

## On Newton-Like Method for Solving Generalized Nonlinear Operator Equations in Banach Spaces

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**Abstract.** The purpose of this paper is to prove existence and uniqueness theorem for solving an operator equation of the form  $F(x) + G(x) = 0$ , where  $F$  is a Gâteaux differentiable operator and  $G$  is a Lipschitzian operator defined on an open convex subset of a Banach space. Our result extends and improves the previously known results in recent literature.

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### 1. Introduction

Many applied problems can be formulated to fit the model of nonlinear equations. Undoubtedly, Newton's methods are the most popular methods for solving such equations. There are numerous generalizations of Newton's method for solving nonlinear operator equation

$$(1.1) \quad F(x) = 0.$$

Details can be found in Argyros [6], Wu and Zhao [11] and references therein.

Özban [7] and Weerakoon and Fernando [10] obtained interesting variants of Newton's method for solving operator equation (1.1) in case of real-valued functions, under the strong assumption that at least the third derivative of  $F$  exists. However, neither Özban [7] nor Weerakoon and Fernando [10] specified the size of the interval containing the iterates converging to the solution of operator equation (1.1).

In this paper, a generalized Newton-like method in a Banach space is formulated and a semi-local convergence theorem is proved. Here we are concerned with the problem of approximating a locally unique solution of the generalized operator equation

$$(1.2) \quad F(x) + G(x) = 0,$$

where  $F$  and  $G$  are defined on an open convex subset  $D$  of a Banach space  $X$  with values in a Banach space  $Y$ . Let  $x_0 \in D$  be fixed and define the Newton-like method by

$$(1.3) \quad x_{n+1} = x_n - (\lambda_n F'_{x_n} + (1 - \lambda_n) F'_{z_n})^{-1} (F(x_n) + G(x_n)), \quad n = 0, 1, 2, \dots,$$

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where  $F'_x$  denotes the Gâteaux derivative of  $F$  evaluated at  $x \in D$ .

Since our assumptions on the nonlinear operator  $F$  involving in operator equation (1.2) are fairly general, our main result (Theorem 2.1) covers a wide variety of nonlinear operator equations and it significantly improves the corresponding results of Vijesh and Subrahmanyam ([1, 2]), Argyros [3] and Weerakoon and Fernando [10].

## 2. Convergence analysis

Before presenting main result, we need the following lemma:

**Lemma 2.1.** (see Rall [8, p. 50]) *Let  $L$  be a bounded linear operator on a Banach space  $X$ . Then the following are equivalent:*

- (a) *There is a bounded linear operator  $M$  on  $X$  such that  $M^{-1}$  exists, and*

$$\|M - L\| < \frac{1}{\|M^{-1}\|}.$$

- (b)  *$L^{-1}$  exists.*

$$\text{Further, if } L^{-1} \text{ exists then } \|L^{-1}\| \leq \frac{\|M^{-1}\|}{1 - \|M^{-1}L\|} \leq \frac{\|M^{-1}\|}{1 - \|M^{-1}\|\|M - L\|}.$$

Now we are ready to present the main result of this paper.

**Theorem 2.1.** *Let  $F$  and  $G$  be two operators defined on an open convex subset  $D$  of a Banach space  $X$  with values in another Banach space  $Y$  such that  $F$  has Gâteaux derivative at each point in some neighbourhood  $U(x_0, r) = \{x \in X : \|x - x_0\| < r\}$  of  $x_0 \in D$  and  $G$  is Lipschitzian on  $D$  with Lipschitz constant  $k$ . Assume that  $\{\lambda_n\}_{n=0}^\infty$  is a sequence in  $[0, 1]$ . Assume further that*

- (i)  $(F'_{x_0})^{-1} \in L(Y, X)$ , the space of bounded linear operators from  $Y$  to  $X$ ;  
(ii) for some  $z_0 \in U(x_0, r)$  and  $\eta > 0$ ,  $(\lambda_0 F'_{x_0} + (1 - \lambda_0) F'_{z_0})^{-1} \in L(Y, X)$  and

$$\|(\lambda_0 F'_{x_0} + (1 - \lambda_0) F'_{z_0})^{-1} (F(x_0) + G(x_0))\| \leq \eta;$$

- (iii) for some  $\varepsilon > 0$ ,  $\|(F'_{x_0})^{-1} (F'_x - F'_{x_0})\| \leq \varepsilon$  whenever  $x \in U(x_0, r)$ ;

- (iv)  $F'_x$  is piecewise-hemicontinuous at each  $x \in U(x_0, r)$  and  $\bar{U}(x_0, r) \subseteq D$ .

Set  $c = \frac{(2 - \lambda_0)\varepsilon + \mu}{1 - \varepsilon}$  and  $d = \frac{2\varepsilon + \mu}{1 - \varepsilon}$  such that  $(1 + \frac{c}{1 - d})\eta < r$ , where  $\mu = k\|(F'_{x_0})^{-1}\|$  and  $\varepsilon \in (0, \frac{1 - \mu}{3})$ . Then, for any sequence  $\{z_n\}$  in  $U(x_0, r)$ , the sequence of iterates  $\{x_n\}$  generated by (1.3) is well defined remains in  $U(x_0, r)$  and converges strongly to a point  $x^* \in \bar{U}(x_0, r)$ . Further, if  $F$  is continuous at  $x^*$ , then  $x^*$  is the unique solution of operator equation (1.2). Moreover, the following error-estimates hold:

$$(E_1): \quad \|x_{n+1} - x_n\| \leq d^{n-1} c \eta,$$

$$(E_2): \quad \|x_n - x^*\| \leq \frac{d^{n-1} c \eta}{1 - d}.$$

*Proof.* Set  $L_n = \lambda_n F'_{x_n} + (1 - \lambda_n) F'_{z_n}$ . For  $n = 0$ , we have

$$\begin{aligned} \|x_1 - x_0\| &= \|(\lambda_0 F'_{x_0} + (1 - \lambda_0) F'_{z_0})^{-1} (F(x_0) + G(x_0))\| \\ &\leq \eta < r. \end{aligned}$$

Therefore,  $x_1 \in U(x_0, r)$ . Observe that

$$\begin{aligned} \|I - (F'_{x_0})^{-1}L_1\| &= \|I - (F'_{x_0})^{-1}(\lambda_1 F'_{x_1} + (1 - \lambda_1)F'_{z_1})\| \\ &= \|(F'_{x_0})^{-1}(F'_{x_0} - (\lambda_1 F'_{x_1} + (1 - \lambda_1)F'_{z_1}))\| \\ &= \|(F'_{x_0})^{-1}(\lambda_1 F'_{x_0} + (1 - \lambda_1)F'_{x_0} - \lambda_1 F'_{x_1} - (1 - \lambda_1)F'_{z_1})\| \\ &\leq \lambda_1 \|(F'_{x_0})^{-1}(F'_{x_0} - F'_{x_1})\| + (1 - \lambda_1) \|(F'_{x_0})^{-1}(F'_{x_0} - F'_{z_1})\| \\ &\leq \lambda_1 \varepsilon + (1 - \lambda_1)\varepsilon = \varepsilon < 1. \end{aligned}$$

Hence, by Lemma 2.1,  $(F'_{x_0})^{-1}L_1$  is invertible and  $\|((F'_{x_0})^{-1}L_1)^{-1}\| \leq \frac{1}{1-\varepsilon}$ .

By assumption  $(F'_{x_0})^{-1}$  is invertible, it follows that  $L_1$  is invertible. From the definition of  $x_2$  and by condition (iv), we have

$$\begin{aligned} x_2 - x_1 &= -L_1^{-1}(F(x_1) + G(x_1)) \\ &= -L_1^{-1}(F(x_1) - F(x_0) + F(x_0) + G(x_0) + G(x_1) - G(x_0)) \\ &= -L_1^{-1} \left[ \int_0^1 F'_{tx_1+(1-t)x_0}(x_1 - x_0)dt - L_0(x_1 - x_0) + G(x_1) - G(x_0) \right] \\ &= -L_1^{-1} \left[ \int_0^1 (F'_{tx_1+(1-t)x_0} - L_0)(x_1 - x_0)dt + G(x_1) - G(x_0) \right] \\ &= -((F'_{x_0})^{-1}L_1)^{-1} \left[ \int_0^1 (F'_{x_0})^{-1}(F'_{tx_1+(1-t)x_0} - (\lambda_0 F'_{x_0} + (1 - \lambda_0)F'_{z_0}))(x_1 - x_0)dt \right. \\ &\quad \left. + (F'_{x_0})^{-1}(G(x_1) - G(x_0)) \right] \\ &= -((F'_{x_0})^{-1}L_1)^{-1} \left[ \lambda_0 \int_0^1 (F'_{x_0})^{-1}(F'_{tx_1+(1-t)x_0} - F'_{x_0})(x_1 - x_0)dt \right. \\ &\quad \left. + (1 - \lambda_0) \int_0^1 (F'_{x_0})^{-1}(F'_{tx_1+(1-t)x_0} - F'_{z_0})(x_1 - x_0)dt + (F'_{x_0})^{-1}(G(x_1) - G(x_0)) \right]. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|x_2 - x_1\| &\leq \|(F'_{x_0})^{-1}L_1\|^{-1} \left\| \left[ \lambda_0 \int_0^1 (F'_{x_0})^{-1}(F'_{tx_1+(1-t)x_0} - F'_{x_0})(x_1 - x_0)dt \right] \right\| \\ &\quad + (1 - \lambda_0) \left\| \int_0^1 (F'_{x_0})^{-1}(F'_{tx_1+(1-t)x_0} - F'_{z_0})(x_1 - x_0)dt \right\| \\ &\quad + \|(F'_{x_0})^{-1}\| k \|x_1 - x_0\| \\ &\leq \|(F'_{x_0})^{-1}L_1\|^{-1} (\lambda_0 \varepsilon + (1 - \lambda_0)2\varepsilon + \mu) \|x_1 - x_0\| \\ &\leq \frac{(2 - \lambda_0)\varepsilon + \mu}{1 - \varepsilon} \|x_1 - x_0\| \\ &\leq c\eta. \end{aligned}$$

Consequently, we have  $\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq c\eta + \eta < r$ . Thus,  $x_2 \in U(x_0, r)$ . Assume that  $x_k \in U(x_0, r)$  and  $\|x_k - x_{k-1}\| \leq d^{k-2}c\eta$ , for  $k = 2, 3, 4, \dots, n - 1$ . In view of the hypothesis (iii), we obtain

$$\begin{aligned} \|I - (F'_{x_0})^{-1}L_k\| &= \|(F'_{x_0})^{-1}(F'_{x_0} - L_k)\| \\ &= \|(F'_{x_0})^{-1}(F'_{x_0} - (\lambda_k F'_{x_k} + (1 - \lambda_k)F'_{z_k}))\| \end{aligned}$$

$$\begin{aligned}
&= \|(F'_{x_0})^{-1}(\lambda_k F'_{x_0} + (1 - \lambda_k)F'_{x_0} - (\lambda_k F'_{x_k} + (1 - \lambda_k)F'_{z_k}))\| \\
&\leq \lambda_k \|(F'_{x_0})^{-1}(F'_{x_0} - F'_{x_k})\| + (1 - \lambda_k) \|(F'_{x_0})^{-1}(F'_{x_0} - F'_{z_k})\| \\
&\leq \lambda_k \varepsilon + (1 - \lambda_k) \varepsilon \\
&= \varepsilon < 1.
\end{aligned}$$

Hence again by Lemma 2.1,  $L_k^{-1}$  exists and  $\|((F'_{x_0})^{-1}L_k)^{-1}\| \leq \frac{1}{1-\varepsilon}$ . From the definition of  $x_{k+1}$  and by condition (iv), we have

$$\begin{aligned}
x_{k+1} - x_k &= -L_k^{-1}(F(x_k) + G(x_k)) \\
&= -L_k^{-1}(F(x_k) - F(x_{k-1}) + F(x_{k-1}) + G(x_{k-1}) + G(x_k) - G(x_{k-1})) \\
&= -L_k^{-1} \left[ \int_0^1 F'_{\theta x_k + (1-\theta)x_{k-1}}(x_k - x_{k-1})d\theta - L_{k-1}(x_k - x_{k-1}) \right. \\
&\quad \left. + G(x_k) - G(x_{k-1}) \right] \\
&= -L_k^{-1} \left[ \int_0^1 (F'_{\theta x_k + (1-\theta)x_{k-1}} - L_{k-1})(x_k - x_{k-1})d\theta + G(x_k) - G(x_{k-1}) \right] \\
&= -L_k^{-1} \left[ \int_0^1 (F'_{\theta x_k + (1-\theta)x_{k-1}} - \lambda_{k-1}F'_{x_{k-1}} - (1 - \lambda_{k-1})F'_{z_{k-1}})(x_k - x_{k-1})d\theta \right. \\
&\quad \left. + G(x_k) - G(x_{k-1}) \right] \\
&= -L_k^{-1}F'_{x_0} \left[ \lambda_{k-1} \int_0^1 F'_{x_0}{}^{-1}(F'_{\theta x_k + (1-\theta)x_{k-1}} - F'_{x_{k-1}})(x_k - x_{k-1})d\theta \right. \\
&\quad \left. + (1 - \lambda_{k-1}) \int_0^1 F'_{x_0}{}^{-1}(F'_{\theta x_k + (1-\theta)x_{k-1}} - F'_{z_{k-1}})(x_k - x_{k-1})d\theta \right. \\
&\quad \left. + F'_{x_0}{}^{-1}(G(x_k) - G(x_{k-1})) \right].
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x_{k+1} - x_k\| &\leq [\lambda_{k-1}2\varepsilon + (1 - \lambda_{k-1})2\varepsilon + \mu] \|(F'_{x_0})^{-1}L_k)^{-1}\| \|x_k - x_{k-1}\| \\
&\leq \frac{2\varepsilon + \mu}{1 - \varepsilon} \|x_k - x_{k-1}\| \\
&\leq \frac{2\varepsilon + \mu}{1 - \varepsilon} d^{k-2}c\eta \\
&= d^{k-1}c\eta.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|x_{k+1} - x_0\| &\leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\| \\
&\leq d^{k-1}c\eta + d^{k-2}c\eta + \cdots + c\eta + \eta \\
&= \left(1 + \frac{c(1-d^k)}{1-d}\right) \eta \\
&\leq \eta \left(1 + \frac{c}{1-d}\right) < r.
\end{aligned}$$

Hence, the sequence  $\{x_n\}$  is well defined and remains in  $U(x_0, r)$ .

Now, let  $k \geq 2$  and  $m \in \mathbb{N}$ , we have

$$\begin{aligned} \|x_{k+m} - x_k\| &\leq \|x_{k+m} - x_{k+m-1}\| + \|x_{k+m-1} - x_{k+m-2}\| + \dots + \|x_{k+1} - x_k\| \\ &\leq d^{k+m-2}c\eta + d^{k+m-3}c\eta + \dots + d^{k-1}c\eta \\ &\leq \frac{1-d^m}{1-d}d^{k-1}c\eta \\ &\leq \frac{d^{k-1}c\eta}{1-d}. \end{aligned}$$

Since  $0 < d < 1$ , it follows that  $\{x_n\}$  is a Cauchy sequence in  $U(x_0, r)$  and hence converges strongly to an element  $x^*$  in  $\bar{U}(x_0, r)$ . From the hypothesis (iii) and using triangle inequality it follows that  $\|L_n\| \leq M$ , where  $M = \left( \frac{\varepsilon}{\|(F'_{x_0})^{-1}\|} + \|(F'_{x_0})^{-1}\| \right)$ .

From equation (1.3), we get

$$F(x_n) + G(x_n) = -L_n(x_{n+1} - x_n).$$

Therefore, we have

$$(2.1) \quad \|F(x_n) + G(x_n)\| \leq \|L_n\| \|x_{n+1} - x_n\| \leq M \|x_{n+1} - x_n\|.$$

By letting  $n \rightarrow \infty$  in (2.1) and using the continuity of  $F$  and  $G$  it follows from the convergence of  $\{x_n\}$  to  $x^*$  that  $F(x^*) + G(x^*) = 0$ .

If  $x^*$  and  $y^*$  be two solution of (1.2) in  $\bar{U}(x_0, r)$ , then

$$\begin{aligned} \|x^* - y^*\| &= \|x^* - y^* - (F'_{x_0})^{-1}(F(x^*) + G(x^*) - F(y^*) - G(y^*))\| \\ &\leq \left\| \int_0^1 [I - (F'_{x_0})^{-1}F'_{\theta x^* + (1-\theta)y^*}](x^* - y^*)d\theta \right\| \\ &\quad + \|(F'_{x_0})^{-1}(G(x^*) - G(y^*))\| \\ &\leq \varepsilon \|x^* - y^*\| + \mu \|x^* - y^*\| \\ &= (\varepsilon + \mu) \|x^* - y^*\| < \|x^* - y^*\|. \end{aligned}$$

This contradiction implies that  $x^* = y^*$ . Hence the theorem holds. ■

**Remark 2.1.** Theorem 2.1 is more general in nature. It generalizes the corresponding results of Vijesh and Subrahmanyam ([1], Theorem 2.1), Weerakoon and Fernando [10] and Argyros ([3], Theorem 1). Theorem 2.1 also extends the results of Özban [7] from real line to Banach spaces.

**Remark 2.2.** For the choice,  $\lambda_n = 0$  and  $z_n \equiv x_0$ , Theorem 2.1 reduces to the modified Newton’s method (see e.g. [9]).

Set  $\lambda_n = \lambda$  in Theorem 2.1, we have:

**Corollary 2.1.** Suppose  $F$  and  $G$  are continuous and satisfy all conditions of Theorem 2.1. Then, Newton’s method defined by

$$x_{n+1} = x_n - (\lambda F'_{x_n} + (1 - \lambda)F'_{z_n})^{-1}(F(x_n) + G(x_n)),$$

where  $z_n \in U(x_0, r)$ , converges strongly to a unique solution of generalized operator equation (1.2) in  $\overline{U}(x_0, r)$ .

**Remark 2.3.** For the choice of  $G \equiv 0$ , Corollary 2.1 reduces to the result of Vijesh and Subrahmanyam ([2], Theorem 2.1).

For the choice  $z_n = \frac{x_n + y_n}{2}$ , for all  $n = 0, 1, \dots$ , in Theorem 2.1, we derive the following two point Newton method.

**Theorem 2.2.** Suppose  $F$  and  $G$  are continuous and satisfy conditions of Theorem 2.1. Then, the two-point Newton's method defined by

$$\begin{cases} y_n = x_n - (F'_{x_n})^{-1}(F(x_n) + G(x_n)), \\ x_{n+1} = x_n - (\lambda_n F'_{x_n} + (1 - \lambda_n) F'_{\frac{x_n + y_n}{2}})^{-1}(F(x_n) + G(x_n)), \end{cases}$$

converges strongly to a unique solution of (1.2) in  $\overline{U}(x_0, r)$ .

**Remark 2.4.** Theorem 2.2 is more general than results studied by Argyros [4] and Argyros and Uko [5] in Banach space setting.

### 3. Numerical examples

The following example shows that the operator  $F$  is not necessarily continuous in its domain:

**Example 3.1.** Let  $X = Y = \mathbb{R}$ ,  $D = (-1, 1)$  and consider the problem of solving the operator equation (1.2), where  $F$  and  $G$  are operators on  $D$  defined by

$$F(x) = \begin{cases} e^x, & \text{for } x \in D - \{\frac{1}{2}\} \\ \frac{1}{2}, & \text{for } x = \frac{1}{2} \end{cases} \text{ and } G(x) = \frac{x}{13} - 1, \forall x \in D,$$

respectively. Clearly, for  $x_0 = 0.05$  and  $r = 0.12$ ,  $F$  is Gâteaux differentiable in  $U(0.05, 0.12)$ , where  $U(0.05, 0.12)$  is an open interval of radius 0.12 centered at 0.05, and  $G$  is Lipschitzian on  $D$  with Lipschitz constant  $k = \frac{1}{13}$ . The Gâteaux derivative of  $F$  at any point  $x$  of  $U(0.05, 0.12)$  is given by  $F'_x(y) = e^x y, \forall y \in D$ . The inverse operator  $F'_{x_0}{}^{-1}$  of  $F'_{x_0}$  at the point  $x_0$  also exists with  $F'_{x_0}{}^{-1}(y) = \frac{1}{e^{x_0}} y, \forall y \in D$ ,  $\|F'_{x_0}{}^{-1}\| = \frac{1}{1.051271096}$  and  $\mu = 0.07317149$ . For  $\lambda_n = \frac{3n}{3n+1}, n \geq 1$ , choose  $\eta = 0.0721867438$ . For  $x \in U(0.05, 0.12)$  take  $\varepsilon = 0.1424$ , we have

$$\begin{aligned} \|F'_x - F'_{x_0}\| &= \sup \{ |(F'_x - F'_{x_0})(y)| : \|y\| \leq 1 \} \\ &= \sup \{ |(e^x - e^{x_0})(y)| : \|y\| \leq 1 \} \\ &\leq \sup \{ |(1.185304851 - 1.051271096)(y)| : \|y\| \leq 1 \} \\ &\leq 0.134033755 \\ &< 0.1424 \times 1.051271096 = \frac{\varepsilon}{\|F'_{x_0}{}^{-1}\|}. \end{aligned}$$

Thus,  $\|F'_{x_0}{}^{-1}(F'_x - F'_{x_0})\| < \varepsilon, \forall x \in U(x_0, r)$ . Since  $c = 0.29287721, d = \frac{2\varepsilon + \mu}{1 - \varepsilon} = 0.41741079$  and hence  $(1 + \frac{c}{1-d})\eta = 0.10847621$ . For  $z_n = 0.07, \forall n \in \mathbb{N} \cup \{0\}$ , we have

$$\|(\frac{3}{4}F'_{x_0} + \frac{1}{4}F'_{z_0})^{-1}(F(x_0) + G(x_0))\| = 0.0721867438.$$

Since  $(1 + \frac{c}{1-d})\eta < r$  and  $\|(\frac{3}{4}F'_{x_0} + \frac{1}{4}F'_{z_0})^{-1}(F(x_0) + G(x_0))\| \leq \eta$ , all the assumptions of Theorem 2.1 are verified. Therefore, the operator equation (1.2) has a unique solution in  $\bar{U}(0.05, 0.12)$ . The following table shows numerically the convergence of iteration scheme defined by (1.3) corresponding to operator equation defined in this example.

$n$	$x_n$
1	0.0500000000000000
2	-0.002165695968407
3	0.000144601776272
4	-0.000009992612488
5	0.000000709022355
6	-0.000000051095256
7	0.000000003721214
8	-0.00000000273083
9	0.00000000020156
10	-0.00000000001494
11	0.00000000000111
12	-0.00000000000008
13	0.00000000000001
14	0.00000000000000

We now consider the following example in which the operator  $F$  is not necessarily Fréchet differentiable:

**Example 3.2.** Let  $X = Y = \mathbb{R}^2$  and consider the problem of solving the operator equation (1.2), where  $F$  and  $G$  are operators on  $X$  defined by

$$F(x, y) = \begin{cases} (3x^2y, \frac{xy}{x^2+y}), & \text{if } x^2 + y \neq 0; \\ (0, 0), & \text{if } x^2 + y = 0 \end{cases}$$

and

$$G(x, y) = (x, x + y - 1), \forall (x, y) \in \mathbb{R}^2,$$

respectively. Clearly,  $F$  is not Fréchet differentiable on  $X$ , whereas,  $F$  is Gâteaux differentiable on  $X$  and  $G$  is Lipschitzian on  $X$  with Lipschitz constant  $k = 1$ . The Gâteaux derivative of  $F$  at any point  $\bar{x} = (x, y)$  of  $X$  is given by

$$F'_{\bar{x}} = \begin{cases} \left[ \begin{array}{cc} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{array} \right], & \text{if } (x, y) \neq (0, 0); \\ \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right], & \text{if } (x, y) = (0, 0), \end{cases}$$

where  $f_1$  and  $f_2$  are coordinate maps of  $F$ . The zeros of (1.2) are given in the following Figure 1.

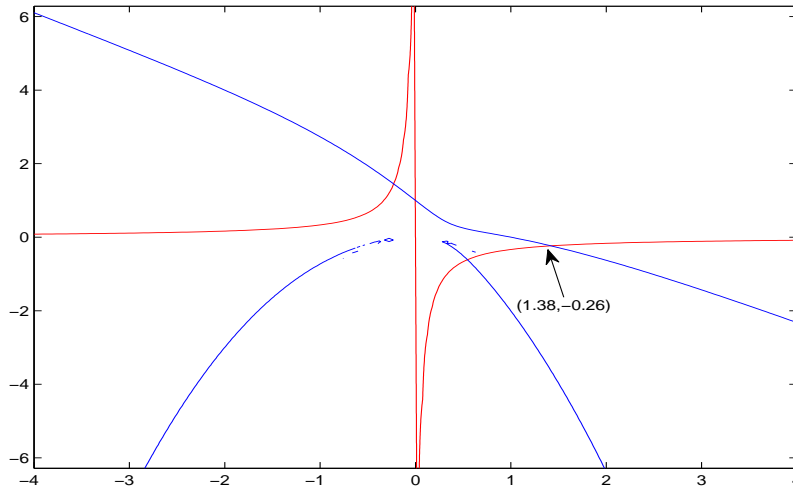


Figure 1. Location of zeros of  $F(x) + G(x) = 0$

One can easily see that neighbourhood of point  $(1.38, -0.26)$  contains a solution of operator equation (1.2). For computing this solution, let us consider  $x_0 = (1.38, -0.26)$  and  $\forall n$ , take  $\lambda_n = 0.75$  and  $z_n = (1.2, -0.8)$ . We can easily compute the sequence  $\{x_n\}$  defined in (1.3) as follows:

$$x_1 = \begin{pmatrix} 1.407873288046822 \\ -0.224477670475158 \end{pmatrix}, x_2 = \begin{pmatrix} 1.417003908114665 \\ -0.232948021427251 \end{pmatrix}, \dots$$

Some iterations can be seen in the following table, which shows the convergence of  $\{x_n\}$  to the unique solution of (1.2) in a neighbourhood of the point  $(1.38, -0.26)$ .

$n$	$x_n$	$n$	$x_n$	$n$	$x_n$
1	1.3800000000000000 -0.2600000000000000	12	1.421238490090396 -0.234537148206760	23	1.421238614480936 -0.234537205742015
2	1.407873288046822 -0.224477670475158	13	1.421238575450872 -0.234537187689247	24	1.421238614481183 -0.234537205742130
3	1.417003908114665 -0.232948021427251	14	1.421238602234630 -0.234537200077662	25	1.421238614481261 -0.234537205742165
4	1.419915895174425 -0.233890563737492	15	1.421238610638632 -0.234537203964813	26	1.421238614481285 -0.234537205742177
5	1.420823040135105 -0.234348279928150	16	1.421238613275575 -0.234537205184493	27	1.421238614481293 -0.234537205742180
6	1.421108278840281 -0.234476610644721	17	1.421238614102974 -0.234537205567194	28	1.421238614481295 -0.234537205742181
7	1.421197713800660 -0.234518317039062	18	1.421238614362590 -0.234537205687276	29	1.421238614481296 -0.234537205742182
8	1.421225781523298 -0.234531267282175	19	1.421238614444049 -0.234537205724954	30	1.421238614481296 -0.234537205742182
9	1.421234587810309 -0.234535343525011	20	1.421238614469609 -0.234537205736776	31	1.421238614481297 -0.234537205742182
10	1.421237351028290 -0.234536621325773	21	1.421238614477629 -0.234537205740486	32	1.421238614481297 -0.234537205742182
11	1.421238218044333 -0.234537022378350	22	1.421238614480146 -0.234537205741650	33	1.421238614481297 -0.234537205742182



The following Figure 2 shows the convergence of the iterates  $\{x_n\}$  defined by (1.3) with starting from  $x_0 = (1.38, -0.26)$  for different sequence  $\{\lambda_n\}$ , where  $\lambda_n = a, \forall n$  and  $a \in [0, 1]$ .

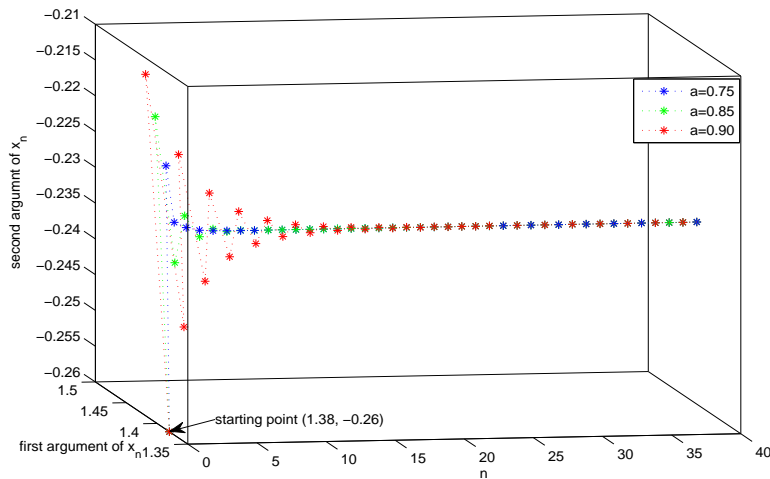


Figure 2. Convergence of (1.3) for different sequence  $\{\lambda_n\}$

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