BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(3) (2013), 691–697*

Furuta Type Inequalities Dealing with Pedersen-Takesaki Type Operator Equations

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Abstract. In this paper, we study the existence of solutions of some Pedersen-Takesaki type operator equations via Furuta type inequalities. Firstly, we investigate characterizations of operator order $A \geqslant B$ and chaotic operator order $\log A \geqslant \log B$ for positive definite operators A and B in terms of Pedersen-Takesaki type operator equations, which extend the related results before. Finally, we give some applications of complete form of Furuta inequality to Pedersen-Takesaki type operator equations and prove the related characterizations of the solutions.

2010 Mathematics Subject Classification: 47A63

Keywords and phrases: Furuta inequality, operator order, chaotic operator order, operator equation, complete form of Furuta inequality.

1. Introduction

A bounded linear operator A on a Hilbert space $\mathscr{H}$ is said to be positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. $A \geqslant O$ and $A > O$ mean a positive semidefinite operator and a positive definite operator, respectively. If $X > O$, $\log X$ is defined as $\log X = lim_{a\to+0}(X^a - I)/a$. We call the relation $\log A \geqslant \log B$ chaotic operator order (in symbol: $A \gg B$). The celebrated Löwner-Heinz inequality (see [7] and [4]): $A \geqslant B \geqslant O \Rightarrow A^\alpha \geqslant B^\alpha$ for any $\alpha \in [0,1]$, assures that $A \geqslant B > O \Rightarrow A \gg B$.

In 1987 and 1995, Furuta showed the following operator inequalities:

Theorem 1.1. (Furuta Inequality) [2]. *If $A \geqslant B \geqslant O$, then for each $r \geqslant 0$,*

$$\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}, \tag{1.1}$$

$$\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \tag{1.2}$$

hold for $p \geqslant 0$, $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

Communicated by Mohammad Sal Moslehian.
Received: August 15, 2011; *Revised:* November 8, 2011.

Theorem 1.2. (Grand Furuta Inequality) [3]. *If $A \geqslant B \geqslant O$ with $A > O$, then for $t \in [0,1]$ and $p \geqslant 1$,*

$$A^{1-t+r} \geqslant \left(A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}}\right)^{s}A^{\frac{r}{2}}\right)^{\frac{1-t+r}{(p-t)s+r}} \tag{1.3}$$

for $s \geqslant 1$ and $r \geqslant t$.

In 1996, Tanahashi proved the conditions for p and q in Furuta inequality are the best possible if $r \geqslant 0$ in [9]; In 2000, he proved the outer exponent value of grand Furuta inequality is the best possible in [10]. In 2008, Yuan *et al.* proved a new operator inequality, which is called complete form of Furuta inequality:

Theorem 1.3. (Complete Form of Furuta Inequality) [13]. *Let $A \geqslant B \geqslant O$, $r \geqslant 0$, $p > p_0 > 0$, and $s = \min\{p, 2p_0 + \min\{1,r\}\}$. Then*

$$\left(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\right)^{\frac{s+r}{p_0+r}} \geqslant \left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)^{\frac{s+r}{p+r}}. \tag{1.4}$$

Yuan *et al.* also proved the optimality of the outer exponent of complete form of Furuta inequality under the condition of $\{2p_0 + \min\{1,r\}\} \geq p$ or $r \geq 1$ with $2p_0 + 1 < p$ in [13]. There were many related studies on complete form of Furuta inequality, such as [14] and [15].

Recently, some kinds of operator equations, which extend the form of Pedersen-Takesaki operator equation $THT = K$ (see [8]), have been shown and given deep discussion via Furuta inequality and grand Furuta inequality (see [5, 11]). In [5], Lin proved the following two theorems, which the characterizations of operator order $A \geqslant B > O$ and chaotic operator order $A \gg B$ for positive definite operator A and B in terms of operator equations are discussed, respectively:

Theorem 1.4. [5]. *For $r \geqslant 0$ and a nonnegative integer $n \geqslant 0$ such that $(1+r)(n+1) = p+r$ (so, $p \geqslant 1$), the following assertions are equivalent.*

- (A1) $A \geqslant B > O$;
- (A2) $A^{1+r} \geqslant (A^{r/2}B^{p}A^{r/2})^{1/(n+1)}$;
- (A3) $(B^{r/2}A^{p}B^{r/2})^{1/(n+1)} \geqslant B^{1+r}$;
- (A4) *There exists a unique $S > O$ with $\|S\| \leqslant 1$ such that* $B^{p} = A^{1/2}S(A^{1+r}S)^{n}A^{1/2} = A^{1/2}(SA^{1+r})^{n}SA^{1/2}$;
- (A5) *There exists a unique $S > O$ with $\|S\| \leqslant 1$ such that* $A^{p} = B^{1/2}S^{-1}(B^{1+r}S^{-1})^{n}B^{1/2} = B^{1/2}(S^{-1}B^{1+r})^{n}S^{-1}B^{1/2}$.

Theorem 1.5. [5]. *For $p, r > 0$ and any integer $n \geqslant 1$ such that $r(n+1) = p+r$, the following assertions are equivalent.*

- (B1) $A \gg B$;
- (B2) $A^{r} \geqslant (A^{r/2}B^{p}A^{r/2})^{1/(n+1)}$;
- (B3) $(B^{r/2}A^{p}B^{r/2})^{1/(n+1)} \geqslant B^{r}$;
- (B4) *There exists a unique $T > O$ with $\|T\| \leqslant 1$ such that* $B^{p} = T(A^{r}T)^{n} = (TA^{r})^{n}T$;
- (B5) *There exists a unique $T > O$ with $\|T\| \leqslant 1$ such that* $A^{p} = T^{-1}(B^{r}T^{-1})^{n} = (T^{-1}B^{r})^{n}T^{-1}$.

In this paper, we will show generalized characterizations in terms of Pedersen-Takesaki type operator equations due to Theorem 1.4 and Theorem 1.5. Then, we will give some applications of complete form of Furuta inequality to Pedersen-Takesaki type operator equations.

In order to prove our results, let's recall two useful theorems:

Theorem 1.6. [12]. *For $A,B>O$, $A\geqslant B$ if and only if*

$$A^{1+t+r}\geqslant\left(A^{\frac{r}{2}}\left(A^{\frac{t}{2}}B^pA^{\frac{t}{2}}\right)^sA^{\frac{r}{2}}\right)^{\frac{1+t+r}{(p+t)s+r}}$$

holds for $p\geqslant 1$, $t\geqslant 0$, $r\geqslant 0$ and $s\geqslant(1+t)/(p+t)$.

Theorem 1.7. [12]. *For $A,B>O$, $\log A\geqslant\log B$ if and only if*

$$A^{t+r}\geqslant\left(A^{\frac{r}{2}}\left(A^{\frac{t}{2}}B^pA^{\frac{t}{2}}\right)^sA^{\frac{r}{2}}\right)^{\frac{t+r}{(p+t)s+r}}$$

holds for $p>0$, $t\geqslant 0$, $r\geqslant 0$ and $s\geqslant t/(p+t)$.

2. The characterizations of operator order $A\geq B>O$ and chaotic operator order $A\gg B$

In this section, we will show further development of characterizations of operator order $A\geqslant B>O$ and chaotic operator order $\log A\geqslant\log B$ for positive definite operators A and B in terms of operator equations due to Lin's results in [5].

Throughout this section we assume that A,B are positive definite operators.

Theorem 2.1. *For $r\geqslant 0$, $t\geqslant 0$, $p\geqslant 1$ and a nonnegative n such that $(p+t)s+r=(n+1)\cdot(1+t+r)$ (so, $s\geqslant(1+t)/(p+t)$), the following assertions are equivalent.*

(C1) $A\geqslant B>O$;

(C2) $A^{1+t+r}\geqslant(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2})^{1/(n+1)}$;

(C3) $(B^{r/2}(B^{t/2}A^pB^{t/2})^sB^{r/2})^{1/(n+1)}\geqslant B^{1+t+r}$;

(C4) *There exists a unique $S>O$ with $\|S\|\leqslant 1$ such that*

$$B^p=A^{-\frac{t}{2}}\left(A^{\frac{1+t}{2}}(SA^{1+t+r})^nSA^{\frac{1+t}{2}}\right)^{\frac{1}{s}}A^{-\frac{t}{2}}=A^{-\frac{t}{2}}\left(A^{\frac{1+t}{2}}S(A^{1+t+r}S)^nA^{\frac{1+t}{2}}\right)^{\frac{1}{s}}A^{-\frac{t}{2}};$$

(C5) *There exists a unique $S>O$ with $\|S\|\leqslant 1$ such that*

$$\begin{aligned}A^p&=B^{-\frac{t}{2}}\left(B^{\frac{1+t}{2}}(S^{-1}B^{1+t+r})^nS^{-1}B^{\frac{1+t}{2}}\right)^{\frac{1}{s}}B^{-\frac{t}{2}}\\&=B^{-\frac{t}{2}}\left(B^{\frac{1+t}{2}}S^{-1}(B^{1+t+r}S^{-1})^nB^{\frac{1+t}{2}}\right)^{\frac{1}{s}}B^{-\frac{t}{2}}.\end{aligned}$$

Proof. (C1) $\Rightarrow$ (C2) is obvious by Theorem 1.6.
(C2) $\Rightarrow$ (C1). Take $n=0$, $t=0$, $r=0$ in $(p+t)s+r=(n+1)\cdot(1+t+r)$ and (C2), then $ps=1$ and $A\geqslant B$.
(C2) $\Leftrightarrow$ (C3) holds by $A\geqslant B>O\Leftrightarrow B^{-1}\geqslant A^{-1}>O$.
(C2) $\Rightarrow$ (C4). According to Douglas's majorization and factorization theorem in [1], $TT^*\leq\lambda HH^*$ for some $\lambda\geq 0\Leftrightarrow$ There exists a bounded operator X such that $T=HX$, where $\|X\|^2=inf\{\mu:TT^*\leq\mu HH^*\}$.

By (C2), there exists C with $\|C\|\leqslant 1$ such that

$$\left(A^{\frac{r}{2}}\left(A^{\frac{t}{2}}B^pA^{\frac{t}{2}}\right)^sA^{\frac{r}{2}}\right)^{\frac{1}{2(n+1)}}=A^{\frac{1+t+r}{2}}C=C^*A^{\frac{1+t+r}{2}}.$$

Taking $S=CC^*$, according to the fact that C is determined by A and B with $\|C\|\leqslant 1$, we have S is unique with $\|S\|\leqslant 1$, and $(A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2})^{1/(n+1)}=A^{(1+t+r)/2}SA^{(1+t+r)/2}$. Therefore, the following equalities hold.

$$A^{\frac{r}{2}}\big(A^{\frac{t}{2}}B^pA^{\frac{t}{2}}\big)^sA^{\frac{r}{2}}=\big(A^{\frac{1+t+r}{2}}SA^{\frac{1+t+r}{2}}\big)^{n+1}=A^{\frac{1+t+r}{2}}(SA^{1+t+r})^nSA^{\frac{1+t+r}{2}}; \tag{2.1}$$

$$A^{\frac{r}{2}}\big(A^{\frac{t}{2}}B^pA^{\frac{t}{2}}\big)^sA^{\frac{r}{2}}=\big(A^{\frac{1+t+r}{2}}SA^{\frac{1+t+r}{2}}\big)^{n+1}=A^{\frac{1+t+r}{2}}S(A^{1+t+r}S)^nA^{\frac{1+t+r}{2}}. \tag{2.2}$$

By (2.1) and (2.2), $(A^{t/2}B^pA^{t/2})^s=A^{(1+t)/2}(SA^{1+t+r})^nSA^{(1+t)/2}=A^{(1+t)/2}S(A^{1+t+r}S)^nA^{(1+t)/2}$, then (C4) is obtained.
(C4) $\Rightarrow$ (C2).

$$\begin{aligned}
&\big(A^{\frac{r}{2}}\big(A^{\frac{t}{2}}B^pA^{\frac{t}{2}}\big)^sA^{\frac{r}{2}}\big)^{\frac{1}{n+1}}\\
&=\big(A^{\frac{r}{2}}\big(A^{\frac{t}{2}}A^{-\frac{t}{2}}\big(A^{\frac{1+t}{2}}(SA^{1+t+r})^nSA^{\frac{1+t}{2}}\big)^{1/s}A^{-\frac{t}{2}}A^{\frac{t}{2}}\big)^sA^{\frac{r}{2}}\big)^{\frac{1}{n+1}}\\
&=\big(A^{\frac{1+t+r}{2}}(SA^{1+t+r})^nSA^{\frac{1+t+r}{2}}\big)^{\frac{1}{n+1}}=\big(\big(A^{\frac{1+t+r}{2}}SA^{\frac{1+t+r}{2}}\big)^{n+1}\big)^{\frac{1}{n+1}}\\
&=A^{\frac{1+t+r}{2}}SA^{\frac{1+t+r}{2}}\leqslant A^{1+t+r}.
\end{aligned}$$

The first equality is due to the equation of (C4), and the last inequality is due to

$$O<S\leqslant\|S\|I\leqslant I.$$

(C3) $\Rightarrow$ (C5). By (C3), we have $B^{-(1+t+r)}\geqslant(B^{-r/2}(B^{-t/2}A^{-p}B^{-t/2})^sB^{-r/2})^{1/(n+1)}$. According to Douglas's majorization and factorization theorem, there exists C such that $\|C\|\leqslant 1$, and $(B^{-r/2}(B^{-t/2}A^{-p}B^{-t/2})^sB^{-r/2})^{1/(2(n+1))}=B^{-(1+t+r)/2}C=C^*B^{-(1+t+r)/2}$. Taking $S=CC^*$, because $\|C\|\leqslant 1$ and C is uniquely determined by A and B, we have $\|S\|\leqslant 1$ and S is unique, and $(B^{-r/2}(B^{-t/2}A^{-p}B^{-t/2})^sB^{-r/2})^{\frac{1}{n+1}}=B^{-(1+t+r)/2}SB^{-(1+t+r)/2}$. Therefore, the following equality holds.

$$\begin{aligned}
&B^{\frac{r}{2}}\big(B^{\frac{t}{2}}A^pB^{\frac{t}{2}}\big)^sB^{\frac{r}{2}}\\
&=\big(B^{-\frac{r}{2}}\big(B^{-\frac{t}{2}}A^{-p}B^{-\frac{t}{2}}\big)^sB^{-\frac{r}{2}}\big)^{-1}=\big(B^{-\frac{1+t+r}{2}}SB^{-\frac{1+t+r}{2}}\big)^{-(n+1)}\\
&=\big(B^{\frac{1+t+r}{2}}S^{-1}B^{\frac{1+t+r}{2}}\big)^{n+1}=B^{\frac{1+t+r}{2}}S^{-1}(B^{1+t+r}S^{-1})^nB^{\frac{1+t+r}{2}}\\
&=B^{\frac{1+t+r}{2}}(S^{-1}B^{1+t+r})^nS^{-1}B^{\frac{1+t+r}{2}}.
\end{aligned}$$

Then (C5) is obtained immediately.
(C5) $\Rightarrow$ (C3).

$$\begin{aligned}
&\big(B^{\frac{r}{2}}\big(B^{\frac{t}{2}}A^pB^{\frac{t}{2}}\big)^sB^{\frac{r}{2}}\big)^{\frac{1}{n+1}}\\
&=\big(B^{\frac{r}{2}}\big(B^{\frac{t}{2}}B^{-\frac{t}{2}}\big(B^{\frac{1+t}{2}}S^{-1}(B^{1+t+r}S^{-1})^nB^{\frac{1+t}{2}}\big)^{\frac{1}{s}}B^{-\frac{t}{2}}B^{\frac{t}{2}}\big)^sB^{\frac{r}{2}}\big)^{\frac{1}{n+1}}\\
&=\big(B^{\frac{1+t+r}{2}}S^{-1}(B^{1+t+r}S^{-1})^nB^{\frac{1+t+r}{2}}\big)^{\frac{1}{n+1}}=\big(\big(B^{\frac{1+t+r}{2}}S^{-1}B^{\frac{1+t+r}{2}}\big)^{n+1}\big)^{\frac{1}{n+1}}\\
&=B^{\frac{1+t+r}{2}}S^{-1}B^{\frac{1+t+r}{2}}\geqslant B^{1+t+r}.
\end{aligned}$$

The first equality is due to (C5), the last inequality is due to $O<S\leqslant\|S\|I\leqslant I\Rightarrow S^{-1}\geqslant I$. ∎

Remark 2.1. The special case $r=0$ and $s=1$ of Theorem 2.1 is just Theorem 1.4. (A2), (A3), (A4) and (A5) in Theorem 1.4 are extended to (C2), (C3), (C4) and (C5) in Theorem 2.1, respectively.

Theorem 2.2. *For $p,\ r>0$, $t\geqslant 0$ and any integer $n\geqslant 1$ such that $(p+t)s+r=(n+1)\cdot(t+r)$, the following assertions are equivalent.*

(D1) $A\gg B$;

(D2) $A^{t+r}\geqslant (A^{r/2}(A^{t/2}B^pA^{t/2})^sA^{r/2})^{1/(n+1)}$;

(D3) $(B^{r/2}(B^{t/2}A^pB^{t/2})^sB^{r/2})^{1/(n+1)}\geqslant B^{t+r}$;

(D4) *There exists a unique $T>O$ with $\|T\|\leqslant 1$ such that*

$$B^p=A^{-\frac{t}{2}}\big(A^{\frac{t}{2}}(TA^{t+r})^nTA^{\frac{t}{2}}\big)^{\frac{1}{s}}A^{-\frac{t}{2}}=A^{-\frac{t}{2}}\big(A^{\frac{t}{2}}T(A^{t+r}T)^nA^{\frac{t}{2}}\big)^{\frac{1}{s}}A^{-\frac{t}{2}};$$

(D5) *There exists a unique $T>O$ with $\|T\|\leqslant 1$ such that*

$$A^p=B^{-\frac{t}{2}}\big(B^{\frac{t}{2}}(T^{-1}B^{t+r})^nT^{-1}B^{\frac{t}{2}}\big)^{\frac{1}{s}}B^{-\frac{t}{2}}=B^{-\frac{t}{2}}\big(B^{\frac{t}{2}}T^{-1}(B^{t+r}T^{-1})^nB^{\frac{t}{2}}\big)^{\frac{1}{s}}B^{-\frac{t}{2}}.$$

Proof. (D1) $\Rightarrow$ (D2) is obvious by Theorem 1.7.
(D2) $\Rightarrow$ (D1). Take $t=0$, $r=1/n$, then $ps=1$ and $A^{1/n}\geqslant (A^{1/(2n)}BA^{1/(2n)})^{1/(n+1)}$. Taking logarithm of both sides and refining, we have $\log A\geqslant n/(n+1)\,\log\,(A^{1/(2n)}BA^{1/(2n)})$. If we take $n\to\infty$, then $\log A\geqslant \log B$ is obtained.
(D2) $\Leftrightarrow$ (D3) is due to the fact that $A\gg B\Leftrightarrow B^{-1}\gg A^{-1}$.
The proofs of (D2) $\Leftrightarrow$ (D4) and (D3) $\Leftrightarrow$ (D5) are similar to the proofs of (C2) $\Leftrightarrow$ (C4) and (C3) $\Leftrightarrow$ (C5), so we omit them here. ∎

Remark 2.2. The special case $t=0$ and $s=1$ of Theorem 2.2 is just Theorem 1.5. (B2), (B3), (B4) and (B5) in Theorem 1.5 are extended to (D2), (D3), (D4) and (D5) in Theorem 2.2, respectively.

3. Applications of complete form of Furuta inequality to operator Equations

In this section, we will introduce some applications of complete form of Furuta inequality to operator equations. Several kinds of operator equations will be researched and related characterizations of solutions will be proved.

Throughout this section we assume A and B are positive definite operators.

Theorem 3.1. *If $1\geqslant r\geqslant 0$, $p>p_0>0$ with $p\geq 2p_0+r$, for a nonnegative integer n such that $p+r=(n+1)((2p_0+r)+r)$, the following assertions are equivalent.*

(3-1) $A\geqslant B>O$;

(3-2) $(A^{r/2}B^{p_0}A^{r/2})^2\geqslant (A^{r/2}B^pA^{r/2})^{1/(n+1)}$;

(3-3) *There exists a unique $S>O$ with $\|S\|\leqslant 1$ such that*

$$B^p=B^{p_0}A^{\frac{r}{2}}S\big(A^{\frac{r}{2}}B^{p_0}A^rB^{p_0}A^{\frac{r}{2}}S\big)^nA^{\frac{r}{2}}B^{p_0}=B^{p_0}A^{\frac{r}{2}}\big(SA^{\frac{r}{2}}B^{p_0}A^rB^{p_0}A^{\frac{r}{2}}\big)^nSA^{\frac{r}{2}}B^{p_0}.$$

Proof. (3-1) $\Rightarrow$ (3-2) is obvious by Theorem 1.3.
(3-2) $\Rightarrow$ (3-1). Take $p_0=1$, $r=1$, $n=0$ in $p+r=(n+1)((2p_0+r)+r)$ and (3-2), then $p=3$ and $(A^{1/2}BA^{1/2})^2\geq A^{1/2}B^3A^{1/2}$. Thus, $A\geq B$ is obtained.
(3-2) $\Rightarrow$ (3-3). According to Douglas's majorization and factorization theorem, there exists an operator C with $\|C\|\leqslant 1$ such that $(A^{r/2}B^pA^{r/2})^{1/(2(n+1))}=(A^{r/2}B^{p_0}A^{r/2})C=C^*(A^{r/2}B^{p_0}A^{r/2})$. Taking $S=CC^*$, then $\|S\|=\|CC^*\|\leqslant 1$ and

$$\big(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\big)^{\frac{1}{n+1}}=\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)S\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big). \tag{3.1}$$

By (3.1), S is unique and the following equality holds.

$$A^{\frac{r}{2}}B^pA^{\frac{r}{2}} = \big(\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)S\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)\big)^{n+1} = A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}S\big(\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^2S\big)^nA^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}$$
$$= A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big(S\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^2\big)^nSA^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}.$$

By above equalities, we can obtain (3.3).

(3-3) $\Rightarrow$ (3-2).

$$\big(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\big)^{\frac{1}{n+1}} = \big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}S\big(A^{\frac{r}{2}}B^{p_0}A^rB^{p_0}A^{\frac{r}{2}}S\big)^nA^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^{\frac{1}{n+1}}$$
$$= \big(\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}SA^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^{n+1}\big)^{\frac{1}{n+1}} = \big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)S\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)$$
$$\leqslant \big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^2.$$

The first equality is due to (3-3), and the inequality is due to $O < S \leqslant \|S\|I \leqslant I$. ∎

Theorem 3.2. *If $r \geqslant 1$, $p > p_0 > 0$ with $p \geq 2p_0+1$, for a nonnegative integer n such that $p+r = (n+1)((2p_0+1)+r)$, the following assertions are equivalent.*

(3-1) $A \geqslant B > O$;

(3-4) $(A^{r/2}B^{p_0}A^{r/2})^{(2p_0+1+r)/(p_0+r)} \geqslant (A^{r/2}B^pA^{r/2})^{1/(n+1)}$;

(3-5) *There exists a unique $S > O$ with $\|S\| \leqslant 1$ such that*

$$B^p = A^{-\frac{r}{2}}\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^{\frac{2p_0+1+r}{2p_0+2r}}S\Big(\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^{\frac{2p_0+1+r}{p_0+r}}S\Big)^n\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^{\frac{2p_0+1+r}{2p_0+2r}}A^{-\frac{r}{2}}$$
$$= A^{-\frac{r}{2}}\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^{\frac{2p_0+1+r}{2p_0+2r}}\Big(S\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^{\frac{2p_0+1+r}{p_0+r}}\Big)^nS\big(A^{\frac{r}{2}}B^{p_0}A^{\frac{r}{2}}\big)^{\frac{2p_0+1+r}{2p_0+2r}}A^{-\frac{r}{2}}.$$

Proof. (3-1) $\Rightarrow$ (3-4) is obvious by Theorem 1.3.

(3-4) $\Rightarrow$ (3-1). Take $n = 0$, $p_0 = 1$, $r = 1$ in $p+r = (n+1)((2p_0+1)+r)$ and (3-4), then $p = 3$ and $(A^{1/2}BA^{1/2})^2 \geq A^{1/2}B^3A^{1/2}$. Thus, $A \geq B$ is obtained.

The proof of (3-4) $\Leftrightarrow$ (3-5) is similar to the proof of (3-2) $\Leftrightarrow$ (3-3) in Theorem 3.1, so we omit it here. ∎

Theorem 3.3. *If $r \geqslant 0$, $2p_0+\min\{1,r\} \geqslant p > p_0 > 0$, for a positive integer n such that $n(p+r) = (n+1)(p_0+r)$, the following assertions are equivalent.*

(3-1) $A \geqslant B > O$;

(3-6) $(A^{r/2}B^{p_0}A^{r/2})^{1+1/n} \geqslant A^{r/2}B^pA^{r/2}$;

(3-7) *There exists a unique $S > O$ with $\|S\| \leqslant 1$ such that* $(A^{r/2}B^{p_0}A^{r/2})^{n+1} = ((A^{r/2}B^pA^{r/2})^{1/2}S^{-1}(A^{r/2}B^pA^{r/2})^{1/2})^n$.

Proof. (3-1) $\Rightarrow$ (3-6) is obvious by Theorem 1.3.

(3-6) $\Rightarrow$ (3-1). Take $n = 1$, $p_0 = 1$, $r = 1$ in $n(p+r) = (n+1)(p_0+r)$ and (3-6), then $p = 3$ and $(A^{1/2}BA^{1/2})^2 \geq A^{1/2}B^3A^{1/2}$. Thus, $A \geq B$ is obtained.

(3-6) $\Rightarrow$ (3-7). By (3-6) we have $A^{-r/2}B^{-p}A^{-r/2} \geqslant (A^{-r/2}B^{-p_0}A^{-r/2})^{(n+1)/n}$. According to Douglas's majorization and factorization theorem, there exists an operator C with $\|C\| \leqslant 1$ such that

$$\big(A^{-\frac{r}{2}}B^{-p_0}A^{-\frac{r}{2}}\big)^{\frac{n+1}{2n}} = \big(A^{-\frac{r}{2}}B^{-p}A^{-\frac{r}{2}}\big)^{\frac{1}{2}}C = C^*\big(A^{-\frac{r}{2}}B^{-p}A^{-\frac{r}{2}}\big)^{\frac{1}{2}}.$$

Taking $S = CC^*$, then $S > O$, $\|S\| \leqslant 1$ and the following equality holds.

$$\big(A^{-\frac{r}{2}}B^{-p_0}A^{-\frac{r}{2}}\big)^{\frac{n+1}{n}} = \big(A^{-\frac{r}{2}}B^{-p}A^{-\frac{r}{2}}\big)^{\frac{1}{2}}S\big(A^{-\frac{r}{2}}B^{-p}A^{-\frac{r}{2}}\big)^{\frac{1}{2}}. \tag{3.2}$$

By (3.2), S is unique and $(A^{r/2}B^{p_0}A^{r/2})^{n+1} = ((A^{r/2}B^pA^{r/2})^{1/2}S^{-1}(A^{r/2}B^pA^{r/2})^{1/2})^n$.
(3-7) $\Rightarrow$ (3-6). Because of (3-7) and the fact that $S > O$ with $\|S\| \leqslant 1 \Rightarrow S^{-1} \geqslant I$, we have $(A^{r/2}B^{p_0}A^{r/2})^{1+1/n} = (A^{r/2}B^pA^{r/2})^{1/2}S^{-1}(A^{r/2}B^pA^{r/2})^{1/2} \geqslant A^{r/2}B^pA^{r/2}$. ∎

The characterizations of the operator inequality $A \geq B \geq C$ are discussed by Lin and Cho [6].

Acknowledgement. We would like to express our cordial appreciation to the referee for giving useful and valuable comments and suggestions for this paper. The referee kindly informed the authors of the existence of [6]. This work is supported by National Natural Science Fund of China (10771011 and 11171013).

References

[1] R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.* **17** (1966), 413–415.
[2] T. Furuta, $A \geq B \geq 0$ assures $(B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, *Proc. Amer. Math. Soc.* **101** (1987), no. 1, 85–88.
[3] T. Furuta, Extension of the Furuta inequality and Ando-Hiai log-majorization, *Linear Algebra Appl.* **219** (1995), 139–155.
[4] E. Heinz, Beiträge zur Stїungstheorie der Spektrallegung, *Math. Ann.* **123** (1951), 415–438.
[5] C.-S. Lin, On operator order and chaotic operator order for two operators, *Linear Algebra Appl.* **425** (2007), no. 1, 1–6.
[6] C.-S. Lin and Y. J. Cho, Characterizations of the operator inequality $A \geq B \geq C$, *Math. Inequal. Appl.* **14** (2011), no. 3, 575–580.
[7] K. Löwner, Über monotone Matrixfunktionen, *Math. Z.* **38** (1934), no. 1, 177–216.
[8] G. K. Pedersen and M. Takesaki, The operator equation $THT = K$, *Proc. Amer. Math. Soc.* **36** (1972), 311–312.
[9] K. Tanahashi, Best possibility of the Furuta inequality, *Proc. Amer. Math. Soc.* **124** (1996), no. 1, 141–146.
[10] K. Tanahashi, The best possibility of the grand Furuta inequality, *Proc. Amer. Math. Soc.* **128** (2000), no. 2, 511–519.
[11] J. Yuan and Z. Gao, The operator equation $K^p = H^{\frac{\delta}{2}}T^{\frac{1}{2}}(T^{\frac{1}{2}}H^{\delta+r}T^{\frac{1}{2}})^{\frac{p-\delta}{\delta+r}}T^{\frac{1}{2}}H^{\frac{\delta}{2}}$ and its applications, *J. Math. Anal. Appl.* **341** (2008), no. 2, 870–875.
[12] J. Yuan and Z. Gao, Classified construction of generalized Furuta type operator functions, *Math. Inequal. Appl.* **11** (2008), no. 2, 189–202.
[13] J. Yuan and Z. Gao, Complete form of Furuta inequality, *Proc. Amer. Math. Soc.* **136** (2008), no. 8, 2859–2867.
[14] J. Yuan, Furuta inequality and q-hyponormal operators, *Oper. Matrices* **4** (2010), no. 3, 405–415.
[15] J. Yuan and G. Ji, Extensions of Kadison's inequality on positive linear maps, *Linear Algebra Appl.* **436** (2012), 747–752.