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On Oscillation of Second-Order Nonlinear Neutral Functional Differential Equations

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Abstract. In this paper, some sufficient conditions are established for the oscillation of second-order neutral functional differential equation

$$\left[r(t)[x(t)+p(t)x(\tau(t))]'\right]'+q(t)f(x(\sigma(t)))=0, \quad t\geq t_0,$$

where $\int_{t_0}^{\infty} \mathrm{d}t/r(t)=\infty$, or $\int_{t_0}^{\infty} \mathrm{d}t/r(t)<\infty$, $0\leq p(t)\leq p_0<\infty$. The results obtained here complement and improve some known results in the literature.

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1. Introduction

This paper is concerned with the oscillatory behavior of the second-order neutral functional differential equation

$$\left[r(t)[x(t)+p(t)x(\tau(t))]'\right]'+q(t)f(x(\sigma(t)))=0, \quad t\geq t_0, \tag{1.1}$$

where $r\in C^1([t_0,\infty),\mathbb{R})$, $p,\ q\in C([t_0,\infty),\mathbb{R})$, $f\in C(\mathbb{R},\mathbb{R})$. Throughout this paper, we assume that the following conditions hold.

(a) $r(t)>0$, $0\leq p(t)\leq p_0<\infty$, $q(t)\geq 0$, and $q(t)$ is not identically zero on any ray of the form $[t_*,\infty)$ for any $t_*\geq t_0$;
(b) $f(u)/u\geq k>0$, for $u\neq 0$, k is a constant;
(c) $\tau\in C^1([t_0,\infty),\mathbb{R})$, $\sigma\in C([t_0,\infty),\mathbb{R})$, $\tau'(t)\geq\tau_0>0$, $\lim_{t\to\infty}\sigma(t)=\infty$, $\tau\circ\sigma=\sigma\circ\tau$, where τ_0 is a constant.

We shall also consider the two cases

$$\int_{t_0}^{\infty}\frac{\mathrm{d}t}{r(t)}=\infty, \tag{1.2}$$

$$\int_{t_0}^{\infty}\frac{\mathrm{d}t}{r(t)}<\infty. \tag{1.3}$$

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We note that second-order neutral functional differential equations have applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems. For further applications and questions concerning existence and uniqueness of solutions of neutral functional differential equations, see Hale [11].

In recent years, there has been an increasing interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions for different classes of differential equations, we refer to the books [1,2] and the papers [3,5,6,8,15,16,23,24,28,30]. Also, the oscillatory behavior of neutral functional differential equations has been the subject of intensive study, see, e.g., [4, 7, 9, 10, 12–14, 17–22, 25–27, 29, 31–35].

In 1985, Grammatikopoulos *et al.* [10] obtained that if $0 \le p(t) \le 1,\ q(t) \ge 0$ and

$$\int_{t_0}^{\infty} q(s)[1 - p(s-\sigma)]\mathrm{d}s = \infty,$$

then second-order neutral differential equation

$$[x(t) + p(t)x(t-\tau)]'' + q(t)x(t-\sigma) = 0$$

is oscillatory. Later, Grace and Lalli [9] considered the second-order nonlinear neutral delay differential equation

$$\left[r(t)(x(t) + p(t)x(t-\tau))'\right]' + q(t)f(x(t-\sigma)) = 0, \tag{1.4}$$

where

$$\frac{f(x)}{x} \ge k, \text{ for some } x \ne 0,\ \int_{t_0}^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty,$$

and showed that if exists a function $\rho \in C^1([t_0,\infty),\mathbb{R})$ such that

$$\int_{t_0}^{\infty} \left[\rho(s)q(s)(1 - p(s-\sigma)) - \frac{(\rho'(s))^2 r(s-\sigma)}{4k\rho(s)}\right] \mathrm{d}s = \infty,$$

then Equation (1.4) oscillates. In [18, 25], the authors established some general oscillation criteria for second-order neutral delay differential equation (1.4) when $\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty,\ 0 \le p(t) \le 1$. Li [17] studied the second-order neutral delay differential equation

$$[x(t) + p(t)x(t-\tau)]'' + q(t)f(x(t-\sigma)) = 0, \tag{1.5}$$

where $f(x)/x \ge k$, for $x \ne 0$, and established some new oscillation criteria for (1.5) under the condition $0 \le p(t) \le 1$.

In [1, 2], the authors obtained some comparison results for the oscillation of even-order neutral functional equation

$$(x(t) + p(t)x(\tau(t)))^{(n)} + q(t)f(x(g(t))) = 0, \quad t \ge t_0,$$

(see, e.g., [1, Theorem 2.14.4, 2.14.5, 2.14.6]). Especially, [1, Theorem 2.14.5] and [1, Theorem 2.14.6] obtained some comparison theorems for the cases when $1 \le p_1 \le p(t) \le p_2$ and $0 \le p(t) < \infty$, respectively. Tanaka [29] studied the even-order neutral delay differential equation

$$[x(t) + h(t)x(t-\tau)]^{(n)} + f(t, x(g(t))) = 0, \tag{1.6}$$

where $0 \le \mu \le h(t) \le \lambda < 1$ or $1 < \lambda \le h(t) \le \mu$. The author established some comparison theorems for the oscillation of Equation (1.6).

In 2008, Xu and Xia [31] studied the second-order neutral delay differential equation (1.5) and showed that, if

$$0 \le p(t) < \infty, \quad q(t) \ge M > 0,$$

then (1.5) is oscillatory. We note that the result obtained in [31] fails to apply the cases when $p(t) = \gamma/t$ or $p(t) = \gamma/t^2$ for $\gamma > 0$.

To the best of our knowledge, under the case (1.3), it seems to have few oscillation results for Equation (1.1). For instance, Xu and Meng [33] considered the second-order neutral delay differential equation

$$\left[r(t)|(x(t)+p(t)x(t-\tau))'|^{\alpha-1}(x(t)+p(t)x(t-\tau))'\right]' + q(t)f(x(\sigma(t))) = 0, \tag{1.7}$$

the authors obtained the sufficient condition [33, Theorem 2.3], which guarantees that every solution x of Equation (1.7) oscillates or $\lim_{t\to\infty} x(t) = 0$.

Ye and Xu [34] studied the second-order quasilinear neutral delay differential equation

$$(r(t)\psi(x(t))|z'(t)|^{\alpha-1}z'(t))' + q(t)f(x(\sigma(t))) = 0, \tag{1.8}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, and established some new oscillation criteria for (1.8).

In 2010, Han *et al.* [13] studied the oscillation of the second-order nonlinear neutral delay differential equations

$$(r(t)\psi(x(t))|Z'(t)|^{\alpha-1}Z'(t))' + q(t)f(x(\sigma(t))) = 0, \quad t \ge t_0 > 0, \tag{1.9}$$

where $Z(t) = x(t) + p(t)x(\tau(t))$ for $t \ge t_0, \alpha > 0$, $\int_{t_0}^{\infty} 1/(r^{\frac{1}{\alpha}}(s))\mathrm{d}s < \infty$, $0 \le p(t) \le 1$, and obtained the sufficient condition [13, Theorem 2.1 and Theorem 2.2], which guarantees that every solution x of Equation (1.9) oscillates.

Regarding the oscillation of (1.1), we refer to the reader the books [1,2] and the articles [12,29,31] when $p(t) > 1$. In this paper, we try to obtain some new oscillation criteria for (1.1). The paper is organized as follows: In the next section, under the cases (1.2) or (1.3), we will utilize the Riccati transformation technique to obtain some sufficient conditions for the oscillation of (1.1). We shall give several examples to illustrate the main results. In Section 3, we give some remarks to compare our results with those in the literature.

In the sequel, for the sake of convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

2. Main results

In this section, we establish some new oscillation criteria for (1.1). For the sake of convenience, we define the following notations

$$Q(t) := \min\{q(t),\, q(\tau(t))\}, \quad (\rho'(t))_+ := \max\{0, \rho'(t)\},$$

$$R(t) := \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)} \quad \text{and} \quad \delta(t) := \int_{t}^{\infty} \frac{\mathrm{d}s}{r(s)}.$$

We start with the following oscillation result.

Theorem 2.1. *Assume that* (1.2) *holds,* $\sigma'(t) > 0$, $\sigma(t) \le t$, *and* $\sigma(t) \le \tau(t)$ *for* $t \ge t_0$. *Further, suppose that there exists a real-valued function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[k\rho(s)Q(s) - \left(1+\frac{p_0}{\tau_0}\right)\frac{r(\sigma(s))((\rho'(s))_+)^2}{4\rho(s)\sigma'(s)}\right]\mathrm{d}s = \infty. \tag{2.1}$$

Then Equation (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$, for all $t \geq t_1$. Define $z(t) = x(t) + p(t)x(\tau(t))$. Then $z(t) > 0$ for $t \geq t_1$. From (1.1), we have

$$(r(t)z'(t))' \leq -kq(t)x(\sigma(t)) \leq 0, \quad t \geq t_1. \tag{2.2}$$

Therefore $r(t)z'(t)$ is a decreasing function. We claim that

$$z'(t) > 0 \tag{2.3}$$

for $t \geq t_1$. If not, there exists $t_2 \geq t_1$ such that $z'(t_2) < 0$. Then from (2.2), we obtain

$$r(t)z'(t) \leq r(t_2)z'(t_2), \quad t \geq t_2.$$

Hence

$$z(t) \leq z(t_2) - [-r(t_2)z'(t_2)] \int_{t_2}^{t} \frac{1}{r(s)} \mathrm{d}s.$$

Letting $t \to \infty$, we get $z(t) \to -\infty$, $t \to \infty$. This contradiction proves that $z'(t) > 0$ for $t \geq t_1$. On the other hand, using definition of z and applying (1.1), for all sufficiently large t,

$$(r(t)z'(t))' + kq(t)x(\sigma(t)) + p_0 kq(\tau(t))x(\sigma(\tau(t))) + \frac{p_0}{\tau'(t)}(r(\tau(t))z'(\tau(t)))' \leq 0.$$

Note that $\tau'(t) \geq \tau_0 > 0$ and $\tau \circ \sigma = \sigma \circ \tau$. Then from (2.2), we get

$$(r(t)z'(t))' + \frac{p_0}{\tau_0}(r(\tau(t))z'(\tau(t)))' + kQ(t)z(\sigma(t)) \leq 0. \tag{2.4}$$

We define a Riccati substitution

$$\omega(t) = \rho(t)\frac{r(t)z'(t)}{z(\sigma(t))}, \ t \geq t_1. \tag{2.5}$$

Clearly, $\omega(t) > 0$. Differentiating (2.5), from (2.2), we have $z'(\sigma(t)) \geq r(t)z'(t)/r(\sigma(t))$, and

$$\begin{aligned} \omega'(t) &\leq \rho(t)\frac{(r(t)z'(t))'}{z(\sigma(t))} + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\sigma'(t)\omega^2(t)}{\rho(t)r(\sigma(t))} \\ &\leq \rho(t)\frac{(r(t)z'(t))'}{z(\sigma(t))} + \frac{(\rho'(t))_+}{\rho(t)}\omega(t) - \frac{\sigma'(t)\omega^2(t)}{\rho(t)r(\sigma(t))}. \end{aligned} \tag{2.6}$$

Similarly, we introduce another Riccati transformation

$$v(t) = \rho(t)\frac{r(\tau(t))z'(\tau(t))}{z(\sigma(t))}, \ t \geq t_1. \tag{2.7}$$

Then $v(t) > 0$. Differentiating (2.7), by (2.2) and $\sigma(t) \leq \tau(t)$, we find

$$z'(\sigma(t)) \geq r(\tau(t))z'(\tau(t))/r(\sigma(t)),$$

and

$$\begin{aligned} v'(t) &\leq \rho(t)\frac{(r(\tau(t))z'(\tau(t)))'}{z(\sigma(t))} + \frac{\rho'(t)}{\rho(t)}v(t) - \frac{\sigma'(t)v^2(t)}{\rho(t)r(\sigma(t))} \\ &\leq \rho(t)\frac{(r(\tau(t))z'(\tau(t)))'}{z(\sigma(t))} + \frac{(\rho'(t))_+}{\rho(t)}v(t) - \frac{\sigma'(t)v^2(t)}{\rho(t)r(\sigma(t))}. \end{aligned} \tag{2.8}$$

It follows from (2.6) and (2.8) that

$$\omega'(t)+\frac{p_0}{\tau_0}\nu'(t)\le\rho(t)\frac{(r(t)z'(t))'}{z(\sigma(t))}+\frac{p_0}{\tau_0}\rho(t)\frac{(r(\tau(t))z'(\tau(t)))'}{z(\sigma(t))}$$
$$+\frac{(\rho'(t))_+}{\rho(t)}\omega(t)-\frac{\sigma'(t)\omega^2(t)}{\rho(t)r(\sigma(t))}+\frac{p_0}{\tau_0}\frac{(\rho'(t))_+}{\rho(t)}\nu(t)-\frac{p_0}{\tau_0}\frac{\sigma'(t)\nu^2(t)}{\rho(t)r(\sigma(t))}.$$

In view of (2.4) and the above inequality, we obtain

$$\begin{aligned}\omega'(t)+\frac{p_0}{\tau_0}\nu'(t)&\le -k\rho(t)Q(t)+\frac{(\rho'(t))_+}{\rho(t)}\omega(t)-\frac{\sigma'(t)\omega^2(t)}{\rho(t)r(\sigma(t))}\\&\quad+\frac{p_0}{\tau_0}\frac{(\rho'(t))_+}{\rho(t)}\nu(t)-\frac{p_0}{\tau_0}\frac{\sigma'(t)\nu^2(t)}{\rho(t)r(\sigma(t))}\\&\le -k\rho(t)Q(t)+\left(1+\frac{p_0}{\tau_0}\right)\frac{r(\sigma(t))((\rho'(t))_+)^2}{4\rho(t)\sigma'(t)}.\end{aligned}$$

Integrating the above inequality from t_1 to t, we get

$$\omega(t)+\frac{p_0}{\tau_0}\nu(t)\le\omega(t_1)+\frac{p_0}{\tau_0}\nu(t_1)-\int_{t_1}^{t}\left[k\rho(s)Q(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{r(\sigma(s))((\rho'(s))_+)^2}{4\rho(s)\sigma'(s)}\right]\mathrm{d}s,$$

which follows that

$$\int_{t_1}^{t}\left[k\rho(s)Q(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{r(\sigma(s))((\rho'(s))_+)^2}{4\rho(s)\sigma'(s)}\right]\mathrm{d}s\le\omega(t_1)+\frac{p_0}{\tau_0}\nu(t_1),$$

which contradicts (2.1). ∎

Choosing $\rho(t)=R(\sigma(t))$. By Theorem 2.1, we have the following results.

Corollary 2.1. *Assume that* (1.2) *holds,* $\sigma'(t)>0,\ \sigma(t)\le t$, *and* $\sigma(t)\le\tau(t)$ *for* $t\ge t_0$. *If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[kR(\sigma(s))Q(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\sigma'(s)}{4r(\sigma(s))R(\sigma(s))}\right]\mathrm{d}s=\infty, \tag{2.9}$$

then Equation (1.1) *oscillates.*

Corollary 2.2. *Assume that* (1.2) *holds,* $\sigma'(t)>0,\ \sigma(t)\le t$, *and* $\sigma(t)\le\tau(t)$ *for* $t\ge t_0$. *If*

$$\liminf_{t\to\infty}\frac{1}{\ln R(\sigma(t))}\int_{t_0}^{t}R(\sigma(s))Q(s)\mathrm{d}s>\frac{1+\frac{p_0}{\tau_0}}{4k}, \tag{2.10}$$

then Equation (1.1) *oscillates.*

Proof. It is not hard to verify that (2.10) yields the existence $\varepsilon>0$ such that for all large t,

$$\frac{1}{\ln R(\sigma(t))}\int_{t_0}^{t}R(\sigma(s))Q(s)\mathrm{d}s\ge\frac{1+\frac{p_0}{\tau_0}}{4k}+\varepsilon,$$

which follows that

$$\int_{t_0}^{t}R(\sigma(s))Q(s)\mathrm{d}s-\frac{1+\frac{p_0}{\tau_0}}{4k}\ln R(\sigma(t))\ge\varepsilon\ln R(\sigma(t)),$$

that is,

$$\begin{aligned}(2.11)\qquad &\int_{t_0}^{t}\left[R(\sigma(s))Q(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\sigma'(s)}{4kr(\sigma(s))R(\sigma(s))}\right]\mathrm{d}s\\ &\geq \varepsilon\ln R(\sigma(t))-\frac{1+\frac{p_0}{\tau_0}}{4k}\ln R(\sigma(t_0)).\end{aligned}$$

Now, it is obvious that (2.11) implies (2.9) and the assertion of Corollary 2.2 follows from Corollary 2.1. ▮

Corollary 2.3. *Assume that* (1.2) *holds,* $\sigma'(t)>0,\ \sigma(t)\leq t$, *and* $\sigma(t)\leq\tau(t)$ *for* $t\geq t_0$. *If*

$$(2.12)\qquad \liminf_{t\to\infty}\frac{Q(t)R^2(\sigma(t))r(\sigma(t))}{\sigma'(t)}>\frac{1+\frac{p_0}{\tau_0}}{4k},$$

then Equation (1.1) *oscillates.*

Proof. It is easy to verify that (2.12) yields the existence $\varepsilon>0$ such that for all large t,

$$\frac{Q(t)R^2(\sigma(t))r(\sigma(t))}{\sigma'(t)}\geq\frac{1+\frac{p_0}{\tau_0}}{4k}+\varepsilon.$$

Multiplying $\sigma'(t)/(R(\sigma(t))r(\sigma(t)))$ on both sides of the above inequality, we have

$$R(\sigma(s))Q(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\sigma'(s)}{4kr(\sigma(s))R(\sigma(s))}\geq\varepsilon\frac{\sigma'(s)}{4kr(\sigma(s))R(\sigma(s))}$$

which implies that (2.9) holds. Therefore, by Corollary 2.1, Equation (1.1) is oscillatory. ▮

Next, choosing $\rho(t)=t$. By Theorem 2.1, we have the following result.

Corollary 2.4. *Assume that* (1.2) *holds,* $\sigma'(t)>0,\ \sigma(t)\leq t$, *and* $\sigma(t)\leq\tau(t)$ *for* $t\geq t_0$. *If*

$$(2.13)\qquad \limsup_{t\to\infty}\int_{t_0}^{t}\left[ksQ(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{r(\sigma(s))}{4s\sigma'(s)}\right]\mathrm{d}s=\infty,$$

then Equation (1.1) *oscillates.*

For an application of Corollary 2.4, we give the following example.

Example 2.1. Consider the second-order neutral differential equation

$$(2.14)\qquad \left[t(x(t)+p(t)x(\lambda_1 t))'\right]'+\frac{\gamma}{t}f(x(\lambda_2 t)))=0,\quad t\geq 1,$$

where $r(t)=t,\ \tau(t)=\lambda_1 t,\ \sigma(t)=\lambda_2 t,\ \lambda_1\geq\lambda_2,\ f(x)=x(1+x^2), 0\leq p(t)\leq p_0<\infty,\ q(t)=\gamma/t$ and $\gamma>0$. Let $k=1$ and $\tau_0=\lambda_1$. Then

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[ksQ(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{r(\sigma(s))}{4s\sigma'(s)}\right]\mathrm{d}s=\limsup_{t\to\infty}\int_{1}^{t}\left[\gamma-\frac{1+\frac{p_0}{\lambda_1}}{4}\right]\mathrm{d}s=\infty,$$

for $\gamma>(1+p_0/\lambda_1)/4$. Hence, from Corollary 2.4, Equation (2.14) is oscillatory for $\gamma>(1+p_0/\lambda_1)/4$.

Next, we will give the following results under the case (1.3).

Theorem 2.2. *Assume that* (1.3) *holds,* $\sigma'(t) > 0,\ \sigma(t) \le \tau(t) \le t$ *for* $t \ge t_0$. *Suppose also that there exists a real-valued function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that* (2.1) *holds, and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[k\delta(s)Q(s) - \frac{1+\frac{p_0}{\tau_0}}{4r(s)\delta(s)} \right] \mathrm{d}s = \infty. \tag{2.15}$$

Then Equation (1.1) *is oscillatory.*

Proof. Suppose that x is a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0,\ x(\tau(t)) > 0,\ x(\sigma(t)) > 0$, for all $t \ge t_1$. Setting z as in Theorem 2.1. From (1.1), $r(t)z'(t)$ is nonincreasing eventually. Consequently, it is easy to conclude that there exist two possible cases of the sign of $z'(t)$, that is, $z'(t) > 0$ or $z'(t) < 0$ for $t \ge t_2 \ge t_1$. If $z'(t) > 0$, then we are back to the case of Theorem 2.1, and we can get a contradiction to (2.1). If $z'(t) < 0$, then we define the function ω by

$$\omega(t) = \frac{r(t)z'(t)}{z(t)}, \quad t \ge t_2. \tag{2.16}$$

Clearly, $\omega(t) < 0$. Noting that $r(t)z'(t)$ is nonincreasing, we get

$$r(s)z'(s) \le r(t)z'(t), \quad s \ge t \ge t_2.$$

Dividing the above by $r(s)$ and integrating it from t to l, we obtain

$$z(l) \le z(t) + r(t)z'(t) \int_t^l \frac{\mathrm{d}s}{r(s)}, \quad l \ge t \ge t_2.$$

Letting $l \to \infty$ in the above inequality, we see that

$$0 \le z(t) + r(t)z'(t)\delta(t), \quad t \ge t_2.$$

Therefore,

$$\frac{r(t)z'(t)}{z(t)}\delta(t) \ge -1, \quad t \ge t_2.$$

From (2.16), we have

$$-1 \le \omega(t)\delta(t) \le 0, \quad t \ge t_2. \tag{2.17}$$

Similarly, we introduce another function ν by

$$\nu(t) = \frac{r(\tau(t))z'(\tau(t))}{z(t)},\ t \ge t_2. \tag{2.18}$$

Obviously, $\nu(t) < 0$. Noting that $r(t)z'(t)$ is nonincreasing, we have $r(\tau(t))z'(\tau(t)) \ge r(t)z'(t)$. Then $\nu(t) \ge \omega(t)$. From (2.17), we obtain

$$-1 \le \nu(t)\delta(t) \le 0,\ t \ge t_2. \tag{2.19}$$

Differentiating (2.16), we get

$$\omega'(t) = \frac{(r(t)z'(t))'}{z(t)} - \frac{\omega^2(t)}{r(t)}. \tag{2.20}$$

Differentiating (2.18), we have

$$\nu'(t) \le \frac{(r(\tau(t))z'(\tau(t)))'}{z(t)} - \frac{\nu^2(t)}{r(t)}. \tag{2.21}$$

In view of (2.20) and (2.21), we can obtain

$$\omega'(t)+\frac{p_0}{\tau_0}\nu'(t)\le\frac{(r(t)z'(t))'}{z(t)}+\frac{p_0}{\tau_0}\frac{(r(\tau(t))z'(\tau(t)))'}{z(t)}-\frac{\omega^2(t)}{r(t)}-\frac{p_0}{\tau_0}\frac{\nu^2(t)}{r(t)}. \tag{2.22}$$

On the other hand, proceed as in the proof of Theorem 2.1, we have that (2.4) holds. Therefore, by (2.4) and (2.22), we have

$$\omega'(t)+\frac{p_0}{\tau_0}\nu'(t)\le -kQ(t)-\frac{\omega^2(t)}{r(t)}-\frac{p_0}{\tau_0}\frac{\nu^2(t)}{r(t)}. \tag{2.23}$$

Multiplying (2.23) by $\delta(t)$, and integrating on $[t_2,t]$ implies

$$\delta(t)\omega(t)-\delta(t_2)\omega(t_2)+\int_{t_2}^{t}\frac{\omega(s)}{r(s)}\mathrm{d}s+\int_{t_2}^{t}\frac{\omega^2(s)\delta(s)}{r(s)}\mathrm{d}s+\frac{p_0}{\tau_0}\delta(t)\nu(t)-\frac{p_0}{\tau_0}\delta(t_2)\nu(t_2)$$
$$+\frac{p_0}{\tau_0}\int_{t_2}^{t}\frac{\nu(s)}{r(s)}\mathrm{d}s+\frac{p_0}{\tau_0}\int_{t_2}^{t}\frac{\nu^2(s)\delta(s)}{r(s)}\mathrm{d}s+k\int_{t_2}^{t}\delta(s)Q(s)\mathrm{d}s\le 0.$$

From the above inequality, we obtain

$$\delta(t)\omega(t)-\delta(t_2)\omega(t_2)+\frac{p_0}{\tau_0}\delta(t)\nu(t)-\frac{p_0}{\tau_0}\delta(t_2)\nu(t_2)$$
$$+k\int_{t_2}^{t}\delta(s)Q(s)\mathrm{d}s-\frac{1+\frac{p_0}{\tau_0}}{4}\int_{t_2}^{t}\frac{\mathrm{d}s}{r(s)\delta(s)}\le 0.$$

Thus, it follows from the above inequality that

$$\delta(t)\omega(t)+\frac{p_0}{\tau_0}\delta(t)\nu(t)+\int_{t_2}^{t}\left[k\delta(s)Q(s)-\frac{1+\frac{p_0}{\tau_0}}{4r(s)\delta(s)}\right]\mathrm{d}s\le\delta(t_2)\omega(t_2)+\frac{p_0}{\tau_0}\delta(t_2)\nu(t_2).$$

By (2.17) and (2.19), we obtain a contradiction with (2.15). ∎

Corollary 2.5. *Assume that* (1.3) *holds,* $\sigma'(t)>0$, $\sigma(t)\le\tau(t)\le t$ *for* $t\ge t_0$. *Furthermore, assume that one of conditions* (2.9), (2.10), (2.12) *and* (2.13) *holds, and one has* (2.15). *Then Equation* (1.1) *is oscillatory.*

For an application of Corollary 2.5, we will give the following example.

Example 2.2. Consider the second-order neutral delay differential equation

$$\left[t^2(x(t)+p(t)x(t-\tau))'\right]'+tf(x(t-\sigma))=0,\quad t\ge 1, \tag{2.24}$$

where $r(t)=t^2$, $0\le p(t)\le p_0<\infty$, $q(t)=t$, $\tau(t)=t-\tau$, $\sigma(t)=t-\sigma$, $\sigma\ge\tau$, $f(x)=x(1+x^2)$. Therefore, the condition (1.3) holds, $\tau'(t)=1$, $\sigma'(t)=1$, $Q(t)=t-\tau$, $R(t)=1-1/t$ and $\delta(t)=1/t$. Take $k=1$ and $\tau_0=1$, we obtain

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[kR(\sigma(s))Q(s)-(1+\frac{p_0}{\tau_0})\frac{\sigma'(s)}{4r(\sigma(s))R(\sigma(s))}\right]\mathrm{d}s$$
$$=\limsup_{t\to\infty}\int_{1}^{t}\left[(s-\tau)(1-\frac{1}{s-\sigma})-\frac{1+p_0}{4(s-\sigma)^2\left(1-\frac{1}{s-\sigma}\right)}\right]\mathrm{d}s=\infty,$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[k\delta(s)Q(s)-\frac{1+\frac{p_0}{\tau_0}}{4r(s)\delta(s)}\right]\mathrm{d}s=\limsup_{t\to\infty}\int_{1}^{t}\left[1-\frac{\tau}{s}-\frac{1+p_0}{4s}\right]\mathrm{d}s=\infty.$$

Then, we have that (2.9) and (2.15) hold. Hence, by Corollary 2.5, Equation (2.24) is oscillatory.

Theorem 2.3. *Assume that* (1.3) *holds,* $\sigma'(t) > 0,\ \sigma(t) \le \tau(t) \le t$ *for* $t \ge t_0$. *Furthermore, suppose that there exists a real-valued function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that* (2.1) *holds, and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \delta^2(s)Q(s)\mathrm{d}s = \infty. \tag{2.25}$$

Then Equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0,\ x(\tau(t)) > 0,\ x(\sigma(t)) > 0$, for all $t \ge t_1$. Define z as in Theorem 2.1. By (1.1), $r(t)z'(t)$ is nonincreasing eventually. Consequently, it is easy to conclude that there exist two possible cases of the sign of $z'(t)$, that is, $z'(t) > 0$ or $z'(t) < 0$ for $t \ge t_2 \ge t_1$. If $z'(t) > 0$, then we are back to the case of Theorem 2.1, and we can get a contradiction to (2.1). If $z'(t) < 0$, then we define ω and ν as in Theorem 2.2. Then proceed as in the proof of Theorem 2.2, we obtain (2.17), (2.19) and (2.23). Multiplying (2.23) by $\delta^2(t)$, and integrating on $[t_2,t]$ yields

$$\begin{aligned}
&\delta^2(t)\omega(t) - \delta^2(t_2)\omega(t_2) + 2\int_{t_2}^{t} \frac{\omega(s)\delta(s)}{r(s)}\mathrm{d}s \\
&+ \int_{t_2}^{t} \frac{\omega^2(s)\delta^2(s)}{r(s)}\mathrm{d}s + \frac{p_0}{\tau_0}\delta^2(t)\nu(t) - \frac{p_0}{\tau_0}\delta^2(t_2)\nu(t_2) \\
&+ 2\frac{p_0}{\tau_0}\int_{t_2}^{t} \frac{\nu(s)\delta(s)}{r(s)}\mathrm{d}s + \frac{p_0}{\tau_0}\int_{t_2}^{t} \frac{\nu^2(s)\delta^2(s)}{r(s)}\mathrm{d}s + k\int_{t_2}^{t} \delta^2(s)Q(s)\mathrm{d}s \le 0.
\end{aligned} \tag{2.26}$$

It follows from (1.3) and (2.17) that

$$\left|\int_{t_2}^{\infty} \frac{\omega(s)\delta(s)}{r(s)}\mathrm{d}s\right| \le \int_{t_2}^{\infty} \frac{|\omega(s)\delta(s)|}{r(s)}\mathrm{d}s \le \int_{t_2}^{\infty} \frac{1}{r(s)}\mathrm{d}s < \infty,$$

$$\int_{t_2}^{t} \frac{\omega^2(s)\delta^2(s)}{r(s)}\mathrm{d}s \le \int_{t_2}^{\infty} \frac{1}{r(s)}\mathrm{d}s < \infty.$$

In view of (2.19), we have

$$\left|\int_{t_2}^{\infty} \frac{\nu(s)\delta(s)}{r(s)}\mathrm{d}s\right| \le \int_{t_2}^{\infty} \frac{|\nu(s)\delta(s)|}{r(s)}\mathrm{d}s \le \int_{t_2}^{\infty} \frac{1}{r(s)}\mathrm{d}s < \infty,$$

$$\int_{t_2}^{t} \frac{\nu^2(s)\delta^2(s)}{r(s)}\mathrm{d}s \le \int_{t_2}^{\infty} \frac{1}{r(s)}\mathrm{d}s < \infty.$$

From (2.26), we get

$$\limsup_{t\to\infty} \int_{t_0}^{t} \delta^2(s)Q(s)\mathrm{d}s < \infty,$$

which is a contradiction with (2.25). ∎

Corollary 2.6. *Assume that* (1.3) *holds,* $\sigma'(t) > 0,\ \sigma(t) \le \tau(t) \le t$ *for* $t \ge t_0$. *Suppose also that one of conditions* (2.9)*,* (2.10)*,* (2.12) *and* (2.13) *holds, and one has* (2.25)*. Then Equation* (1.1) *is oscillatory.*

Remark 2.1. It is easy to see that Corollary 2.6 can be applied to Equation (2.24).

In the following, we give some new oscillation results for Equation (1.1) when $\sigma(t) \geq \tau(t)$ for $t \geq t_0$.

Theorem 2.4. *Assume that* (1.2) *holds,* $\tau(t) \leq t$, *and* $\sigma(t) \geq \tau(t)$ *for* $t \geq t_0$. *Moreover, suppose that there exists a real-valued function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[k\rho(s)Q(s) - \left(1+\frac{p_0}{\tau_0}\right)\frac{r(\tau(s))((\rho'(s))_+)^2}{4\rho(s)\tau_0}\right] \mathrm{d}s = \infty. \tag{2.27}$$

Then Equation (1.1) *is oscillatory.*

Proof. Suppose that x is a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, for all $t \geq t_1$. Define z as in Theorem 2.1. Similar to the proof of Theorem 2.1, there exists $t_2 \geq t_1$ such that (2.3) and (2.4) hold for $t \geq t_2$. Define a Riccati transformation

$$\omega(t) = \rho(t)\frac{r(t)z'(t)}{z(\tau(t))},\ t \geq t_2. \tag{2.28}$$

Then, $\omega(t) > 0$. Differentiating (2.28), by (2.2), we get $z'(\tau(t)) \geq r(t)z'(t)/r(\tau(t))$, and

$$\begin{aligned}\omega'(t) &\leq \rho(t)\frac{(r(t)z'(t))'}{z(\tau(t))} + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\tau_0\omega^2(t)}{\rho(t)r(\tau(t))}\\ &\leq \rho(t)\frac{(r(t)z'(t))'}{z(\tau(t))} + \frac{(\rho'(t))_+}{\rho(t)}\omega(t) - \frac{\tau_0\omega^2(t)}{\rho(t)r(\tau(t))}.\end{aligned} \tag{2.29}$$

Next, we introduce another function ν by

$$\nu(t) = \rho(t)\frac{r(\tau(t))z'(\tau(t))}{z(\tau(t))}, \quad t \geq t_2. \tag{2.30}$$

Note that $\sigma(t) \geq \tau(t)$. The rest of the proof is similar to that of Theorem 2.1, and so is omitted. ∎

Choosing $\rho(t) = R(\tau(t))$. By Theorem 2.4, we have the following oscillation criteria.

Corollary 2.7. *Assume that* (1.2) *holds,* $\tau(t) \leq t$ *and* $\sigma(t) \geq \tau(t)$ *for* $t \geq t_0$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[kR(\tau(s))Q(s) - \left(1+\frac{p_0}{\tau_0}\right)\frac{\tau_0}{4r(\tau(s))R(\tau(s))}\right] \mathrm{d}s = \infty, \tag{2.31}$$

then Equation (1.1) *is oscillatory.*

Corollary 2.8. *Assume that* (1.2) *holds,* $\tau(t) \leq t$ *and* $\sigma(t) \geq \tau(t)$ *for* $t \geq t_0$. *If*

$$\liminf_{t\to\infty} \frac{1}{\ln R(\tau(t))}\int_{t_0}^{t} R(\tau(s))Q(s)\mathrm{d}s > \frac{1+\frac{p_0}{\tau_0}}{4k}, \tag{2.32}$$

then Equation (1.1) *oscillates.*

Proof. By Corollary 2.7, the proof is similar to that of Corollary 2.2, we omit the details. ∎

Corollary 2.9. *Assume that* (1.2) *holds,* $\tau(t) \leq t$ *and* $\sigma(t) \geq \tau(t)$ *for* $t \geq t_0$. *If*

$$\liminf_{t\to\infty} \frac{Q(t)R^2(\tau(t))r(\tau(t))}{\tau_0} > \frac{1+\frac{p_0}{\tau_0}}{4k}, \tag{2.33}$$

then Equation (1.1) *oscillates.*

Proof. By Corollary 2.7, the proof is similar to that of Corollary 2.3, and so is omitted. ∎

Next, choosing $\rho(t) = t$. From Theorem 2.4, we have the following result.

Corollary 2.10. *Assume that* (1.2) *holds,* $\tau(t) \le t$ *and* $\sigma(t) \ge \tau(t)$ *for* $t \ge t_0$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[ksQ(s) - \left(1 + \frac{p_0}{\tau_0}\right) \frac{r(\tau(s))}{4s\tau_0} \right] \mathrm{d}s = \infty, \tag{2.34}$$

then Equation (1.1) *oscillates.*

For an application of Corollary 2.10, we give the following example.

Example 2.3. Consider the second-order neutral differential equation

$$\left[t(x(t) + p(t)x(\lambda_1 t))'\right]' + q(t)f(x(\lambda_2 t))) = 0, \quad t \ge 1, \tag{2.35}$$

where $r(t) = t$, $\tau(t) = \lambda_1 t$, $\sigma(t) = \lambda_2 t$, $\lambda_1 \le \lambda_2$, $f(x) = x(1+x^2)$, $0 \le p(t) \le p_0 < \infty$, $q(t) = \gamma/t$ and $\gamma > 0$. Let $k = 1$ and $\tau_0 = \lambda_1$. It is easy to see that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[ksQ(s) - \left(1 + \frac{p_0}{\tau_0}\right) \frac{r(\tau(s))}{4s\tau_0} \right] \mathrm{d}s = \limsup_{t\to\infty} \int_{1}^{t} \left[\gamma - \frac{1 + \frac{p_0}{\lambda_1}}{4} \right] \mathrm{d}s = \infty,$$

for $\gamma > (1 + (p_0/\lambda_1))/4$. Hence, by Corollary 2.10, Equation (2.35) is oscillatory for $\gamma > (1 + (p_0/\lambda_1))/4$.

Next, we will give the following results under the case when (1.3) and $\sigma(t) \ge \tau(t)$.

Theorem 2.5. *Assume that* (1.3) *holds,* $\tau(t) \le \sigma(t) \le t$ *for* $t \ge t_0$. *Further, suppose that there exists a real-valued function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that* (2.27) *holds. Suppose also that one of* (2.15) *and* (2.25) *holds. Then Equation* (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, for all $t \ge t_1$. Setting z as in Theorem 2.1. In view of (1.1), $r(t)z'(t)$ is nonincreasing eventually. Consequently, it is easy to conclude that there exist two possible cases of the sign of $z'(t)$. That is, $z'(t) > 0$ or $z'(t) < 0$ for $t \ge t_2 \ge t_1$.

If $z'(t) > 0$, then we are back to the case of Theorem 2.4, and we can get a contradiction to (2.27).

If $z'(t) < 0$, then by the proof of Theorem 2.2 or Theorem 2.3, we can obtain a contradiction to (2.15) or (2.25), respectively. ∎

Corollary 2.11. *Assume that* (1.3) *holds,* $\tau(t) \le \sigma(t) \le t$ *for* $t \ge t_0$. *Suppose also that one of conditions* (2.31), (2.32), (2.33) *and* (2.34) *holds, and one has* (2.15) *or* (2.25). *Then Equation* (1.1) *is oscillatory.*

For an application of Corollary 2.11, we will give the following example.

Example 2.4. Consider the second-order neutral delay differential equation

$$\left[t^2(x(t) + p(t)x(t-\tau))'\right]' + tf(x(t-\sigma)) = 0, \quad t \ge 1, \tag{2.36}$$

where $r(t) = t^2$, $0 \le p(t) \le p_0 < \infty$, $q(t) = t$, $\tau(t) = t - \tau$, $\sigma(t) = t - \sigma$, $\sigma \le \tau$ and $f(x) = x(1+x^2)$. Hence, the condition (1.3) holds, $\tau'(t) = 1$, $Q(t) = t - \tau$, $R(t) = 1 - 1/t$ and $\delta(t) = 1/t$. Take $k = 1$ and $\tau_0 = 1$, we get

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[kR(\tau(s))Q(s) - \left(1 + \frac{p_0}{\tau_0}\right) \frac{\tau_0}{4r(\tau(s))R(\tau(s))} \right] \mathrm{d}s$$

$$= \limsup_{t\to\infty} \int_1^t \left[(s-\tau)\left(1-\frac{1}{s-\tau}\right) - \frac{1+p_0}{4(s-\tau)^2\left(1-\frac{1}{s-\tau}\right)} \right] \mathrm{d}s = \infty,$$

and

$$\limsup_{t\to\infty} \int_{t_0}^t \left[k\delta(s)Q(s) - \frac{1+\frac{p_0}{\tau_0}}{4r(s)\delta(s)} \right] \mathrm{d}s = \limsup_{t\to\infty} \int_1^t \left[1 - \frac{\tau}{s} - \frac{1+p_0}{4s} \right] \mathrm{d}s = \infty.$$

That is, we get (2.31) and (2.15). Thus, by Corollary 2.11, Equation (2.36) is oscillatory.

3. Remarks

In this section, we will give some remarks to illustrate our results.

Remark 3.1. The condition $\tau \circ \sigma = \sigma \circ \tau$ in this paper means that the deviating arguments τ and δ are of the same form, for example, if $\tau(t) = t - a$, then at the same time $\delta(t) = t - b$ or $\delta(t) = t + b$, where a and b are nonnegative constants.

Remark 3.2. In the last few years, many authors studied the oscillatory nature of the second-order neutral delay differential equation (1.4). Most of established results are based on the conditions $0 \le p(t) < 1$ and $\int_{t_0}^{\infty}(\mathrm{d}t/r(t)) = \infty$, we refer the reader to the papers [9, 10, 17, 18, 25]. Thus, our results complement these results when $\int_{t_0}^{\infty}(\mathrm{d}t/r(t)) < \infty$. Also, when $0 \le p(t) < 1$, we note that Theorem 2.4 can be applied to the equation

$$\left[r(t)[x(t) + p(t)x(t-\tau)]'\right]' + q(t)f(x(t+\delta)) = 0, \quad t \ge t_0,$$

where $\delta > 0$ is a constant.

Remark 3.3. Recently, Xu and Xia [31] considered the oscillation of equation (1.5) under the case when $0 \le p(t) < \infty$. For example, they gave the following theorem.

[31, Theorem 3.1] Suppose that $0 \le p(t) < \infty$, $f(x)/x \ge k > 0$ for $x \ne 0$, $q(t) \ge M > 0$. Then equation (1.5) is oscillatory.

Hence, in some sense, Corollary 2.4 and Corollary 2.10 improve the results given in [31].

Remark 3.4. Han *et al.* [12] investigated the oscillatory behavior of equation (1.1), and established some sufficient conditions which ensure that every solution of equation (1.1) is oscillatory. For example

[12, Theorem 2.1] Assume that (a), (b), (c) and (1.2) hold. If

$$\int_{t_0}^{\infty} Q(t)\mathrm{d}t = \infty,$$

then every solution of (1.1) is oscillatory.

Therefore, Corollary 2.4 and Corollary 2.4 improve results of [12, Theorem 2.1]. Also, our results complement their results when $\int_{t_0}^{\infty}(\mathrm{d}t/r(t)) < \infty$.

Remark 3.5. It would be interesting to find another method to study equation (1.1) for the case when $\tau \circ \sigma \not\equiv \sigma \circ \tau$ and $\int_{t_0}^{\infty}(\mathrm{d}t/r(t)) < \infty$.

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