BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(3) (2013), 709–716*

Central Automorphisms of Semidirect Products

[1]HAMID MOUSAVI AND [2]AMIR SHOMALI
[1,2]Department of Mathematics, University of Tabriz, P. O. Box 51666-17766, Tabriz, Iran
[1]hmousavi@tabrizu.ac.ir, [2]a_shomali@tabrizu.ac.ir

Abstract. In this paper we describe the structure of $\mathrm{Aut}_N^Z(G)$ for a group $G = HK$, where K is a normal subgroup of G and $N = H \cap K$ is $\mathrm{Aut}^Z(G)$-invariant, in particular, if N = 1, this amounts to a description of the central automorphism group of the semi-direct product $G = K \rtimes H$. We also show that if $N \trianglelefteq G$ and $\mathscr{C}_K(H/N) = N$, then $\mathrm{Aut}_N^Z(G)$ is a split extension. Particular if G is solvable, then $\mathrm{Aut}_N^Z(G)$ is an abelian by abelian split extension. This description of the group of central automorphisms of semidirect products is of great importance, because any solvable group has a splitting quotient.

2010 Mathematics Subject Classification: 20D15, 20D45

Keywords and phrases: Central automorphism, solvable group, semidirect.

1. Introduction

Let G be a group and let M and N be normal subgroups of G. By $\mathrm{Aut}^N(G)$ we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms centralizing G/N and by $\mathrm{Aut}_M(G)$ we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms centralizing M. We denote $\mathrm{Aut}^N(G) \cap \mathrm{Aut}_M(G)$ by $\mathrm{Aut}_M^N(G)$. Clearly $\alpha \in \mathrm{Aut}^N(G)$ if and only if $g^{-1}\alpha(g) \in N$ for all g in G.

Given a group G, the subgroup $\mathrm{Aut}^Z(G)$ is called the central automorphism group of G when $Z = Z(G)$. Hence if $\sigma \in \mathrm{Aut}^Z(G)$ then $g^{-1}\sigma(g)$ lies in the center $Z(G)$ of G for all g in G. It is easily seen that $\mathrm{Aut}^Z(G) = \mathscr{C}_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$. The group of central automorphisms of a finite group G is of great importance in investigating of $\mathrm{Aut}(G)$, and has been studied by several authors (see, for example, [1–7]).

A non-abelian group G that has no non-trivial abelian direct factor is said to be purely non-abelian. In [1] Adney and Yen has shown that if G is a finite purely non-abelian group, then $|\mathrm{Aut}^Z(G)| = |\mathrm{Hom}(G/G', Z(G))|$. In [6] Jamali and Jafari introduced some special subgroups of $\mathrm{Aut}^Z(G)$ in order to find the structure of $\mathrm{Aut}^Z(G)$ for a group $G = K \times H$, where K is purely non-abelian and H an abelian subgroup of G. Also in [5] they investigated the nilpotency and solubility of the central automorphisms group of a finite group.

Let $G = HK$, where K is a normal subgroup of G and $N = H \cap K$ is invariant under the central automorphism of G. We shall show that if $N \lhd G$ and $\mathscr{C}_K(H/N) = N$ then $\mathrm{Aut}_N^Z(G)$

Communicated by Shum Kar Ping.
Received: July 7, 2011; *Revised:* October 13, 2011.

is a split extension. The significance of this result is that every finite solvable group has a splitting quotient. Let G be a group and let N be an $\mathrm{Aut}^Z(G)$-invariant subgroup of G. Then the natural action of $\mathrm{Aut}^Z(G)$ on N has $\mathrm{Aut}^Z_N(G)$ as its kernel and $\mathrm{Aut}^Z(G)/\mathrm{Aut}^Z_N(G)$ can be embedded in $\mathrm{Aut}^Z(N)$. Now according to the exact sequence

$$1 \longrightarrow \mathrm{Aut}^Z_N(G) \longrightarrow \mathrm{Aut}^Z(G) \longrightarrow \mathrm{Aut}^Z(G)/\mathrm{Aut}^Z_N(G) \longrightarrow 1,$$

the study of structure of $\mathrm{Aut}^Z_N(G)$ becomes more important. Clearly if all central automorphisms of G fix N pointwise, then $\mathrm{Aut}^Z_N(G) = \mathrm{Aut}^Z(G)$.

For an especial example, let

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3,$$

the group 48#51 as quoted in "Small Group" library of GAP [10]. Then (by using GAP),

$$\mathrm{Aut}^Z(G) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes PSL(3,2).$$

Consider the maximal subgroup $M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of G and set $N = \mathrm{Core}_G(M) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $\mathrm{Aut}^Z_N(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and

$$\mathrm{Aut}^Z(G)/\mathrm{Aut}^Z_N(G) \cong PSL(3,2) \cong \mathrm{Aut}^Z(N).$$

Throughout the paper our notation is standard, and can be found in [9], for example.

2. Preliminaries

The aim of this section is to collect several facts and basic results that will be used in the rest of the paper. Let σ be a central automorphism of G. Clearly the map $f_\sigma : x \longmapsto x^{-1}\sigma(x)$ defines a homomorphism from G into $Z(G)$. On the other hand, the map $\sigma_f : x \longmapsto xf(x)$ defines an endomorphism of G for all f in $\mathrm{Hom}(G,Z(G))$. This endomorphism is a central automorphism if and only if $f(x) \neq x^{-1}$ for every x in $G - \{1\}$, because, $x \in \mathrm{Ker}(\sigma_f)$ if and only if $xf(x) = 1$ or $f(x) = x^{-1}$, so $x = 1$. Also $xf(x) = yf(y)$ implies the contradictory equality $f(x^{-1}y) = (x^{-1}y)^{-1}$, and so the set $\{xf(x) \,|\, x \in G\}$ is G. As any homomorphism $f : G \longrightarrow Z(G)$ induces a homomorphism $f : G/G' \longrightarrow Z(G)$, and vice versa, we see that

$$|\mathrm{Hom}(G,Z(G))| = |\mathrm{Hom}(G/G',Z(G))|.$$

Throughout the paper G is a finite group, and π_H, π_K are the projection maps from $G = K \rtimes H$ into H and K, respectively. Also σ_H, σ_K are corresponding restrictions of σ on H and K, where $\sigma_H : H \longrightarrow G$ and $\sigma_K : K \longrightarrow G$. Now we set

$$R = \{\sigma_H \,|\, \sigma \in \mathrm{Aut}^Z(G)\}, \qquad S = \{\sigma_K \,|\, \sigma \in \mathrm{Aut}^Z(G)\},$$
$$T = \{\pi_K\sigma_H \,|\, \sigma \in \mathrm{Aut}^Z(G)\}, \quad U = \{\pi_H\sigma_K \,|\, \sigma \in \mathrm{Aut}^Z(G)\}.$$

Let N be a normal subgroup of G such that $G/N = K/N \rtimes H/N$ for some subgroups K and H of G, and let $L \leqslant Z(G)$. If $N \leqslant \mathrm{Ker} f$ then every element f of $\mathrm{Hom}(H,L)$ induces an element $\hat{f}$ of $\mathrm{Hom}(G,L)$, where $\hat{f}(kh) = f(h)$. Also if $N[H,K] \leqslant \mathrm{Ker} f$, then every element f of $\mathrm{Hom}(K,L)$ induces an element $\hat{f}$ of $\mathrm{Hom}(G,L)$, in the case we have

$$\hat{f}(h_1k_1h_2k_2) = \hat{f}(h_1h_2[h_2,k_1^{-1}]k_1k_2) = f([h_2,k_1^{-1}]k_1k_2) = f(k_1)f(k_2) = \hat{f}(k_1h_1)\hat{f}(k_2h_2).$$

Lemma 2.1. *Let G be a finite group and $N \trianglelefteq G$ such that $G/N = K/N \rtimes H/N$. For any subgroup $L \leqslant Z(G)$ we set*

$$A = \{f \in \mathrm{Hom}(H,L), N \leqslant \mathrm{Ker} f\},$$
$$B = \{f \in \mathrm{Hom}(K,L), N[H,K] \leqslant \mathrm{Ker} f\},$$

$$A' = \{\sigma_{\hat{f}} |\ f \in \mathrm{Hom}(H,L), N \leqslant \mathrm{Ker}\, f\},$$
$$B' = \{\sigma_{\hat{f}} |\ f \in \mathrm{Hom}(K,L), N[H,K] \leqslant \mathrm{Ker}\, f\}.$$

Then we have

(i) *if* G *is purely non-abelian then* A' *and* B' *are subgroups of* $\mathrm{Aut}^Z(G)$ *of order* $|A|$ *and* $|B|$ *respectively,*

(ii) *if* $L \leqslant K \cap Z(G)$ *then* A' *is a subgroup of* $\mathrm{Aut}^Z(G)$ *isomorphic to* A*,*

(iii) *if* $L \leqslant H \cap Z(G)$ *then* B' *is a subgroup of* $\mathrm{Aut}^Z(G)$ *isomorphic to* B*.*

Proof. (i) Since G is a purely non-abelian, then $\sigma_{\hat{f}} \in \mathrm{Aut}^Z(G)$ in all cases. So we can easily see that A' and B' are subgroups of $\mathrm{Aut}^Z(G)$ of order $|A|$ and $|B|$ respectively.

(ii) Let $g = kh$ for some $k \in K$ and $h \in H$. Then $\hat{f}(g) = g^{-1}$ implies that $f(h) = h^{-1}k^{-1}$, so $h \in N$ and $f(h) = 1$, therefore $g = 1$. Hence we can assume that $\hat{f}(g) \neq g^{-1}$ for all $1 \neq g \in G$, then $\sigma_{\hat{f}} \in \mathrm{Aut}^Z(G)$. Also

$$\begin{aligned}\sigma_{\hat{f}_1}\sigma_{\hat{f}_2}(g) = \sigma_{\hat{f}_1}(g\hat{f}_2(g)) &= g\hat{f}_2(g)\hat{f}_1(g)\hat{f}_1(\hat{f}_2(g)) \\ &= g\hat{f}_1(g)\hat{f}_2(g), \quad (\hat{f}_1(hk) = f_1(h)\ \&\ \hat{f}_2(g) \in L \leqslant K) \\ &= g(\hat{f}_1.\hat{f}_2)(g),\end{aligned}$$

hence $\sigma_{\hat{f}_1}\sigma_{\hat{f}_2} = \sigma_{\hat{f}_1.\hat{f}_2} \in A'$ for all $\sigma_{\hat{f}_1}, \sigma_{\hat{f}_2} \in A'$, therefore A' is a subgroup of $\mathrm{Aut}^Z(G)$. Now the mapping $f \longmapsto \sigma_{\hat{f}}$ is an isomorphism from A into A', and the result follows.

(iii) Follows similarly. ∎

The following lemma is similar to that of [6, lemma 2.4]

Lemma 2.2. *Let* G *be a finite group with a normal subgroup* N *such that* $G/N = K/N \rtimes H/N$*. Suppose that* $A \leqslant H \cap Z(G)$ *and* $B \leqslant K \cap Z(G)$ *with* $A \cap B = 1$*. We set*

$$R_1 = \{\sigma_{\hat{f}} | f \in \mathrm{Hom}(H,A), N \leqslant \mathrm{Ker}\, f,\ f(x) \neq x^{-1},\ x \in H\backslash\{1\}\},$$
$$S_1 = \{\sigma_{\hat{f}} | f \in \mathrm{Hom}(K,B), N[H,K] \leqslant \mathrm{Ker}\, f,\ f(x) \neq x^{-1},\ x \in K\backslash\{1\}\},$$
$$T_1 = \{\sigma_{\hat{f}} | f \in \mathrm{Hom}(H,B), N \leqslant \mathrm{Ker}\, f\ \},$$
$$U_1 = \{\sigma_{\hat{f}} | f \in \mathrm{Hom}(K,A), N[H,K] \leqslant \mathrm{Ker}\, f\ \}.$$

Then

(i) R_1*,* S_1*,* T_1 *and* U_1 *are all subgroups of* $\mathrm{Aut}^Z(G)$ *having mutually trivial intersections,*

(ii) T_1 *and* U_1 *are abelian and* $[R_1, S_1] = 1$*,*

(iii) $R_1 \cup S_1 \subseteq \mathscr{N}_{\mathrm{Aut}^Z(G)}(T_1) \cap \mathscr{N}_{\mathrm{Aut}^Z(G)}(U_1)$.

Proof. (i) We only show that $R_1 \leqslant \mathrm{Aut}^Z(G)$, others are similar. At first, we prove that $\hat{f}(g) \neq g^{-1}$, which implies $\sigma_{\hat{f}} \in \mathrm{Aut}^Z(G)$. If $\hat{f}(g) = g^{-1}$ for some $1 \neq g = hk$, then $f(h) = k^{-1}h^{-1}$ which implies $hf(h) = k^{-1} \in N$. Now, from $f(hf(h)) = 1$ we have $f(f(h)) = (f(h))^{-1}$, which contradicts $1 \neq f(h) \in A \leqslant H$. Now we show that R_1 is closed under composition. To see that, let $\tau = \sigma_{\hat{f}_1}\sigma_{\hat{f}_2}$. So $\tau(x) = x\hat{f}_1(x)\hat{f}_2(x)\hat{f}_1(\hat{f}_2(x))$, hence $x^{-1}\tau(x) \in A$. Now, define $\hat{f} : G \to A$ with $\hat{f}(x) = x^{-1}\tau(x)$ and let f be the restriction of $\hat{f}$ to H. Since for all $n \in N$, $\tau(n) = n$, then $f(n) = \hat{f}(n) = n^{-1}\tau(n) = 1$, and so $N \leqslant \mathrm{Ker}(f)$. Also, if $h^{-1} = f(h)$ then $h^{-1} = \hat{f}(h) = h^{-1}\tau(h)$ hence $h = 1$. Therefore $\tau = \sigma_{\hat{f}} \in R_1$.

For the second part, let $X \in \{A,B\}$ and $f \in \mathrm{Hom}(H,X)$, then $\sigma_{\hat{f}}(k) = kf(1) = k$. Also for any $f \in \mathrm{Hom}(K,X)$, $\sigma_{\hat{f}}(h) = hf(1) = h$. So for all σ in $R_1 \cap S_1$, $R_1 \cap U_1$ or $S_1 \cap T_1$, $\sigma(hk) = \sigma(h)\sigma(k) = hk$. Now, for $\sigma_{\hat{f}} \in R_1 \cap T_1$, $f(h) \in A \cap B = 1$. Thus, $\sigma(h) = h$ and $\sigma_{\hat{f}}(k) = kf(1) = k$, hence $\sigma_{\hat{f}}$ is an identity map. Similarly, one can show that $S_1 \cap U_1 = 1$.

(ii) Let $f_1, f_2 \in \mathrm{Hom}(H,B)$. Since $B \leqslant K$ then for all $x \in G$, $\hat{f}_1(\hat{f}_2(x)) = \hat{f}_2(\hat{f}_1(x)) = 1$. Hence,

$$\sigma_{\hat{f}_1}\sigma_{\hat{f}_2}(x) = x\hat{f}_1(x)\hat{f}_2(x) = x\hat{f}_2(x)\hat{f}_1(x) = \sigma_{\hat{f}_2}\sigma_{\hat{f}_1}(x).$$

So T_1 is abelian. Similarly one can see that U_1 is abelian. Let $f \in \mathrm{Hom}(H,A)$ and $g \in \mathrm{Hom}(K,B)$. Then, $\hat{f}(\hat{g}(hk)) = \hat{f}(g(k)) = 1$ and $\hat{g}(\hat{f}(hk)) = \hat{g}(f(h)) = 1$. Therefore, $\sigma_{\hat{f}}\sigma_{\hat{g}}(x) = x\hat{g}(x)\hat{f}(x) = \sigma_{\hat{g}}\sigma_{\hat{f}}(x)$. Now, we have $\sigma_{\hat{f}}\sigma_{\hat{g}} = \sigma_{\hat{g}}\sigma_{\hat{f}}$.

(iii) Let $\sigma_{\hat{f}} \in R_1, \sigma_{\hat{\gamma}} \in T_1$, and set $\mu = \sigma_{\hat{f}}^{-1}\sigma_{\hat{\gamma}}\sigma_{\hat{f}}$. We have $K \leqslant \mathrm{Ker} f_\mu$, because $\sigma_{\hat{f}}$ and $\sigma_{\hat{\gamma}}$ fix K elementwise. Now $\mu(x) = \sigma_{\hat{f}}^{-1}(\hat{\gamma}(\sigma_{\hat{f}}(x)))x = \hat{\gamma}(\sigma_{\hat{f}}(x))x$ for all $x \in G$, and $f_\mu(x) = x^{-1}\mu(x) \in B$. Thus $f_\mu \in \mathrm{Hom}(H,B)$ and $N \leqslant \mathrm{Ker} f_\mu$. Hence $\mu = \sigma_{\hat{f}_\mu} \in T_1$ and so R_1 normalizes T_1. Next let $\sigma_{\hat{f}} \in R_1, \sigma_{\hat{\gamma}} \in U_1$, and set $\mu = \sigma_{\hat{f}}\sigma_{\hat{\gamma}}\sigma_{\hat{f}}^{-1}$. We find that $\mu(h) = h$ for all $h \in H$ because $\sigma_{\hat{\gamma}}$ fixes H elementwise. Therefore $H \leqslant \mathrm{Ker} f_\mu$. Now $\mu(x) = f(\gamma(\sigma_{\hat{f}}^{-1}(x)))\gamma(\sigma_{\hat{f}}^{-1}(x))x$ for all $x \in G$, and $f_\mu(x) = x^{-1}\mu(x) \in A$. Thus $f_\mu \in \mathrm{Hom}(K,A)$ and $N[H,K] \leqslant \mathrm{Ker} f_\mu$. Hence $\mu = \sigma_{\hat{f}_\mu} \in U_1$ and so R_1 normalizes U_1, which completes the proof. ∎

Lemma 2.3. *If $G = K \rtimes H$ then $Z(G) \leqslant \mathscr{C}_K(H)Z(H)$. Furthermore, if $Z(H)$ acts trivially on K then $Z(H) \leqslant Z(G)$ and $Z(G) = \mathscr{C}_{Z(K)}(H) \times Z(H)$.*

The proof of above Lemma is obvious. Note that since $Z(H)$ acts trivially on K, we have $\mathscr{C}_{Z(K)}(H) = K \cap Z(G)$ and $Z(H) = H \cap Z(G)$.

3. Main results

In this section we give the order and the structure of $\mathrm{Aut}^Z(G)$, where G is a semidirect product.

Theorem 3.1. *Let $G = K \rtimes H$ be a semidirect product of groups H and K, where $(|H|,|K|) = 1$. Then $\sigma_H \in \mathrm{Aut}^Z(H)$ and $\sigma_K \in \mathrm{Aut}^Z(K)$ for all σ in $\mathrm{Aut}^Z(G)$ and $\mathrm{Aut}^Z(G) \cong R \times S$ (defined as in page 3). Furthermore, if $Z(H)$ acts trivially on K then $\mathrm{Aut}^Z(G) \cong \mathrm{Aut}^Z(H) \times S$. In particular, if $G = K \times H$ then $\mathrm{Aut}^Z(G) \cong \mathrm{Aut}^Z(H) \times Aut^Z(K)$.*

Proof. Let $\sigma \in \mathrm{Aut}^Z(G)$ and $k \in K$. Then $\sigma_K \in \mathrm{Aut}(K)$ and $k^{-1}\sigma(k) \in Z(G) \cap K \leqslant Z(K)$. Hence $\sigma_K \in \mathrm{Aut}^Z(K)$. Let $\sigma \in \mathrm{Aut}^Z(G)$ and $h \in H$. We may write $\sigma(h) = h'k$, where $h' \in H$ and $k \in K$. Then $h^{-1}h' \in Z(H)$ and $k \in \mathscr{C}_K(H)$. On letting $|K| = t$, we see that $\sigma(h^t) = h'^t$ for some $h' \in H$. Since for all $h \in H$ there exists an element $h_1 \in H$ such that $h = h_1^t$, we have $\sigma(h) \in H$. Hence H is an invariant subgroup under the central automorphism σ of G. Then $\sigma_H \in \mathrm{Aut}(H)$ and $h^{-1}\sigma(h) \in Z(G) \cap H \leqslant Z(H)$, hence $\sigma_H \in \mathrm{Aut}^Z(H)$.

On taking $A = Z(G) \cap H, B = Z(G) \cap K, N = 1$ and using Lemma 2.2, we find that $T_1 = 1$ and $U_1 = 1$. Also the subgroups R_1 and S_1 of $\mathrm{Aut}^Z(G)$ have trivial intersection. Now we show that $R \cong R_1$ and $S \cong S_1$. We define a map $\varphi : R_1 \to R$ by $\varphi(\sigma_{\hat{f}}) = \sigma_{\hat{f}}|_H$. Then φ is well-defined and is a homomorphism. Also if $\varphi(\sigma_{\hat{f}}) = 1$ then $\sigma_{\hat{f}}(hk) = \sigma_{\hat{f}}(h)\sigma_{\hat{f}}(k) = hk$ and

$\sigma_{\hat{f}} = 1$. Furthermore, φ is surjective, because if $\sigma \in \mathrm{Aut}^Z(G)$, then on setting $\tau = \sigma_H \in R$, we have $f_\tau \in \mathrm{Hom}(H, Z(G) \cap H)$ and $f_\tau(h) \neq h^{-1}$ for every $h \in H - \{1\}$. Thus $\sigma_{\hat{f}_\tau} \in R_1$ and $\varphi(\sigma_{\hat{f}_\tau}) = \sigma_{\hat{f}_\tau}|_H = \tau$. It follows that φ is an isomorphism. Similarly we can see that $S \cong S_1$. Now it is easy to observe that the map $\theta : \mathrm{Aut}^Z(G) \to R \times S$ defined by $\theta(\sigma) = (\sigma_H, \sigma_K)$ is a monomorphism. We show that θ is surjective. To do this, we consider $\delta, \tau \in \mathrm{Aut}^Z(G)$ for which we have $\delta_H \in \mathrm{Aut}^Z(H)$ and $\tau_K \in \mathrm{Aut}^Z(K)$, since $(|H|, |K|) = 1$. Therefore $\delta_H \in R$ and $\tau_K \in S$. Now we define $\sigma(hk) = \delta(h)\tau(k)$, we can easily see that $\sigma \in \mathrm{Aut}^Z(G)$ and $\theta(\sigma) = (\delta, \tau)$. So θ is an isomorphism.

If $Z(H)$ acts trivially on K then $Z(H) \leqslant Z(G)$ and $R = \mathrm{Aut}^Z(H)$, so $\mathrm{Aut}^Z(G) \cong \mathrm{Aut}^Z(H) \times S$. In particular, if $G = K \times H$ then $R = \mathrm{Aut}^Z(H)$, $S = \mathrm{Aut}^Z(K)$ and $\mathrm{Aut}^Z(G) \cong \mathrm{Aut}^Z(H) \times \mathrm{Aut}^Z(K)$. ∎

Theorem 3.2. *Let $G = K \rtimes H$ and $\mathscr{C}_K(H) = 1$. Then $\sigma_H \in \mathrm{Aut}^Z(H)$ and $\pi_H \sigma_K \in \mathrm{Hom}(K, Z(G) \cap H)$ for all σ in $\mathrm{Aut}^Z(G)$. Moreover, $\mathrm{Aut}^Z(G) \cong U \rtimes R$.*

Proof. Let $\sigma \in \mathrm{Aut}^Z(G)$ and $h \in H$. Then $\sigma(h) = h_1 k_1$ and $h^{-1}\sigma(h) = h^{-1}h_1 k_1 \in Z(G)$. Therefore from Lemma 2.3, $h^{-1}h_1 \in Z(H)$ and $k_1 \in \mathscr{C}_K(H) = 1$, and hence $\sigma_H \in \mathrm{Aut}^Z(H)$. Let $k \in K$, since $\mathscr{C}_K(H) = 1$, we have $\sigma(k) = hk$ and $\pi_H \sigma_K \in \mathrm{Hom}(K, Z(G) \cap H)$, where $h \in H$.

By taking $A = Z(G) \cap H$, $N = 1$ and using Lemma 2.2, we find that $T_1 = 1$ and $S_1 = 1$. Also the subgroups R_1 and U_1 of $\mathrm{Aut}^Z(G)$ have trivial intersection and $R_1 \leqslant \mathscr{N}_{\mathrm{Aut}^Z(G)}(U_1)$. By a similar argument given for Theorem 3.1, we have $R \cong R_1$. It is now sufficient to show $U \cong U_1$. We define the map $\varphi : U_1 \to U$ by $\varphi(\sigma_{\hat{f}}) = \pi_H \sigma_{\hat{f}}|_K$. Clearly φ is well-defined. Furthermore, φ is a homomorphism because

$$\begin{aligned}\varphi(\sigma_{\hat{f}_1}\sigma_{\hat{f}_2})(k) &= \pi_H(\sigma_{\hat{f}_1}\sigma_{\hat{f}_2})(k) = \pi_H(k\hat{f}_1(k)\hat{f}_2(k)\hat{f}_1(\hat{f}_2(k)))\\ &= \hat{f}_1(k)\hat{f}_2(k) = (\varphi(\sigma_{\hat{f}_1})\varphi(\sigma_{\hat{f}_2}))(k).\end{aligned}$$

It is easy to check that φ is one-to-one. Also φ is surjective because if $\sigma \in \mathrm{Aut}^Z(G)$, on setting $g = \pi_H \sigma_K \in U$, we have $g \in \mathrm{Hom}(K, Z(G))$ and $[H, K] \leqslant \mathrm{Ker}\, g$. Thus $\sigma_{\hat{g}} \in U_1$ and $\varphi(\sigma_{\hat{g}}) = \pi_H \sigma_{\hat{g}}|_K = g$. Therefore φ is an isomorphism.

Now the map $\psi : U_1 \rtimes R_1 \to \mathrm{Aut}^Z(G)$ defined by $(\sigma_{\hat{\gamma}}, \sigma_{\hat{f}}) \mapsto \sigma_{\hat{f}}\sigma_{\hat{\gamma}}$ is one-to-one, and so $|R_1||U_1| \leq |\mathrm{Aut}^Z(G)|$.

Next we consider the map $\phi : \mathrm{Aut}^Z(G) \to U \rtimes R$ defined by $\sigma \mapsto (\pi_H \sigma_K, \sigma_H)$. If $\phi(\sigma) = \phi(\tau)$ then $\pi_H \sigma_K = \pi_H \tau_K$ and $\sigma_H = \tau_H$. Thus for any $h \in H$ and $k \in K$, we have $\sigma(h) = \tau(h)$ and $\sigma(k) = \tau(k)$. Therefore ϕ is one-to-one and so $|\mathrm{Aut}^Z(G)| \leq |R||U| = |R_1||U_1|$. Hence $|\mathrm{Aut}^Z(G)| = |R||U|$ and $\mathrm{Aut}^Z(G) \cong U \rtimes R$. ∎

Theorem 3.3. *Let $G = K \rtimes H$, where K is purely non-abelian and $Z(H)$ acts trivially on K. Then $\mathrm{Aut}^Z(G) \cong RTUS$.*

Proof. We only show that $\pi_H \sigma_H \in \mathrm{Aut}^Z(H)$ and $\pi_K \sigma_K \in \mathrm{Aut}^Z(K)$, similarly we can see that $\pi_H \sigma_K \in \mathrm{Hom}(K, Z(H))$ and $\pi_K \sigma_H \in \mathrm{Hom}(H, Z(G) \cap K)$.

Let $\sigma \in \mathrm{Aut}^Z(G)$ and $k \in K$. We may write $\sigma(k) = k_1 h_1$, where $k_1 \in K$ and $h_1 \in H$. Hence $k^{-1}k_1 h_1 \in Z(G)$. We have by Lemma 2.3, $k^{-1}k_1 \in Z(K)$ and $h_1 \in Z(H)$. We have $\pi_K \sigma_K \in \mathrm{Hom}(K, K)$ and $k^{-1}\pi_K \sigma(k) \in Z(G) \cap K \leqslant Z(K)$. Now define $f_\sigma : K \to Z(G) \cap K$ by

$f_\sigma(k) = k^{-1}\pi_K\sigma(k)$. Clearly f_σ is well-defined. Furthermore, f_σ is a homomorphism since

$$\begin{aligned}
f_\sigma(kk') &= k'^{-1}k^{-1}\pi_K\sigma(kk') = k'^{-1}k^{-1}\pi_K(\sigma(k)\sigma(k')) \\
&= k'^{-1}k^{-1}\pi_K(h_1k_1h_2k_2), \quad (\text{where } \sigma(k) = k_1h_1 \text{ and } \sigma(k') = k_2h_2), \\
&= k'^{-1}k^{-1}\pi_K(h_1h_2k_1^{h_2}k_2) = k'^{-1}k^{-1}k_1^{h_2}k_2 \\
&= k'^{-1}k^{-1}k_1k_2, \quad (\text{by Lemma 2.3 and this fact } h_2 \in Z(H)), \\
&= k^{-1}k_1k'^{-1}k_2 = k^{-1}\pi_K\sigma(k)k'^{-1}\pi_K\sigma(k') = f_\sigma(k)f_\sigma(k').
\end{aligned}$$

Therefore, $\sigma_{f_\sigma} \in \mathrm{Aut}^Z(K)$ because K is purely non-abelian. Thus $\pi_K\sigma_K = \sigma_{f_\sigma} \in \mathrm{Aut}^Z(K)$. Next let $\sigma \in \mathrm{Aut}^Z(G)$ and $h \in H$. We may write $\sigma(h) = h_1k_1$, where $k_1 \in K$ and $h_1 \in H$. Then $h^{-1}h_1 \in Z(H)$, and so $h^{-1}\pi_H\sigma(h) \in Z(H)$. Suppose that $\mathrm{Ker}\,\pi_H\sigma_H \neq 1$; so $\pi_H\sigma(h) = 1$, for some $h \in H - \{1\}$. Therefore, there is a non-trivial element k in K, such that $\sigma(h) = k$. It follows that k belongs to the kernel of $\pi_K{\sigma^{-1}}_K$ contrary to the first part of the proof. Hence $\pi_H\sigma_H \in \mathrm{Aut}^Z(H)$. Similarly $\pi_H\sigma_K \in \mathrm{Hom}(K, Z(H))$ and $\pi_K\sigma_H \in \mathrm{Hom}(H, Z(G)\cap K)$.

By taking $A = Z(H)$, $B = Z(G)\cap K$, $N = 1$ and using Lemma 2.2, we find that R_1, S_1, T_1 and U_1 being subgroups of $\mathrm{Aut}^Z(G)$ have mutually trivial intersections. Also $R_1 \cup S_1 \subseteq \mathscr{N}_{\mathrm{Aut}^Z(G)}(T_1) \cap \mathscr{N}_{\mathrm{Aut}^Z(G)}(U_1)$. It is easy to see that $R \cong R_1$, $S \cong S_1$, $T \cong T_1$ and $U \cong U_1$.

We now consider the map $\phi : R_1 \times T_1 \times U_1 \times S_1 \longrightarrow \mathrm{Aut}^Z(G)$ defined by

$$(\sigma_{\hat f}, \sigma_{\hat g}, \sigma_u, \sigma_v) \mapsto \sigma_{\hat f}\sigma_{\hat g}\sigma_u\sigma_v.$$

Then ϕ is one-to-one. To see this, let $h \in H$ and $\sigma_{\hat f}\sigma_{\hat g}\sigma_{\hat u}\sigma_{\hat v}(h) = \sigma_{\hat{f'}}\sigma_{\hat{g'}}\sigma_{\hat{u'}}\sigma_{\hat{v'}}(h)$. Then we have $\sigma_{\hat f}(h\sigma_{\hat g}(h)) = \sigma_{\hat{f'}}(h\sigma_{\hat{g'}}(h))$, which implies that

$$\hat{f'}^{-1}(h)\hat f(h) = \hat g(h)\hat{g'}^{-1}(h) \in H \cap K = 1.$$

Hence $\hat f = \hat{f'}$ and $\hat g = \hat{g'}$. Therefore $\sigma_{\hat f} = \sigma_{\hat{f'}}$ and $\sigma_{\hat g} = \sigma_{\hat{g'}}$. Now let $k \in K$ and $\sigma_{\hat f}\sigma_{\hat g}\sigma_{\hat u}\sigma_{\hat v}(k) = \sigma_{\hat{f'}}\sigma_{\hat{g'}}\sigma_{\hat{u'}}\sigma_{\hat{v'}}(k)$. It follows that $\sigma_{\hat u}\sigma_{\hat v}(k) = \sigma_{\hat{u'}}\sigma_{\hat{v'}}(k)$. So

$$\hat v(k)\hat{v'}^{-1}(k) = \hat{u'}(k)\hat{u'}(\hat{v'}(k))\hat u(\hat v(k))^{-1}\hat u^{-1}(k) \in H \cap K = 1.$$

Therefore, $\sigma_{\hat u} = \sigma_{\hat{u'}}$ and $\sigma_{\hat v} = \sigma_{\hat{v'}}$, so $|R_1||T_1||U_1||S_1| \leqslant |\mathrm{Aut}^Z(G)|$.

Now we define a map $\varphi : \mathrm{Aut}^Z(G) \longrightarrow R \times T \times U \times S$ by

$$\sigma \mapsto (\pi_H\sigma_H, \pi_K\sigma_H, \pi_H\sigma_K, \pi_K\sigma_K).$$

Clearly φ is well-defined. Also if $\varphi(\sigma) = \varphi(\tau)$ then

$$(\pi_H\sigma_H, \pi_K\sigma_H, \pi_H\sigma_K, \pi_K\sigma_K) = (\pi_H\tau_H, \pi_K\tau_H, \pi_H\tau_K, \pi_K\tau_K).$$

Let $h \in H$, $\sigma(h) = h_1k_1$ and $\tau(h) = h_2k_2$. Then $h_1 = h_2$, $k_1 = k_2$ and $\sigma(h) = \tau(h)$. Similarly, we have $\sigma(k) = \tau(k)$ for all $k \in K$. Thus $\sigma = \tau$ and φ is one-to-one, and hence $|\mathrm{Aut}^Z(G)| \leq |R||T||U||S|$. Therefore, $|\mathrm{Aut}^Z(G)| = |R||T||U||S|$ and $\mathrm{Aut}^Z(G) \cong RTUS$. ∎

Theorem 3.4. *Let $G = HK$, where K is a normal subgroup of G and $N = H \cap K$ is invariant under the central automorphisms of G. If $N \trianglelefteq G$ and $\mathscr{C}_K(H/N) = N$, then $\mathrm{Aut}_N^Z(G) \cong U' \rtimes R'$, where $R' = \{\sigma_H | \sigma \in \mathrm{Aut}_N^Z(G)\}$ and $U' = \{f_{\sigma_K} | \sigma \in \mathrm{Aut}_N^Z(G)\}$.*

Proof. Since $\sigma(N) \leqslant N$, every $\sigma \in \mathrm{Aut}^Z(G)$ induces an automorphism $\sigma^* \in \mathrm{Aut}^Z(G/N)$. Also we have $\mathscr{C}_{K/N}(H/N) = N$. Now by Theorem 3.2, $\sigma^*|_{H/N} \in \mathrm{Aut}^Z(H/N)$, and we see that $\sigma_H \in \mathrm{Aut}^Z(H)$ and $\sigma^*(kN) = kNhN$. So $\sigma(k)N = khN$ and $k^{-1}\sigma(k) = hn \in H$. The map $f_{\sigma_K} : K \longrightarrow Z(G) \cap H$ defined by $k \longmapsto k^{-1}\sigma(k)$ is a homomorphism.

On taking $A = Z(G) \cap H$ and using Lemma 2.2, R_1 and U_1 being subgroups of $\mathrm{Aut}^Z(G)$ have trivial intersection. We have $R' \cong R_1$ and $U' \cong U_1$. It is easy to see that the map

$$\psi : U_1 \rtimes R_1 \to \mathrm{Aut}^Z_N(G)$$

defined by $(\sigma_{\hat{f}}, \sigma_{\hat{\gamma}}) \mapsto \sigma_{\hat{f}}\sigma_{\hat{\gamma}}$ is one-to-one, so that $|R_1||U_1| \leq |\mathrm{Aut}^Z_N(G)|$. Next we consider the map $\phi : \mathrm{Aut}^Z_N(G) \to U' \rtimes R'$ defined by $\sigma \mapsto (f_{\sigma_K}, \sigma_H)$. If $\phi(\sigma) = \phi(\tau)$, then $f_{\sigma_K} = f_{\tau_K}$ and $\sigma_H = \tau_H$. Thus for every $h \in H$ and $k \in K$, we have $\sigma(h) = \tau(h)$ and $\sigma(k) = \tau(k)$. Therefore, ϕ is one-to-one, and hence $|\mathrm{Aut}^Z_N(G)| \leq |R'||U'| = |R_1||U_1|$. Therefore, $|\mathrm{Aut}^Z(G)| = |R'||U'|$ and $\mathrm{Aut}^Z_N(G) \cong U' \rtimes R'$. ∎

Corollary 3.1. *Let N be a normal subgroup of G such that $G/N = K/N \rtimes H/N$ and $(|H/N|, |K/N|) = 1$. If N is invariant under $\mathrm{Aut}^Z(G)$, then $\sigma_H \in \mathrm{Aut}^Z(H)$ and $\sigma_K \in \mathrm{Aut}^Z(K)$ for all σ in $\mathrm{Aut}^Z(G)$. Moreover, $\mathrm{Aut}^Z_N(G) \cong R' \times S'$, where $R' = \{\sigma_H | \sigma \in \mathrm{Aut}^Z_N(G)\}$ and $S' = \{\sigma_K | \sigma \in \mathrm{Aut}^Z_N(G)\}$.*

Proof. Since N is invariant under $\mathrm{Aut}^Z(G)$, every $\sigma \in \mathrm{Aut}^Z(G)$ induces an automorphism $\sigma^* \in \mathrm{Aut}^Z(G/N)$. We have $(|H/N|, |K/N|) = 1$, now by Theorem 3.1, $\sigma^*|_{H/N} \in \mathrm{Aut}^Z(H/N)$ and $\sigma^*|_{K/N} \in \mathrm{Aut}^Z(K/N)$, and we see that $\sigma_H \in \mathrm{Aut}^Z(H)$ and $\sigma_K \in \mathrm{Aut}^Z(K)$. By a similar argument given for Theorem 3.1, we have $\mathrm{Aut}^Z_N(G) \cong R' \times S'$. ∎

4. Applications

Let G be a finite solvable group and M be a non-normal maximal subgroup of G. Then $G/C = L/C \rtimes M/C$, where $C = \mathrm{Core}_G(M)$ and L/C is an elementary abelian group. If $(|M/C|, |C|) = 1$, then $G = L \rtimes M_1$, where $M_1 \leqslant M$. Furthermore, if $(|M/C|, |L/C|) = 1$, then $(|L|, |M_1|) = 1$. Now by Theorem 3.1, we have $\mathrm{Aut}^Z(G) \cong R \times S$.

Let G be a finite solvable group. By [8, Proposition 4.9], we see that G has a maximal subgroup M such that in all must all cases, M/C becomes abelian, and hence $(|M/C|, |L/C|) = 1$ holds.

Theorem 4.1. *Let G be a finite solvable group and let M be a non-normal maximal subgroup of G such that M/C is abelian, where $C = \mathrm{Core}_G(M)$. If $(|L/C|, |C|) = 1$, then $\mathrm{Aut}^Z(G) \cong R$, where $R = \{\sigma_M | \sigma \in \mathrm{Aut}^Z(G)\}$.*

Proof. We have $G' \leqslant L$ because $G/L \cong M/C$ is abelian, also $L/C = [L/C, M/C]$ hence $L = [L, M]C \leqslant G'C \leqslant L$, therefore $L = CG'$. Since $G' \cap M \leqslant G' \cap C$,

$$G/(G' \cap C) = (G'M)/(G' \cap C) \cong G'/(G' \cap C) \rtimes M/(G' \cap C).$$

Now since p does not divide $|M/C|$ and $|C|$, so $p \nmid |M|$ also $G'/(G' \cap C) \cong L/C$ is of order of a power of p, therefore $(|G'/(G' \cap C)|, |M/(G' \cap C)|) = 1$, and we see that $\mathrm{Aut}^Z(G) \cong R$. ∎

Theorem 4.2. *Let G be a finite solvable group and let M be a non-normal maximal subgroup of index p^t. Then*

(i) $\mathrm{Aut}^Z_C(G) \cong U' \rtimes R'$, *where $C = \mathrm{Core}_G(M)$,*

(ii) *if M/C is abelian, then* $\mathrm{Aut}_C^Z(G) \cong R' \times S'$*, where R' is abelian and S' is elementary abelian of order $|\Omega_1(P)|^t$, where P is a Sylow p-subgroup of $Z(G)$.*

Proof. (i) We have $G/C = L/C \rtimes M/C$, where $C = \mathrm{Core}_G(M)$ and L/C is elementary abelian. Thus C is invariant under $\mathrm{Aut}^Z(G)$ and $\mathscr{C}_{L/C}(M/C) = C$. On taking $M = H$, $L = K$ and $C = N$, now Theorem 3.4 completes the proof.

(ii) Since M/C is abelian, $(|M/C|, |L/C|) = 1$. By taking

$$R_1 = \{\sigma_{\hat{f}} \mid f \in \mathrm{Hom}(M, Z(G)), C \leqslant \mathrm{Ker} f,\ f(m) \neq m^{-1},\ m \in M \backslash \{1\}\},$$

$$S_1 = \{\sigma_{\hat{f}} \mid f \in \mathrm{Hom}(L, Z(G)), C[M, L] \leqslant \mathrm{Ker} f,\ f(l) \neq l^{-1},\ l \in L \backslash \{1\}\},$$

and using Corollary 3.1, we have $R_1 \cong R'$, $S_1 \cong S'$ and $\mathrm{Aut}_C^Z(G) \cong R' \times S'$. The subgroup R_1 is abelian because for all $\sigma_{\hat{f}}, \sigma_{\hat{h}} \in R_1$ and $g = ml \in G$, where $m \in M$ and $l \in L$, we have

$$\begin{aligned}\sigma_{\hat{f}}\sigma_{\hat{h}}(g) &= \sigma_{\hat{f}}\sigma_{\hat{h}}(ml) = \sigma_{\hat{f}}(ml\,h(m)) = ml\,h(m)f(m) \quad (\text{since } h(m) \in Z(G) \leqslant C)\\ &= \sigma_{\hat{h}}\sigma_{\hat{f}}(g).\end{aligned}$$

Similarly the subgroup S_1 is abelian. Also for all $\sigma_{\hat{f}} \in S_1$ and $g = ml \in G$, where $m \in M$ and $l \in L$, we have $(\sigma_{\hat{f}})^p(ml) = ml(\hat{f}(ml))^p = mlf(l^p) = ml$ because L/C is elementary abelian. Hence R' is abelian and S' is elementary abelian. ∎

Example 4.1. Let $G = ((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9) \rtimes \mathbb{Z}_3$, consider the maximal subgroup $M = \mathbb{Z}_9 \rtimes \mathbb{Z}_3$ of G. Then (by using GAP),

$$C = \mathrm{Core}_G(M) = \mathbb{Z}_3 \times \mathbb{Z}_3,\ Z(G) = \mathbb{Z}_3,\ \ M/C = \mathbb{Z}_3 \textit{ and } L/C = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Since $(|L/C|, |Z(G)|) = 1$, $S' \cong S_1 = \{1\}$. Furthermore $R' \cong R_1 \cong \mathbb{Z}_3$ because,

$$\mathrm{Hom}(M/C, Z(G)) = \mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_3) \cong \mathbb{Z}_3.$$

The group G satisfies the conditions of Theorem 4.2(ii). That is $\mathrm{Aut}^Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathrm{Aut}_C^Z(G) \cong \mathbb{Z}_3$ and $\mathrm{Aut}^Z(G)/\mathrm{Aut}_C^Z(G) \cong \mathbb{Z}_3$.

References

[1] J. E. Adney and T. Yen, Automorphisms of a p-group, *Illinois J. Math.* **9** (1965), 137–143.

[2] M. S. Attar, c-characteristically simple groups, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 1, 147–154.

[3] M. J. Curran and D. J. McCaughan, Central automorphisms that are almost inner, *Comm. Algebra* **29** (2001), no. 5, 2081–2087.

[4] W. Guo, X. Feng and J. Huang, New characterizations of some classes of finite groups, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 3, 575–589.

[5] M.-H. Jafari and A.-R. Jamali, On the nilpotency and solubility of the central automorphism group of a finite group, *Algebra Colloq.* **15** (2008), no. 3, 485–492.

[6] M.-H. Jafari and A.-R. Jamali, On the occurrence of some finite groups in the central automorphism group of finite groups, *Math. Proc. R. Ir. Acad.* **106A** (2006), no. 2, 139–148 (electronic).

[7] A.-R. Jamali and H. Mousavi, On the central automorphism groups of finite p-groups, *Algebra Colloq.* **9** (2002), no. 1, 7–14.

[8] A.-R. Jamali and H. Mousavi, A note on the σ-covers of finite soluble groups, *Algebra Colloq.* **12** (2005), no. 4, 691–697.

[9] D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, 80, Springer, New York, 1982.

[10] The GAP Group. GAP - Groups, Algorithms and Programing, Version 4.4; (2005).