)$ [23, Theorem XV.6.7]. Evidently, $T \in \operatorname{loc}(G_m) \Rightarrow T \in (G_m)$. It is known, [27, Proposition 2], that if the Banach space $\mathbb{X}$ is reflexive (in particular, a Hilbert space), then operators $T \in \operatorname{loc}(G_m)$ satisfy Dunford's condition (C). Hence $\operatorname{loc}(G_m)$ operators $T \in \mathbf{L}(\mathbb{X})$ such that $\mathbb{X}$ is reflexive have SVEP, which implies that both T and T^* satisfy a-Browder's theorem. If $T \in \operatorname{loc}(G_m) \cap \mathbf{L}(\mathbb{X})$, $\mathbb{X}$ is reflexive. Duggal [21] proved that $f(T)$ satisfies Weyl's theorem and $f(T^*)$ satisfies a-Weyl's theorem for every function $f \in \mathrm{Hol}(\sigma(T))$.

An operator $T \in \mathbf{L}(\mathbb{X})$ is isoloid (respectively a-isoloid) if every isolated point of $\sigma(T)$ is an eigenvalue of T (respectively every isolated point of $\sigma_a(T)$ is an eigenvalue of T). Evidently, polaroid operators are isoloid and a-polaroid operators are a-isoloid. Recall from [19, Theorem 3.8] that a (necessary and) sufficient condition for $T \in g\mathscr{W}$ is that $T \in gB$ and $H_0(T - \lambda I) = \ker(T - \lambda)^n$, for some $n \in \mathbb{N}$, at points $\lambda \in E(T)$, and that if T is a-isoloid and $T \in \mathrm{ga}\mathscr{W}$, then $f(T) \in \mathrm{ga}\mathscr{W}$ for every $f \in \mathrm{Hol}(\sigma(T))$ [19, Theorem 3.14].

The following (essentially known) lemma proves that operators $T \in \mathrm{loc}(G_m)$ are isoloid, i.e., points $\lambda \in \mathrm{iso}\,\sigma(T)$ are eigenvalues of T. Recall [20] that T is polaroid if every $\lambda \in \mathrm{iso}\,\sigma(T)$ is a pole (no restriction on rank) of the resolvent of T. Polaroid operators are isoloid. In the sequel we assume that $\mathbb{X}$ is a reflexive Banach space.

Lemma 2.1. [21] *Operators $T \in \mathrm{loc}(G_m)$ are polaroid.*

The fact that operators $T \in \mathrm{loc}(G_m)$ have SVEP (recall that our Banach space $\mathbb{X}$ is reflexive) implies that T and T^* satisfy a-Browder's [20, Lemma 2.18], hence also Browder's theorem. More is true.

Theorem 2.1. *If $T \in \mathrm{loc}(G_m)$, then $f(T)$ satisfies generalized Weyl's theorem and $f(T^*)$ satisfies generalized a-Weyl's theorem for every $f \in \mathrm{Hol}(\sigma(T))$.*

Proof. First, we prove that generalized Weyl's theorem holds for T. Since T has SVEP, from [9, Proposition 2.3] it suffices to show that $E(T) = \pi(T)$. But this follows from the fact that operators in $\mathrm{loc}(G_m)$ are polaroid. Hence $T \in g\mathscr{W}$.

Let $f \in \mathrm{Hol}(\sigma(T))$. Under the hypotheses and from [9, Theorem 2.1] and the first part of proof, we conclude that

$$\sigma_{BW}(f(T)) = f(\sigma_{BW}(T)) \quad \text{and} \quad \sigma(T) \setminus E(T) = \sigma_{BW}(T).$$

Hence

$$\sigma_{BW}(f(T)) = f(\sigma_{BW}(T)) = f(\sigma(T) \setminus E(T)).$$

Since T is isoloid, then by [34, Theorem 2.2] we have

$$\sigma_{BW}(f(T)) = f(\sigma_{BW}(T)) = f(\sigma(T) \setminus E(T)) = \sigma(f(T)) \setminus E(f(T)).$$

Thus generalized Weyl's theorem holds for $f(T)$.

Obverse that SVEP implies

$$\sigma(T) = \sigma(T^*) = \sigma_a(T^*), E^a(T^*) = E(T^*)$$

and the polaroid property of T, and therefore of T^*, implies that

$$E(T^*) = \pi(T^*) = \pi(T) = E(T).$$

Recall from proof of [21, Proposition 3.2] that

$$\sigma_{SBF_+^-}(T^*) = \sigma_{BW}(T^*) = \sigma_{BW}(T).$$

Hence

$$\sigma(T) \setminus E(T) = \sigma_{BW}(T) \Rightarrow \sigma_a(T^*) \setminus E^a(T^*) = \sigma_{SBF_+^-}(T^*).$$

That is, T^* satisfies generalized a-Weyl's theorem. Since T^* is a-isoloid, $f(T^*)$ satisfies generalized a-Weyl's theorem for every $f \in \mathrm{Hol}(\sigma(T))$. ∎

An operator $R \in \mathbf{L}(\mathbb{X})$ is a Riesz operator if $R - \lambda I$ is Fredholm for every non-zero $\lambda \in \mathbb{C}$; equivalently, R is Riesz if and only if the essential spectral radius

$$re(R) = \lim_{n\to\infty} \|\phi(R^n)\|^{\frac{1}{n}} = 0,$$

where $\phi : \mathbf{L}(\mathbb{X}) \to \mathbf{L}(\mathbb{X})/\mathbf{K}(\mathbb{X})$ is the Calkin map and $\mathbf{K}(\mathbb{X}) \subset \mathbf{L}(\mathbb{X})$ is the ideal of compact operators. Note that every quasinilpotent operator is a Riesz operator.

Lemma 2.2. *Let $T \in \mathbf{L}(\mathbb{X})$ be such that $\Re(T^n)$ is closed for some n and let Q be a quasinilpotent operator commuting with T. Then $\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T)$.*

Proof. Let $\lambda \notin \sigma_{SBF_+^-}(T)$. Then $T - \lambda I \in SBF_+^-(\mathbb{X}))$, so there exists an $n \in \mathbb{N}$ such that the induced operator T_n is upper semi-Fredholm, $\Re(T_n)$ is closed and $\operatorname{ind}(T_n) = \operatorname{ind}(T - \lambda I) \leq 0$. We shall use the following well-known fact from [32]: if $S \in SF_+^-(\mathbb{X})$, R is a Riesz operator and $RS = SR$, then $S + tR \in SF_+^-(\mathbb{X})$ for all $t \in \mathbb{C}$. Since every quasinilpotent operator is a Riesze operator, $T_nQ_n = Q_nT_n$, and the semi-Fredholm index is continuous function, so it follows from the proof of [21, Theorem 4.1] that $\operatorname{ind}(T_n + Q_n) = \operatorname{ind}(T_n)$ and

$$T_n - \lambda I \in SF_+^-(\mathbb{X}) \Leftrightarrow T_n + Q_n - \lambda I \in SF_+^-(\mathbb{X}),$$

this proves $\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T)$. ∎

Theorem 2.2. *Let $T \in \mathbf{L}(\mathbb{X})$ and let N be a nilpotent operator commuting with T. If generalized a-Weyls theorem holds for T then it also holds for $T+N$.*

Proof. By assumptions and [16, Lemma 3.1], and the fact that $\sigma_a(T+N) = \sigma_a(T)$, we conclude that

$$E^a(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T+N) = \sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N).$$

That is, generalized a-Weyls theorem holds for $T+N$. ∎

Theorem 2.3. *Suppose that $T \in \operatorname{loc}(G_m)$, $N \in \mathbf{L}(\mathbb{X})$ a nilpotent operator commuting with T. Then $f(T^* + N^*)$ satisfies generalized a-Weyl's theorem for every $f \in \operatorname{Hol}(\sigma(T))$.*

Proof. By hypothesis T has SVEP and hence $T+N$ has SVEP, see [3, Corollary 2.45]. The SVEP implies that

$$\sigma(T+N) = \sigma(T) = \sigma(T^*+N^*) = \sigma(T^*) = \sigma_a(T^*+N^*) = \sigma_a(T^*)$$

and by [16, Lemma 3.1] that

$$E^a(T^*+N^*) = E^a(T^*) = E(T^*+N^*) = E(T^*)$$

and the polaroid property of $T+N$, see [5, Theorem 2.10], and therefore of T^*+N^*, implies that

$$E(T^*+N^*) = E(T^*) = \pi(T^*+N^*) = \pi(T+N) = \pi(T) = E(T+N) = E(T),$$

and hence

$$E^a(T^*+N^*) = \pi^a(T^*+N^*) = \pi^a(T^*).$$

Since T^*+N^* satisfies generalized a-Browder's theorem, then

$$\sigma_a(T^*+N^*) \setminus \pi^a(T^*+N^*) = \sigma_{SBF_+^-}(T^*+N^*).$$

Therefore,

$$\sigma_a(T^*+N^*) \setminus \sigma_{SBF_+^-}(T^*+N^*) = E^a(T^*+N^*).$$

That is, T^*+N^* satisfies generalized a-Weyl's theorem. Since T^*+N^* is a-isoloid, $f(T^*+N^*)$ satisfies generalized a-Weyl's theorem for every $f \in \mathrm{Hol}(\sigma(T))$. ∎

Example 2.1. The SVEP itself generally is not enough to guarantee that an operator satisfy either the generalized Weyl's theorem or generalized a-Weyl's theorem. Let T defined on ℓ^2 by

$$T(x_1,x_2,\cdots) = \left(\frac{1}{2}x_2, \frac{1}{3}x_3, \cdots\right).$$

Then T has the SVEP and $\sigma(T) = \sigma_{BW}(T) = E(T) = \{0\}$. Thus T does not obey generalized Weyl's theorem (and nor generalized a-Weyl's theorem).

Lemma 2.3. *If $T \in \mathrm{loc}(G_m)$ and if $F \in \mathbf{L}(\mathbb{X})$ is a finite rank operator commuting with T, then $T+F$ is polaroid operator.*

Proof. It follows from [15, Lemma 3.9] that $\lambda \in \mathrm{acc}\,\sigma(T) \Leftrightarrow \lambda \in \mathrm{acc}\,\sigma(T+F)$, where $\mathrm{acc}\,\sigma(T)$ is the set of the accumulation points of $\sigma(T)$. Since operators in $\mathrm{loc}(G_m)$ being polaroid, then $\sigma_D(T) = \mathrm{acc}\,\sigma(T) = \mathrm{acc}\,\sigma(T+F)$. Since F commutes with T, from [13, Theorem 2.7], we have $\sigma_D(T) = \sigma_D(T+F)$. So $\sigma_D(T+F) = \mathrm{acc}\,\sigma(T+F)$ and $T+F$ is polaroid. ∎

Theorem 2.4. *If $T \in \mathrm{loc}(G_m)$ and if $F \in \mathbf{L}(\mathbb{X})$ is a finite rank operator commuting with T, then $T+F$ satisfies generalized Weyl's theorem.*

Proof. From Lemma 2.3 $T+F$ is polaroid. Then $E(T+F) = \pi(T+F)$. Since T satisfies generalized Weyl's theorem then $\sigma_{BW}(T) = \sigma_D(T)$. Since F is a finite rank operator, then from [11, Theorem 4.3] we have $\sigma_{BW}(T) = \sigma_{BW}(T+F)$. As F commutes with T, from [12, Theorem 2.7] we have $\sigma_D(T) = \sigma_D(T+F)$. So $\sigma_{BW}(T+F) = \sigma_D(T+F)$. Since $E(T+F) = \pi(T+F)$, then from [15, Theorem 2.9] $T+F$ satisfies generalized Weyl's theorem. ∎

In general generalized a-Weyl's theorem and generalized Weyl's theorem are not transmitted from an operator to a commuting finite rank perturbation as the following example shows.

Example 2.2. Let $A : \ell^2 \to \ell^2$ be an injective quasinilpotent operator which is not nilpotent. We define T on the Banach space $\mathbb{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = I \oplus A$ where I is the identity operator on $\ell^2(\mathbb{N})$. Then $\sigma(T) = \sigma_a(T) = \{0,1\}$ and $E^a(T) = \{1\}$. It follows from [14, Example 2] that $\sigma_{BW}(T) = \{0\}$. This implies that $\sigma_{SBF_+^-}(T) = \{0\}$. Hence $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T) = \{1\}$ and T satisfies generalized a-Weyls theorem, so it satisfies generalized Weyls theorem.

We define V on $\ell^2(\mathbb{N})$ by $V(x_1,x_2,\cdots) = (-x_1,0,0,\cdots)$ and $F = V \oplus 0$ on the Banach space $\mathbb{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Then F is a finite rank operator commuting with T. On the other hand, $\sigma(T+F) = \sigma_a(T+F) = \{0,1\}$ and $E^a(T+F) = \{0,1\}$. As $\sigma_{SBF_+^-}(T+F) = \sigma_{SBF_+^-}(T) = \{0\}$, then $\sigma_a(T+F) \setminus \sigma_{SBF_+^-}(T+F) = \{1\} \neq E^a(T+F) = \{0,1\}$ and $T+F$ does not satisfy generalized a-Weyl's theorem. Not that $E^a(T+F) \cap \sigma_a(T) \nsubseteq E^a(T)$. Moreover, $E(T+F) = \{0,1\}$, and as by [11, Theorem 4.3] we have $\sigma_{BW}(T+F) = \sigma_{BW}(T) = \{0\}$, then $T+F$ does not satisfy generalized Weyls theorem.

Proposition 2.1. *If $T \in \mathrm{loc}(G_m)$ and $R \in \mathbf{L}(\mathbb{X})$ is a Riesz operator which commutes with T , then $f(T+R)$ satisfies generalized a-Browder's theorem for every $f \in \mathrm{Hol}_c(\sigma(T+R))$, where $\mathrm{Hol}_c(\sigma(T))$ denote the class of functions $f \in \mathrm{Hol}(\sigma(T))$ such that f is non-constant on each connected component of U of $\sigma(T)$.*

Proof. Since operators in $\mathrm{loc}(G_m)$ have SVEP then the result follows now from [21, Proposition 3.1]. ∎

More can be said in the case in which the Riesz operator R is a quasi-nilpotent.

Proposition 2.2. *If an operator $T \in \mathrm{loc}(G_m)$ and commutes with a quasi-nilpotent $Q \in \mathbf{L}(\mathbb{X})$, then $f(T+Q) \in \mathrm{ga}B$ for all $f \in \mathrm{Hol}(\sigma(T+Q))$ and $f(T^*+Q^*) \in \mathrm{ga}B$ for all $f \in \mathrm{Hol}_c(\sigma(T+Q)$.*

Proof. Evidently, $T+Q$ and T are quasi-nilpotent equivalent. Since T has SVEP, $T+Q$ has SVEP [28, Proposition 3.4.11]. This implies that $f(T+Q)$ has SVEP for all $f \in \mathrm{Hol}(\sigma(T+Q))$ [3, Theorem 2.40]; hence $f(T+Q) \in \mathrm{ga}B$ for all $f \in \mathrm{Hol}(\sigma(T+Q))$ and $f(T^*+Q^*) \in \mathrm{ga}B$ for all $f \in \mathrm{Hol}_c(\sigma(T+Q)$. Hence the result. (It is known that if an operator S has SVEP, then both S and S^* satisfy $\mathrm{ga}B$.) ∎

Recall that $T \in \mathbf{L}(\mathbb{X})$ is called finite a-isoloid (respectively, finite isoloid) operator if an isolated point of $\sigma_a(T)$ is an eigenvalue of finite multiplicity (respectively an isolated point of $\sigma(T)$ is an eigenvalue of finite multiplicity). Clearly, finite a-isoloid implies a-isoloid and finite isoloid, but the converse is not true in general.

Theorem 2.5. *Let $T \in \mathbf{L}(\mathbb{X})$. If $T \in \mathrm{loc}(G_m)$ which is finitely isoloid and commutes with an injective quasi-nilpotent operator $Q \in \mathbf{L}(\mathbb{X})$ and, then $T+Q \in g\mathscr{W}$ and $T^*+Q^* \in \mathrm{ga}\mathscr{W}$.*

Proof. Since T is finitely isoloid. Then it follows from the proof of [20, Proposition 3.3] that $\mathrm{iso}\,\sigma(T) = \mathrm{iso}\,\sigma(T+Q) = \emptyset$. Since $T+Q \in \mathrm{ga}B$ by Proposition 2.2; hence $T+Q \in gB$. That is, $\sigma(T+Q) \setminus \sigma_{BW}(T+Q) = \emptyset$. Since $\pi(T+Q) \subseteq E(T+Q)$ and since $\lambda \in E(T+Q)$ implies $\lambda \in \mathrm{iso}\,\sigma(T+Q) = \emptyset$, it follows that $\sigma(T+Q) \setminus \sigma_{BW}(T+Q) = E(T+Q)$. That is, $T+Q \in g\mathscr{W}$. Since $T^*+Q^* \in \mathrm{ga}B$ by Proposition 2.2 implies that $\sigma_a(T^*+Q^*) \setminus \sigma_{SBF_+^-}(T^*+Q^*) = \pi^a(T^*+Q^*) \subseteq E^a(T^*+Q^*) = E(T^*+Q^*)$. Since $\lambda \in E(T^*+Q^*)$ implies $\lambda \in \mathrm{iso}\,\sigma(T+Q) = \emptyset$, it follows that

$$\sigma_a(T^*+Q^*) \setminus \sigma_{SBF_+^-}(T^*+Q^*) = E^a(T^*+Q^*),$$

thus $T^*+Q^* \in \mathrm{ga}\mathscr{W}$. ∎

$T+Q$ may fail to satisfy generalized Weyl's theorem, and T^*+Q^* may fail to satisfy generalized a-Weyl's theorem , in the absence of the hypothesis that T is finitely isoloid.

Example 2.3. Let $S \in \ell^2(\mathbb{N})$ be the weighted unilateral shift with the weight sequence $\frac{1}{n+1}$. Then S is an injective quasi-nilpotent such that range of Q^n is not closed for every $n \in \mathbb{N}$. Define $Q \in \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $Q = S \oplus S$. Let $T \in \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ be defined by $T = (I-S) \oplus 0$. Then T has SVEP and commutes with Q. It is seen that $\sigma(T+Q) = \sigma_{BW}(T+Q) = \{0,1\}$ and $E(T+Q) = \{1\}$. Evidently, T is not finitely isoloid, $T+Q$ does not satisfy generalized Weyl's theorem. Again, since $\sigma_a(T^*+Q^*) = \sigma_{SBF_+^-}(T^*+Q^*) = \{0,1\}$ and $E^a(T^*+Q^*) = \{1\}$, T^*+Q^* does not satisfy generalized a-Weyl's theorem.

Example 2.4. This example shows that the commutativity hypothesis in Theorem 2.5 is essential. Let S be the injective quasi-nilpotent of Example 2.3, and let $T \in \ell^2(\mathbb{N})$ be the nilpotent defined by

$$T(x_1, x_2, \cdots) = \left(0, -\frac{1}{2}x_1, 0, \cdots\right).$$

Then T and S do not commute, $\sigma(T+S) = \sigma_{BW}(T+S) = E(T+S) = \{0\}$, and $T+S$ does not satisfy generalized Weyl's theorem.

Example 2.5. Generally, generalized a-Weyl's theorem does not extend to a quasinilpotent perturbation: Define on the Banach space $\ell^2(\mathbb{N})$ the operator $T = 0$ and the quasinilpotent operator Q defined by

$$Q(x_1, x_2, \cdots) = \left(\frac{1}{2}x_2, \frac{1}{3}x_3, \cdots\right).$$

Then $\sigma_a(T) = \{0\}$ and $\sigma_{SBF_+^-}(T) = \emptyset$. Moreover we have $E^a(T) = \{0\}$. Hence T satisfies generalized a-Weyl's theorem. But generalized a-Weyl's theorem does not hold for $T + Q = Q$, since $\sigma_{SBF_+^-}(T+Q) = \sigma_a(T+Q) = \{0\}$ and $E^a(T+Q) = \{0\}$.

Definition 2.1. *A bounded linear operator T is said to be algebraic if there exists a non-trivial polynomial h such that $h(T) = 0$.*

From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. A nilpotent operator is a trivial example of an algebraic operator. Also finite rank operators K are algebraic; more generally, if K^n is a finite rank operator for some $n \in \mathbb{N}$ then K is algebraic. Clearly, if T is algebraic then its dual T^* is algebraic, as well as T' in the case of Hilbert space operators.

Theorem 2.6. *Let $T \in \mathbf{L}(\mathbb{X})$ be such that $p(T) \in \operatorname{loc}(G_m)$, for some non-constant polynomial $p(.)$ and let $K \in \mathbf{L}(\mathbb{X})$ be an algebraic operator which commutes with T. Then $f(T+K) \in g\mathscr{W}$ for every $f \in \operatorname{Hol}(\sigma(T+K))$ and $f(T^* + K^*) \in \mathrm{ga}\mathscr{W}$ for every $f \in \operatorname{Hol}_c(\sigma(T+K))$.*

Proof. If $p(T) \in \operatorname{loc}(G_m)$ and K is an algebraic, then $T + K$ has SVEP and polaroid [21, Theorem 4.10]. Then it follows from Theorem 2.1 that $f(T+K) \in g\mathscr{W}$ for every $f \in \operatorname{Hol}(\sigma(T+K))$ and $f(T^* + K^*) \in \mathrm{ga}\mathscr{W}$ for every $f \in \operatorname{Hol}_c(\sigma(T+K))$, see also [20, Corollary 3.7]. ∎

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