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# A Generalization of the $G'/G$-Expansion Method and Its Application to Jimbo–Miwa Equation

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**Abstract.** A generalization of the $G'/G$-expansion method combined with Liu's theorem is proposed to construct exact solutions of the (3+1)-dimensional Jimbo–Miwa equation. As a result, more general travelling wave solutions with parameters are obtained including hyperbolic function solutions, trigonometric function solutions and rational solutions. Some of the obtained hyperbolic function solutions and trigonometric function solutions contain an explicit external linear function of $\{x,y,z,t\}$. It is shown that the $G'/G$-expansion method with the help of symbolic computation may provide us with a straightforward, effective and alternative mathematical tool for solving nonlinear evolution equations in mathematical physics.

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## 1. Introduction

The investigation of exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena and gradually becomes one of the most important and significant tasks. In the past several decades, both mathematicians and physicists have made many significant work in this direction and presented some effective methods for obtaining exact solutions of NLEEs, such as the inverse scattering method [1], Hirota's bilinear method [2], Darboux transformation [3], Painlevé expansion [4], sine-cosine method [5], homogeneous balance method [6], tanh-function method [7–9], Jacobi elliptic function expansion method [10–12], $F$-expansion method [13–15], auxiliary equation method [16–18], rational function expansion method [19–21], variational iteration method [22–24], and exp-function method [25–27].

With the development of computer science, recently, directly searching for exact travelling wave solutions of NLEEs has attracted much attention. This is due to the availability of symbolic computation systems like *Mathematica* or *Maple* which enable us to perform the complex and tedious computation on computers. Wang *et al.* [28] introduced a new direct method called the $G'/G$-expansion method to look for travelling wave solutions of

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NLEEs. The $G'/G$-expansion method is a special case of the idea using integrable ODEs in [8], which is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $(G'/G)$, and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation (ODE):

$$G'' + \lambda G' + \mu G = 0, \tag{1.1}$$

where $G' = (\mathrm{d}G(\xi))/(\mathrm{d}\xi)$, $G'' = (\mathrm{d}^2G(\xi))/(\mathrm{d}\xi^2)$, $\xi = x - Vt$, $V$ is a constant. The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given NLEE. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. It was shown that the method present a wider applicability for handling many kinds of NLEEs [28–36], such as, high-dimensional equations, variable-coefficient equations, differential-difference equations.

The present paper is motivated by the desire to propose a generalization of the $G'/G$-expansion method, combined with Liu's theorem [37], for constructing more general solutions of the following (3+1)-dimensional Jimbo–Miwa equation [21,38–41]:

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0, \tag{1.2}$$

which passes the Painlevé test only for a subclass of solutions and its symmetry algebra does not have a Kac–Moody–Virasoro structure.

The rest of this paper is organized as follows. In Section 2, we propose a generalization of the $G'/G$-expansion method for more general solutions. In Section 3, we use the method to solve the (3+1)-dimensional Jimbo–Miwa equation (1.2). In Section 4, some conclusions and discussions are given.

## 2. Description of a generalization of the $G'/G$-expansion method

For a given NLEE, say, in four variables $x$, $y$, $z$ and $t$:

$$P(x, y, z, t, u, u_x, u_y, u_z, u_t, \ldots,) = 0, \tag{2.1}$$

where $u = u(x, y, z, t)$, $P$ is a polynomial about $u$ and its derivatives. We use the following transformation:

$$u = u(\xi), \quad \xi = ax + by + cz - \omega t, \tag{2.2}$$

where $a$, $b$, $c$ and $\omega$ are constants, then equation (2.1) is reduced into an ODE [21]:

$$Q(x, y, z, t, u^{(r)}, u^{(r+1)}, \ldots) = 0, \tag{2.3}$$

where $u^{(r)} = (\mathrm{d}^r u)/(\mathrm{d}\xi^r)$, $u^{(r+1)} = (\mathrm{d}^{r+1}u)/(\mathrm{d}\xi^{r+1})$, $r \geq 1$, and $r$ is the least order of derivatives in the equation. To keep the solution process as simple as possible, the function $Q$ should not be a total $\xi$-derivative of another function. Otherwise, taking integration with respect to $\xi$ further reduces the transformed equation.

We next further introduce

$$u^{(r)}(\xi) = v(\xi) = \sum_{i=1}^{m} \alpha_i \left(\frac{G'}{G}\right)^i + \alpha_0, \quad \alpha_m \neq 0, \tag{2.4}$$

where $G=G(\xi)$ satisfies equation (1.1), while $\alpha_0$, $\alpha_i$ $(i=1,2,\cdots,m)$ are constants to be determined later, a direct computation gives

$$u^{(r+1)}(\xi)=v'(\xi)=-\sum_{i=1}^{m} i\alpha_i\left[\left(\frac{G'}{G}\right)^{i+1}+\lambda\left(\frac{G'}{G}\right)^{i}+\mu\left(\frac{G'}{G}\right)^{i-1}\right], \tag{2.5}$$

$$\begin{aligned} u^{(r+2)}(\xi)&=v''(\xi)\\ &=\sum_{i=1}^{m} i\alpha_i\left[(i+1)\left(\frac{G'}{G}\right)^{i+2}+(2i+1)\lambda\left(\frac{G'}{G}\right)^{i+1}+i(\lambda^2+2\mu)\left(\frac{G'}{G}\right)^{i}\right.\\ &\quad\left.+(2i-1)\lambda\mu\left(\frac{G'}{G}\right)^{i-1}+(i-1)\mu^2\left(\frac{G'}{G}\right)^{i-2}\right], \end{aligned} \tag{2.6}$$

and so on, here the prime denotes the derivative with respective to $\xi$.

To determine $u$ explicitly, we take the following four steps:

*Step 1.* Determine the integer $m$ by substituting equation (2.4) along with equation (1.1) into equation (2.3), and balancing the highest order nonlinear term(s) and the highest order partial derivative.

*Step 2.* Substitute equation (2.4) given the value of $m$ determined in Step 1 along with equation (1.1) into equation (2.3) and collect all terms with the same order of $(G'/G)$ together, the left-hand side of equation (2.3) is converted into a polynomial in $(G'/G)$. Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for $a$, $b$, $c$, $\omega$, $\alpha_0$ and $\alpha_i$.

*Step 3.* Solve the system of algebraic equations obtained in Step 2 for $a$, $b$, $c$, $\omega$, $\alpha_0$ and $\alpha_i$ by use of *Mathematica*.

*Step 4.* Use the results obtained in above steps to derive a series of fundamental solutions $v(\xi)$ of equation (2.3) depending on $(G'/G)$, since the solutions of equation (1.1) have been well known for us, then we can obtain exact solutions of equation (2.1) by integrating each of the obtained fundamental solutions $v(\xi)$ with respect to $\xi$, $r$ times:

$$u=u(\xi)=\int^{\xi}\int^{\xi_r}\cdots\int^{\xi_2} v(\xi_1)\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{r-1}\mathrm{d}\xi_r+\sum_{j=1}^{r} d_j\xi^{r-j}, \tag{2.7}$$

where $d_j$ are arbitrary constants. Based on these obtained exact solutions, we can get some other solutions by using the following Liu's theorem.

**Theorem 2.1.** [37] *If equation* (2.1) *has a kink-type solution*

$$u=P_k\left(\tanh[A(\xi+\xi_0)]\right), \tag{2.8}$$

*then it has certain the kink-bell-type solution*

$$u=P_k\left(\tanh[2A(\xi+\xi_0)]\pm i\,\mathrm{sech}[2A(\xi+\xi_0)]\right), \tag{2.9}$$

*where $P_k$ is polynomial of degree k, i is the imaginary number unit.*

## 3. Application to the Jimbo–Miwa equation

Let us consider in this section the (3+1)-dimensional Jimbo–Miwa equation (1.2). Using the transformation (2.2), we reduce equation (1.2) into an ODE of the form:

$$a^3bu^{(4)}+6a^2bu'u''-(3ac+2b\omega)u''=0. \tag{3.1}$$

Integrating equation (3.1) once with respect to $\xi$ and setting the integration constant as zero yields

$$a^3bu''' + 3a^2b(u')^2 - (3ac + 2b\omega)u' = 0, \tag{3.2}$$

further letting $r = 1$ and $u' = v$, we have:

$$a^3bv'' + 3a^2bv^2 - (3ac + 2b\omega)v = 0. \tag{3.3}$$

According to Step 1, we get $m + 2 = 2m$, hence $m = 2$. We then suppose that equation (3.3) has the following formal solution:

$$v = \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_2 \neq 0. \tag{3.4}$$

Substituting equation (3.4) along with equation (1.1) into equation (3.3) and collecting all terms with the same order of $(G'/G)$ together, the left-hand side of equation (3.3) is converted into a polynomial in $(G'/G)$. Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for $a$, $b$, $c$, $\omega$, $\alpha_0$, $\alpha_1$ and $\alpha_2$ as follows:

$$\left(\frac{G'}{G}\right)^0 : -3ac\alpha_0 - 2b\omega\alpha_0 + 3a^2b\alpha_0^2 + a^3b\alpha_1\lambda\mu + 2a^3b\alpha_2\mu^2 = 0,$$

$$\left(\frac{G'}{G}\right)^1 : -3ac\alpha_1 - 2b\omega\alpha_1 + 6a^2b\alpha_0\alpha_1 + a^3b\alpha_1\lambda^2 + 2a^3b\alpha_1\mu + 6a^3b\alpha_2\lambda\mu = 0,$$

$$\left(\frac{G'}{G}\right)^2 : 3a^2b\alpha_1^2 - 3ac\alpha_2 - 2b\omega\alpha_2 + 6a^2b\alpha_0\alpha_2 + 3a^3b\alpha_1\lambda + 4a^3b\alpha_2\lambda^2 + 8a^3b\alpha_2\mu = 0,$$

$$\left(\frac{G'}{G}\right)^3 : 2a^3b\alpha_1 + 6a^2b\alpha_1\alpha_2 + 10a^3b\alpha_2\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^4 : 6a^3b\alpha_2 + 3a^2b\alpha_2^2 = 0.$$

Solving the set of algebraic equations by use of *Mathematica*, we have

$$\alpha_2 = -2a, \quad \alpha_1 = -2a\lambda, \quad \alpha_0 = -\frac{1}{3}a(\lambda^2 + 2\mu), \quad \omega = -\frac{a^3b(\lambda^2 - 4\mu) + 3ac}{2b}, \tag{3.5}$$

and

$$\alpha_2 = -2a, \quad \alpha_1 = -2a\lambda, \quad \alpha_0 = -2a\mu, \quad \omega = \frac{a^3b(\lambda^2 - 4\mu) - 3ac}{2b}. \tag{3.6}$$

We, therefore, have

$$v = -2a(\frac{G'}{G})^2 - 2a\lambda(\frac{G'}{G}) - \frac{1}{3}a(\lambda^2 + 2\mu), \quad \omega = -\frac{a^3b(\lambda^2 - 4\mu) + 3ac}{2b}, \tag{3.7}$$

and

$$v = -2a(\frac{G'}{G})^2 - 2a\lambda(\frac{G'}{G}) - 2a\mu, \quad \omega = \frac{a^3b(\lambda^2 - 4\mu) - 3ac}{2b}. \tag{3.8}$$

Substituting the general solutions of equation (1.1) into equation (3.7), we obtain three cases of travelling wave solutions of equation (1.1) as follows:

**Case 3.1(1)**: When $\lambda^2-4\mu>0$, we obtain a hyperbolic function solution:

$$u=-\frac{1}{2}a(\lambda^2-4\mu)\int^{\xi}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)^2 d\xi_1 + \frac{1}{6}a(\lambda^2-4\mu)\xi+d_1, \tag{3.9}$$

where $\xi=ax+by+cz+(a^3b(\lambda^2-4\mu)+3ac)/(2b)t$, $C_1, C_2$ and $d_1$ are arbitrary constants.
If $C_1C_2>0$, then solution (3.9) can be simplified as:

$$u=a\sqrt{\lambda^2-4\mu}\tanh\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\frac{1}{2}\ln\left(\frac{C_2}{C_1}\right)\right]-\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}, \tag{3.10}$$

and therefore from Theorem 2.1 we also have

$$u=a\sqrt{\lambda^2-4\mu}\left\{\tanh\left[\sqrt{\lambda^2-4\mu}\xi+\ln\left(\frac{C_2}{C_1}\right)\right] \pm i\,\mathrm{sech}\left[\sqrt{\lambda^2-4\mu}\xi+\ln\left(\frac{C_2}{C_1}\right)\right]\right\}-\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}, \tag{3.11}$$

where $\xi=ax+by+cz+(a^3b(\lambda^2-4\mu)+3ac)/(2b)t$, $\widetilde{d_1}$ is arbitrary constant.
If $C_1C_2<0$, solution (3.9) can be simplified as:

$$u=a\sqrt{\lambda^2-4\mu}\coth\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\frac{1}{2}\ln\left(-\frac{C_2}{C_1}\right)\right]-\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}, \tag{3.12}$$

and therefore from Theorem 2.1 we also have

$$u=a\sqrt{\lambda^2-4\mu}\left\{\tanh\left[\sqrt{\lambda^2-4\mu}\xi+\ln\left(-\frac{C_2}{C_1}\right)\right] \pm i\,\mathrm{sech}\left[\sqrt{\lambda^2-4\mu}\xi+\ln\left(-\frac{C_2}{C_1}\right)\right]\right\}^{-1}-\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}, \tag{3.13}$$

where $\xi=ax+by+cz+(a^3b(\lambda^2-4\mu)+3ac)/(2b)t$, $\widetilde{d_1}$ is arbitrary constant.
If $C_1=0$, then solution (3.9) gives

$$u=a\sqrt{\lambda^2-4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)-\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}, \tag{3.14}$$

and therefore from Theorem 2.1 we also have

$$u=a\sqrt{\lambda^2-4\mu}\{\tanh[\sqrt{\lambda^2-4\mu}\xi]\pm i\,\mathrm{sech}[\sqrt{\lambda^2-4\mu}\xi]\}^{-1} -\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}, \tag{3.15}$$

where $\xi=ax+by+cz+(a^3b(\lambda^2-4\mu)+3ac)/(2b)t$, $\widetilde{d_1}$ is arbitrary constant.
If $C_2=0$, then solution (3.9) becomes

$$u=a\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)-\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}, \tag{3.16}$$

and therefore from Theorem 2.1 we also have

(3.17) $u=a\sqrt{\lambda^2-4\mu}\{\tanh[\sqrt{\lambda^2-4\mu}\xi]\pm i\text{sech}[\sqrt{\lambda^2-4\mu}\xi]\}-\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}$,

where $\xi=ax+by+cz+(a^3b(\lambda^2-4\mu)+3ac)/(2b)t$, $\widetilde{d_1}$ is arbitrary constant.

**Case 3.1(2)**: When $\lambda^2-4\mu<0$, we obtain a trigonometric function solution:

$$u=-\frac{1}{2}a(4\mu-\lambda^2)\int^{\xi}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)^2\mathrm{d}\xi_1+\frac{1}{6}a(\lambda^2-4\mu)\xi_1+d_1, \tag{3.18}$$

where $\xi=ax+by+cz+(a^3b(\lambda^2-4\mu)+3ac)/(2b)t$, $C_1$, $C_2$ and $d_1$ are constants.
If $C_1\neq 0$, then solution (3.18) can be simplifies as:

$$u=-a\sqrt{4\mu-\lambda^2}\tan\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi-\arctan\left(\frac{C_2}{C_1}\right)\right]-\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}, \tag{3.19}$$

where $\xi=ax+by+cz+(a^3b(\lambda^2-4\mu)+3ac)/(2b)t$, $C_2$ and $\widetilde{d_1}$ are arbitrary constants.
If $C_2\neq 0$, then solution (3.18) can be simplifies as:

$$u=a\sqrt{4\mu-\lambda^2}\cot\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+\arctan\left(\frac{C_1}{C_2}\right)\right]-\frac{1}{3}a(\lambda^2-4\mu)\xi+\widetilde{d_1}, \tag{3.20}$$

where $\xi=ax+by+cz+(a^3b(\lambda^2-4\mu)+3ac)/(2b)t$, $C_1$ and $\widetilde{d_1}$ are arbitrary constants.

**Case 3.1(3)**: When $\lambda^2-4\mu=0$, we obtain a rational solution:

$$u=\frac{2aC_2}{C_1+C_2\xi}+d_1, \tag{3.21}$$

where $\xi=ax+by+cz+(3ac)/(2b)t$, $C_1$, $C_2$ and $d_1$ are arbitrary constants.
Substituting the general solutions of equation (1.1) into equation (3.8), we obtain another three cases of travelling wave solutions of equation (1.1) as follows:

**Case 3.2(1)**: When $\lambda^2-4\mu>0$, we obtain a hyperbolic function solution:

$$u=-\frac{1}{2}a(\lambda^2-4\mu)\int^{\xi}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)^2\mathrm{d}\xi_1+\frac{1}{2}a(\lambda^2-4\mu)\xi+d_1, \tag{3.22}$$

where $\xi=ax+by+cz-(a^3b(\lambda^2-4\mu)-3ac)/(2b)t$, $C_1$, $C_2$ and $d_1$ are arbitrary constants.
If $C_1C_2>0$, then solution (3.22) can be simplified as:

$$u=a\sqrt{\lambda^2-4\mu}\tanh\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\frac{1}{2}\ln\left(\frac{C_2}{C_1}\right)\right]+\widetilde{d_1}, \tag{3.23}$$

and therefore from Theorem 2.1 we also get

$$u = a\sqrt{\lambda^2-4\mu}\left\{\tanh\left[\sqrt{\lambda^2-4\mu}\xi + \ln\left(\frac{C_2}{C_1}\right)\right] \pm i\mathrm{sech}\left[\sqrt{\lambda^2-4\mu}\xi + \ln\left(\frac{C_2}{C_1}\right)\right]\right\} + \widetilde{d_1}, \tag{3.24}$$

where $\xi = ax+by+cz-(a^3b(\lambda^2-4\mu)-3ac)/(2b)t$, $\widetilde{d_1}$ is arbitrary constant.

If $C_1C_2 < 0$, solution (3.22) can be simplified as:

$$u = a\sqrt{\lambda^2-4\mu}\coth\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \frac{1}{2}\ln\left(-\frac{C_2}{C_1}\right)\right] + \widetilde{d_1}, \tag{3.25}$$

and therefore from Theorem 2.1 we also get

$$u = a\sqrt{\lambda^2-4\mu}\left\{\tanh\left[\sqrt{\lambda^2-4\mu}\xi + \ln\left(-\frac{C_2}{C_1}\right)\right] \pm i\mathrm{sech}\left[\sqrt{\lambda^2-4\mu}\xi + \ln\left(-\frac{C_2}{C_1}\right)\right]\right\}^{-1} + \widetilde{d_1}, \tag{3.26}$$

where $\xi = ax+by+cz-(a^3b(\lambda^2-4\mu)-3ac)/(2b)t$, $\widetilde{d_1}$ is arbitrary constant.

If $C_1 = 0$, then solution (3.22) gives

$$u = a\sqrt{\lambda^2-4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + \widetilde{d_1}, \tag{3.27}$$

and therefore from Theorem 2.1 we also get

$$u = a\sqrt{\lambda^2-4\mu}\{\tanh[\sqrt{\lambda^2-4\mu}\xi] \pm i\mathrm{sech}[\sqrt{\lambda^2-4\mu}\xi]\}^{-1} + \widetilde{d_1}, \tag{3.28}$$

where $\xi = ax+by+cz-(a^3b(\lambda^2-4\mu)-3ac)/(2b)t$, $\widetilde{d_1}$ is arbitrary constant.

If $C_2 = 0$, then solution (3.22) becomes

$$u = a\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + \widetilde{d_1}, \tag{3.29}$$

and therefore from Theorem 2.1 we also have

$$u = a\sqrt{\lambda^2-4\mu}\{\tanh[\sqrt{\lambda^2-4\mu}\xi] \pm i\mathrm{sech}[\sqrt{\lambda^2-4\mu}\xi]\} + \widetilde{d_1}, \tag{3.30}$$

where $\xi = ax+by+cz-(a^3b(\lambda^2-4\mu)-3ac)/(2b)t$, $\widetilde{d_1}$ is arbitrary constant.

**Case 3.2(2)**: When $\lambda^2-4\mu < 0$, we obtain a trigonometric function solution:

$$u = -\frac{1}{2}a(4\mu-\lambda^2)\int^{\xi}\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)^2 \mathrm{d}\xi_1 + \frac{1}{2}a(\lambda^2-4\mu)\xi_1 + d_1, \tag{3.31}$$

where $\xi = ax+by+cz-(a^3b(\lambda^2-4\mu)-3ac)/(2b)t$, $C_1$, $C_2$ and $d_1$ are constants.

If $C_1 \neq 0$, then solution (3.31) can be simplifies as:

$$u = -a\sqrt{4\mu - \lambda^2}\tan\left[\frac{\sqrt{4\mu - \lambda^2}}{2}\xi - \arctan\left(\frac{C_2}{C_1}\right)\right] + \widetilde{d_1}, \tag{3.32}$$

where $\xi = ax + by + cz - (a^3b(\lambda^2 - 4\mu) - 3ac)/(2b)t$, $C_2$ and $\widetilde{d_1}$ are arbitrary constants.

If $C_2 \neq 0$, then solution (3.31) can be simplifies as:

$$u = a\sqrt{4\mu - \lambda^2}\cot\left[\frac{\sqrt{4\mu - \lambda^2}}{2}\xi + \arctan\left(\frac{C_1}{C_2}\right)\right] + \widetilde{d_1}, \tag{3.33}$$

where $\xi = ax + by + cz - (a^3b(\lambda^2 - 4\mu) - 3ac)/(2b)t$, $C_1$ and $\widetilde{d_1}$ are arbitrary constants.

**Case 3.2(3)**: When $\lambda^2 - 4\mu = 0$, we obtain a rational solution which is same as the solution (3.21) and is omitted here for simplicity.

It should be noted that if directly using the original version of the $G'/G$-expansion method [28] to solve equation (3.2) we can obtain only solutions (3.21), (3.23), (3.25), (3.27), (3.29), (3.32) and (3.33). If we set $\lambda^2 - 4\mu = -4$ and $\widetilde{d_1} = d$, then solutions (3.19) and (43) with $C_2 = 0$ and solutions (3.20) and (44) with $C_1 = 0$ become Ma and Lee's solutions (3.12) and (3.13) in [21]. If we set $C_1 = 0$ and $C_2 = 1$, then solution (3.21) gives the solution (3.14) in [21]. If we set $a = b = c = 1$, $\widetilde{d_1} = \alpha$ and $\lambda^2 - 4\mu = -(2c+3)$, then solution (40) is equivalent to Wazwaz's solution $u_1$ [39]. Similarly, we can easily see that the other solutions, namely $u_2$–$u_8$ obtained in [39], can be respectively recovered from solutions (3.14)–(3.20) on the condition of choosing some appropriate values of the constant parameters involved. Dai *et al.* [40, 41] obtained some two-wave solutions, none of which has an explicit external linear function of $\{x, y, z, t\}$ as solution (3.10) does. To the best of our knowledge, solutions (3.11), (3.13), (3.15), (3.17), (3.24), (3.26), (3.28) and (3.30) have not been reported in literatures.

**Remark 3.1.** All solutions obtained above have been checked with *Mathematica* by putting them back into the original equation (1.2).

## 4. Conclusions and discussions

In summary, more general travelling wave solutions of the (3+1)-dimensional Jimbo–Miwa equation have been obtained owing to the effective combination of a generalized $G'/G$-expansion method and Liu's theorem. The $G'/G$-expansion method [28] can be used to construct hyperbolic function solutions, trigonometric function solutions and rational solutions of NLEEs. It is easy to see that such types of hyperbolic function solutions and trigonometric function solutions can be reconstructed by means of the exp-function method [25]. We would like to conclude that the $G'/G$-expansion method and its improvements may provide alternative mathematical tools for solving NLEEs. One of the most general method to generating exact solutions especially multiple-wave solutions is Ma and Lee's multiple exp-function method [42], based on Fourier theory. The existence of $N$-soliton solutions often implies the integrability of the considered equations. There is a new solution structure following linear superposition principle for soliton equations with Hirota bilinear form in [43], which is helpful in constructing $N$-wave solutions. Some of the hyperbolic function solutions and trigonometric function solutions obtained in the present paper contain an explicit external linear function of $\{x, y, z, t\}$. It may be important to explain some physical phenomena. Employing the $G'/G$-expansion method and its generalized version to study other NLEEs is our task in the future.

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