Statistical Approximation Properties
of Modified q-Stancu-Beta Operators

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Abstract. In this paper we define the modified q-Stancu-Beta operators and study the weighted statistical approximation by these operators with the help of the Korovkin type approximation theorem. We also establish the rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function. Our results show that rates of convergence of our operators are at least as fast as classical Stancu-Beta operators.

1. Introduction and preliminaries

After the paper of Phillips [18] who generalized the classical Bernstein polynomials based on q-integers, many generalizations of well-known positive linear operators, based on q-integers were introduced and studied by several authors. Recently the statistical approximation properties have also been investigated for q-analogue polynomials. For instance, in [19] q-analogues of Bernstein–Kantorovich operators; in [10] q-Baskakov–Kantorovich operators; in [17] q-Szász–Mirakjan operators; in [4] and [7] q-Bleimann, Butzer and Hahn operators; in [1] and [14] q-analogue of MKZ operators and in [4] q-analogue of Stancu-Beta operators were defined and their statistical approximation properties were investigated.

In this paper, we first introduce a new modification of the operators defined by Aral and Gupta [5] and study the weighted statistical approximation properties of the modified q-Stancu-Beta operators with the help of the Korovkin type approximation theorem. We also estimate the rate of statistical convergence of the sequence of the operators to the function f.

First, we recall certain notations of q-calculus as follows. Details on q-integers can be found in [3]. For each nonnegative integer k, the q-integer $[k]_q$ is defined by

$$[k]_q := \begin{cases} \frac{(1-q^k)}{(1-q)}, & q \neq 1, \\ k, & q = 1, \end{cases} \qquad [k]_q! := \begin{cases} [k]_q [k-1]_q \ldots [1]_q, & k \geq 1, \\ 1, & k = 0, \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

The q-improper integral is defined as (see Koornwinder [12])

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(\frac{q^n}{A}) \frac{q^n}{A}, \ A > 0.$$

The q-Beta integral representations are as follows

$$B_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x,$$

where

$$(a+b)_q^n = \prod_{j=0}^{n-1} (a+q^j b), \quad \text{and} \quad K(A,t+1) = q^t K(A,t), \ A > 0.$$

2. Construction of the operators

Stancu [21] introduced Beta operators L_n of second kind in order to approximate the Lebesgue integrable functions on the interval $(0,\infty)$ as follows:

$$L_n(f;x) = \frac{1}{B(nx, n+1)} \int_0^\infty \frac{t^{nx-1}}{(1+t)nx+n+1} f(t) dt.$$

Aral and Gupta [5] introduced the q-analogue of Stancu-Beta operators as follows:

$$L_n^q(f;x) = \frac{K(A, [n]_q x)}{B_q([n]_q x, [n]_q + 1)} \int_0^{\infty/A} \frac{u^{[n]_q x - 1}}{(1+u)_q^{[n]_q x + [n]_q + 1}} f(q^{[n]_q x} u) d_q u.$$

The following theorem was given by Aral and Gupta [5]:

Theorem 2.1. *Let* $q = (q_n)$ *satisfy* $0 < q_n < 1$ *with* $\lim_{n \to \infty} q_n = 1$. *For each* $f \in C^*_{x^2}[0,\infty)$, *we have*

$$\lim_{n \to \infty} \|(L_n^{q_n}(f);.) - f\|_{x^2} = 0,$$

where $C_{x^2}[0,\infty)$ *denotes the subspace of all continuous functions on* $[0,\infty)$ *such that* $|f(x)| \leq M_f$, *and* $C^*_{x^2}[0,\infty)$ *denotes the spaces of all* $f \in C_{x^2}[0,\infty)$ *such that* $\lim_{x \to \infty} (f(x))/(1+x^2)$ *is finite. The norm on* $C^*_{x^2}[0,\infty)$ *is given by* $\|f\|_{x^2} = \sup_{x \in [0,\infty)} (|f(x)|)/(1+x^2)$.

We define modified q-Stancu-Beta operators as follows:

$$(2.1) \qquad L_n^*(f;q,x) = q \frac{K(A, [n]_q x)}{B_q([n]_q x, [n]_q + 1)} \int_0^{\infty/A} \frac{u^{[n]_q x - 1}}{(1+u)_q^{[n]_q x + [n]_q + 1}} f(q^{[n]_q x} u) d_q u$$

where $x \geq 0, 0 < q \leq 1$. It is easy to verify that if $q = 1$, these operators turn into the classical Stancu-Beta operators.

Remark 2.1. Note that $L_n^*(f;q,x) = L_n^q(f;x)$ and from the Lemma 1 of Aral and Gupta [5], we have $L_n^q(1;x) = 1, L_n^q(t;x) = x, L_n^q(t^2;x) = (([n]_q x + 1)x)/(q([n]_q - 1))$. Hence for $x \geq 0$, $0 < q \leq 1$, we have

(2.2) $$L_n^*(1;q,x) = q, L_n^*(t;q,x) = q\,x \text{ and } L_n^*(t^2;q,x) = \frac{([n]_q x + 1)x}{([n]_q - 1)}.$$

Remark 2.2. Let $q \in (0,1)$ then for $x \in [0,\infty)$, we have

$$L_n^*(t - x;x) = 0 \quad \text{and} \quad L_n^*\left((t-x)^2;x\right) = \frac{([n]_q - q[n]_q + q)x^2 + x}{([n]_q - 1)}.$$

3. Weighted statistical approximation of Korovkin type

In this section we obtain the Korovkin type weighted statistical approximation by the operators defined in (2.1). First we recall the concept of statistical convergence for sequences of real numbers which was introduced by Fast [8] and further studied by many authors.

Let $K \subseteq \mathbb{N}$ and $K_n = \{j \leq n : j \in K\}$. Then the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of the set K_n. A sequence $x = (x_j)$ of real numbers is said to be *statistically convergent* to L provided that for every $\varepsilon > 0$ the set $\{j \in \mathbb{N} : |x_j - L| \geq \varepsilon\}$ has natural density zero, i.e. for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{j \leq n : |x_j - L| \geq \varepsilon\}| = 0.$$

We denote this limit by $st - \lim_n x_n = L$. Note that every convergent sequence is statistically convergent but not conversely.

Recently the idea of statistical convergence has been used in proving some approximation theorems, in particular, Korovkin type approximation theorems [11] by various authors, e.g. [9], [14], [15] and [20] etc.; and it was shown that the statistical versions are stronger than the classical ones. Authors have used many types of classical operators and test functions to study the Korovkin type approximation theorems which further motivates to continue the study. Korovkin type approximation theory has also many useful connections, other than classical approximation theory, in other branches of mathematics (see Altomare and Campiti in [2]).

A real function ρ is called a weight function if it is continuous on $\mathbb{R}$ and $\lim_{|x| \to \infty} \rho(x) = \infty$, $\rho(x) \geq 1$ for all $x \in \mathbb{R}$. Let denote by $B_\rho(\mathbb{R})$ the weighted space of real-valued functions f defined on $\mathbb{R}$ with the property $|f(x)| \leq M_f\,\rho(x)$ for all $x \in \mathbb{R}$, where M_f is a constant depending on the function f. We also consider the weighted subspace $C_\rho(\mathbb{R})$ of $B_\rho(\mathbb{R})$ given by $C_\rho(\mathbb{R}) := \{f \in B_\rho(\mathbb{R}): f \text{ continuous on } \mathbb{R}\}$. Note that $B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are Banach spaces with $\|f\|_\rho := \sup_{x \in \mathbb{R}} |f(x)| / (\rho(x))$. In case of weight function as $\rho_0 = 1 + x^2$, $\|f\|_{\rho_0} := \sup_{x \in \mathbb{R}} |f(x)| / (1 + x^2)$.

Now, we consider a sequence $q = (q_n)$, $q_n \in (0,1)$, such that

(3.1) $$st - \lim_{n \to \infty} q_n = 1.$$

In [6], Doğru gave some examples so that (q_n) is statistically convergent to 1 but it may not convergent to 1 in the ordinary case.

Now we are ready to prove our main result as follows:

Theorem 3.1. *Let (L_n^*) be the sequence of the operators (2.1) and the sequence $q = (q_n)$ satisfies (3.1). Then for any function $f \in C_B[0,\infty)$,*

$$st - \lim_{n \to \infty} \|L_n^*(f;q_n;.) - f\|_{\rho_0} = 0.$$

Proof. Let $e_v = x^v$ where $v = 0, 1, 2$. Since $L_n^*(1;q_n,x) = q_n$. Therefore we can write

$$st - \lim_{n \to \infty} \|L_n^*(1;q_n;x) - 1\|_{\rho_0} = st - \lim_{n \to \infty} \|e_0\|_{\rho_0}|q_n - 1|$$

as

$$\|L_n^*(1;q_n;x) - 1\|_{\rho_0} = \sup_{x \in \mathbb{R}} \frac{|L_n^*(1;q_n;x) - 1|}{1 + x^2} = \|e_0\|_{\rho_0}|q_n - 1| \le |q_n - 1|.$$

By (3.1), it can be observed that

$$st - \lim_{n \to \infty} \|L_n^*(1;q_n;x) - 1\|_{\rho_0} = 0.$$

Similarly

$$\|L_n^*(t;q_n;x) - x\|_{\rho_0} = \|e_1\|_{\rho_0} \frac{[n]_q \Gamma_q([n]_q) \, \Gamma_q([n]_q)}{\Gamma_q([n]_q) \, [n]_q \Gamma_q([n]_q)} (q_n - 1)| \le |q_n - 1| = 1 - q_n.$$

For a given $\varepsilon > 0$, let us define the following sets

$$U = \{n : \|L_n^*(t;q_n;x) - x\|_{\rho_0} \ge \varepsilon\} \quad \text{and} \quad U' = \{n : 1 - q_n \ge \varepsilon\}.$$

It is obvious that $U \subset U'$ and hence

$$\delta(\{k \le n : \|L_n^*(t;q_n;x) - x\|_{\rho_0} \ge \varepsilon\}) \le \delta(\{k \le n : 1 - q_n \ge \varepsilon\}).$$

By using (3.1), we get

$$st - \lim_{n \to \infty}(1 - q_n) = 0.$$

Therefore

$$\delta(\{k \le n : 1 - q_n \ge \varepsilon\}) = 0,$$

and we have

$$st - \lim_{n \to \infty} \|L_n^*(t;q_n;x) - x\|_{\rho_0} = 0.$$

Lastly, we have

$$\|L_n^*(t^2;q_n;x) - x^2\|_{\rho_0} = \|e_2\|_{\rho_0}\left(\frac{[n]_q}{[n]_q - 1}\right) + \|e_1\|_{\rho_0}\left(\frac{1}{[n]_q - 1} - 1\right)$$

(3.2) $$\le \left|\frac{([n]_q + 1)}{([n]_q - 1)} - 1\right|.$$

Also we have that

$$\frac{([n]_{q_n} + 1)}{([n]_{q_n} - 1)} = \frac{1}{q_n^2}\left(\frac{2 + q_n}{[n-1]_{q_n} - 1} + \frac{[n+1]_{q_n}}{[n-1]_{q_n} - 1} - \frac{1 + q_n}{[n-1]_{q_n} - 1}\right).$$

Therefore by (3.2), we get

$$\|L_n^*(t^2;q_n;x) - x^2\|_{\rho_0} \le |\frac{1}{q_n^2}\left(\frac{[n+1]_{q_n}}{[n-1]_{q_n}}\right) - 1| + |\frac{1}{q_n^2}\left(\frac{2 + q_n}{[n-1]_q - 1}\right)| + |\frac{1}{q_n^2}\left(\frac{1 + q_n}{[n-1]_q - 1}\right)|.$$

Now, if we choose

$$\alpha_n = \frac{1}{q_n^2}\left(\frac{[n+1]_{q_n}}{[n-1]_{q_n}}\right) - 1, \quad \beta_n = \frac{1}{q_n^2}\left(\frac{2 + q_n}{[n-1]_q - 1}\right), \quad \gamma_n = \frac{1}{q_n^2}\left(\frac{1 + q_n}{[n-1]_q - 1}\right),$$

then by (3.1), we can write

(3.3) $$st - \lim_{n\to\infty} \alpha_n = st - \lim_{n\to\infty} \beta_n = st - \lim_{n\to\infty} \gamma_n = 0.$$

Now for given $\varepsilon > 0$, we define the following four sets

$$D = \{n : \|L_n^*(t^2; q_n; x) - x^2\|_{\rho_0} \geq \varepsilon\}, \quad D_1 = \{n : \alpha_n \geq \frac{\varepsilon}{3}\},$$

$$D_2 = \{n : \beta_n \geq \frac{\varepsilon}{3}\}, \quad D_3 = \{n : \gamma_n \geq \frac{\varepsilon}{3}\}.$$

It is obvious that $D \subseteq D_1 \cup D_2 \cup D_3$. Then we obtain

$$\delta(\{k \leq n : \|L_n^*(t^2; q_n; x) - x^2\|_{\rho_0} \geq \varepsilon\})$$

$$\leq \delta(\{k \leq n : \alpha_n \geq \frac{\varepsilon}{3}\}) + \delta(\{k \leq n : \beta_n \geq \frac{\varepsilon}{3}\}) + \delta(\{k \leq n : \gamma_n \geq \frac{\varepsilon}{3}\}).$$

Using (3.3), we get

$$st - \lim_{n\to\infty} \|L_n^*(t^2; q_n; x) - x^2\|_{\rho_0} = 0.$$

Since

$$\|L_n^*(f; q_n; x) - f\|_{\rho_0} \leq \|L_n^*(t^2; q_n; x) - x^2\|_{\rho_0} + \|L_n^*(t; q_n; x) - x\|_{\rho_0} + \|L_n^*(1; q_n; x) - 1\|_{\rho_0},$$

we get

$$st - \lim_{n\to\infty} \|L_n^*(f; q_n; x) - f\|_{\rho_0}$$

$$\leq st - \lim_{n\to\infty} \|L_n^*(t^2; q_n; x) - x^2\|_{\rho_0} + st - \lim_{n\to\infty} \|L_n^*(t; q_n; x) - x\|_{\rho_0}$$

$$+ st - \lim_{n\to\infty} \|L_n^*(1; q_n; x) - 1\|_{\rho_0},$$

which implies that

$$st - \lim_{n\to\infty} \|L_n^*(f; q_n; x) - f\|_{\rho_0} = 0. \qquad \blacksquare$$

4. Rates of statistical convergence

In this section, we give the rates of statistical convergence of the operator (2.1) by means of modulus of continuity and Lipschitz type maximal functions. The modulus of continuity for the functions $f \in C_B[0, \infty)$ is defined as

$$w(f; \delta)_{\rho_0} = \sup_{x,t \geq 0, |t-x| < \delta} \frac{|f(t) - f(x)|}{1 + x^{2+\lambda}}$$

where $w(f; \delta)_{\rho_0}$ for $\delta > 0, \lambda \geq 0$ satisfies the following conditions: for every $f \in C_B[0, \infty)$ $\lim_{\delta \to \infty} w(f; \delta)_{\rho_0} = 0$ and

(4.1) $$|f(t) - f(x)| \leq w(f; \delta)_{\rho_0} \left(\frac{|t-x|}{\delta} + 1 \right)$$

Theorem 4.1. *Let the sequence $q = (q_n)$ satisfy the condition (3.1) and $0 < q_n < 1$. Then we have*

$$|L_n^*(f; q_n; x) - f(x)| \leq w(f; \sqrt{\delta_n(x)})_{\rho_0} (1 + q_n),$$

where

(4.2) $$\delta_n(x) = \|e_2\|_{\rho_0} q_n \left(\frac{[n]_q}{[n]_q - 1} - \frac{q_n[n]_{q_n}}{[n]_{q_n} - 1} + \frac{q_n}{[n]_{q_n} - 1} \right) + \|e_1\|_{\rho_0} \frac{q_n}{[n]_{q_n} - 1}.$$

Proof. Since $| L_n^*(f;q_n;x) - f | \leq L_n^*(|f(t) - f(x)|;q_n;x)$, by (4.1) we get

$$| L_n^*(f;q_n;x) - f(x) | \leq w(f;\delta)_{\rho_0} \left(\{L_n^*(1;q_n;x) + \frac{1}{\delta} L_n^*(|t-x|;q_n;x)\} \right).$$

Using Cauchy-Schwartz inequality, we have

$$| L_n^*(f;q_n;x) - f(x) | \leq w(f;\delta_n) \left(q_n + \frac{1}{\delta_n} [(L_n^*(t-x)^2;q_n;x)]^{\frac{1}{2}} [L_n^*(1;q_n;x)]^{\frac{1}{2}} \right)$$

$$\leq w(f;\delta_n)_{\rho_0} (q_n + \frac{1}{\delta_n} \{(\|e_2\|_{\rho_0}) \, q_n \left(\frac{[n]_q}{[n]_q - 1} - \frac{q_n [n]_{q_n}}{[n]_{q_n} - 1} + \frac{q_n}{[n]_{q_n} - 1} \right)$$

$$+ (\|e_1\|_{\rho_0}) \frac{q_n}{[n]_{q_n} - 1} \}^{\frac{1}{2}}.$$

By choosing δ_n as in (4.2), we get the desired result. ∎

Now we will give an estimate concerning the rate of approximation by means of Lipschitz type maximal functions. In [13], Lenze introduced a Lipschitz type maximal function as

$$\tilde{f}_\alpha(x) = \sup_{t>0,t \neq x} \frac{| f(t) - f(x) |}{|t-x|^\alpha}.$$

In [4], the Lipschitz type maximal function space on $E \subset [0,\infty)$ is defined as follows

$$\tilde{W}_\alpha = \{f : \sup(1+x)^\alpha \, \tilde{f}_\alpha(x) \leq M \frac{1}{(1+y)^\alpha}; x \geq 0 \text{ and } y \in E\},$$

where f is bounded and continuous function on $[0,\infty)$, M is a positive constant and $0 < \alpha \leq 1$.

Theorem 4.2. *If L_n^* be defined by (2.1), then for all $f \in \tilde{W}_{\alpha,E}$*

$$(4.3) \qquad | L_n^*(f;q_n;x) - f(x) | \leq M(\eta_n^{\frac{\alpha}{2}} q_n^{\frac{2-\alpha}{2}} + q_n \, d(x,E)),$$

where

$$(4.4) \qquad \eta_n = \|e_2\|_{\rho_0} \left(\frac{[n]_q}{[n]_q - 1} - \frac{q_n [n]_{q_n}}{[n]_{q_n} - 1} + \frac{q_n}{[n]_{q_n} - 1} \right) + \|e_1\|_{\rho_0} \frac{1}{[n]_{q_n} - 1}.$$

Proof. Let $x \geq 0$, $(x,x_0) \in [0,\infty) \times E$. Then we have

$$| f - f(x) | \leq | f - f(x_0) | + | f(x_0) - f(x) |.$$

Since L_n^* is a positive and linear operator, $f \in \tilde{W}_{\alpha,E}$ and using the above inequality

$$| L_n^*(f;q_n;x) - f(x) | \leq | L_n^*(| f - f(x_0) |;q_n;x) - f(x) | + | L_n^*(| f(x_0) - f(x) | L_n^*(1;q_n;x)$$

$$(4.5) \qquad \leq M \left(L_n^*(|t-x_0|^\alpha;q_n;x) + |x-x_0|^\alpha L_n^*(1;q_n;x) \right).$$

Therefore we have

$$L_n^* \left(|t-x_0|^\alpha;q_n;x \right) \leq L_n^*(|t-x|^\alpha;q_n;x) + |x-x_0|^\alpha L_n^*(1;q_n;x).$$

By using the Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$L_n^*((t-x)^\alpha;q_n;x) \leq L_n^* \left((t-x)^2;q_n;x \right)^\alpha 2(L_n^*(1;q_n;x))^{\frac{2-\alpha}{2}} + |x-x_0|^\alpha L_n^*(1;q_n;x)$$

$$= \eta_n^{\frac{\alpha}{2}} q_n^{\frac{2-\alpha}{2}} + |x-x_0|^\alpha q_n.$$

Substituting this in (4.5), we get (4.3). ∎

5. Concluding remarks

(1) Note that in condition (3.3),

$$st - \lim_{n\to\infty} \delta_n = 0.$$

By (4.1) we have

$$st - \lim_{n\to\infty} w(f;\delta_n)_{\rho_0} = 0,$$

which gives us the pointwise rate of statistical convergence of the operator $L_n^*(f;q_n;x)$ to $f(x)$. From the definition of the q-calculus, it can be proved that

$$\sup_{x\geq 0} \delta_n(x) \leq q_n \left(\frac{[n]_q}{[n]_q - 1} - \frac{q_n[n]_{q_n}}{[n]_{q_n} - 1} + \frac{q_n}{[n]_{q_n} - 1} + \frac{1}{[n]_{q_n} - 1} \right).$$

In classical case for $q = 1$, we have

$$\sup \delta_n(x) \leq \frac{1}{n-1} = O(\frac{1}{n})$$

Thus, for every choice of q_n, the rate of convergence of (2.1) to the function f is better than the Stancu-Beta operators.

(2) If we take $E = [0,\infty)$ in Theorem 4.2, since $d(x,E) = 0$, then we obtain the following result: For every $f \in \tilde{W}_{\alpha,[0,\infty)}$

$$|L_n^*(f;q_n;x) - f(x)|_{q_n} \leq M\eta_n^{\frac{\alpha}{2}} q_n^{\frac{2-\alpha}{2}}$$

where η_n is defined as in (4.4).

(3) By using (3.3), It is easy to verify that

$$st - \lim_{n\to\infty} \eta_n = 0.$$

That is, the rate of statistical convergence of operators (2.1) to the function f are estimated by means of Lipschitz type maximal functions.

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