Extremal Bicyclic Graph with Perfect Matching for Different Indices

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Abstract. Let $\mathcal{B}(2m,m)$ be the set of all bicyclic graphs on $2m (m \geq 2)$ vertices with perfect matchings. In this paper, we characterize the bicyclic graphs with minimal number of matchings and maximal number of independent sets in $\mathcal{B}(2m,m)$.

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1. Introduction

Let $G = (V,E)$ be a simple connected graph. Two edges of $G$ are said to be independent if they are not adjacent in $G$. A $k$-matching of $G$ is a set of $k$ mutually independent edges. Denote by $z(G)$ the total number of matchings in a graph $G$, that is, $z(G) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} z(G,k)$, where $z(G,k)$ is the number of $k$-matchings of $G$ for $k \geq 1$ and $z(G,0) = 1$. Two vertices of $G$ are said to be independent if they are not adjacent in $G$. An independent $k$-set is a set of $k$ vertices, no two of which are adjacent. Let $i(G)$ be the total number of independent sets of $G$, then $i(G) = \sum_{k=0}^{n} i(G,k)$, where $i(G,k)$ is the number of $k$-independent sets of $G$ for $k \geq 1$ and $i(G,0) = 1$.

The index $z(G)$ (resp. $i(G)$) is also called Hosoya index (resp. Merrifield-Simmons index) in graphic chemistry. It turned out to be applicable to several questions of molecular chemistry, for example, the connections with physico-chemical properties such as boiling point, entropy or heat of vaporization are well studied [8, 20]. Up to now, many researchers have investigated these graphic invariants. An important direction is to determine the graphs with maximal or minimal number of matchings (or independent sets, resp.) in a given class of graphs. For instance, it was observed in [9, 15] that the star $S_n$ has the minimal number of matchings (or the maximal number of independent sets, resp.) and the path $P_n$ has the maximal number of matchings (or the minimal number of independent sets, resp.) amongst all trees with $n$ vertices, respectively. In [17], Liu et al. studied trees with a prescribed diameter with respect to the number of matchings and independent sets, respectively. Hou [12] characterized the trees with a given size of matching and having minimal and second minimal number of matchings, respectively. In [3], Deng and Chen gave the sharp lower bound.
on the number of matchings of unicyclic graphs. Ou [16] characterized extremal unicyclic molecular graphs with maximal number of matchings. In [14], Li and one of the present authors studied the number of independent sets in unicyclic graphs with a given diameter. Wang and Hua [22] characterized the extremal (maximal and minimal) number of independent sets of unicyclic graphs with a given girth. Xu and Xu [27] determined all the unicyclic graphs of order \( n \) and with given maximum degree maximizing the number of matchings and minimizing the number of independent sets, respectively. Also \( n \)-vertex bicyclic graphs have been the object of study of a series of articles by Deng and coauthors [4, 5, 6, 7]. In particular, Yu and Tian [28] characterized the extremal graphs with minimal number of matchings and maximal number of independent sets, respectively, among all the connected graphs of order \( n \) and size \( n + t - 1 \) with \( 0 \leq t \leq m - 1 \), where \( m \) is the matching number. For further details, we refer readers to survey papers [10, 11, 19, 21, 23, 25, 26, 29, 30], especially, a recent paper by S. Wagner and I. Gutman [24], which is a wonderful survey on this topic, and the cited references therein.

Let \( \mathcal{B}(2m, m) \) be the set of all bicyclic graphs on \( 2m(m \geq 2) \) vertices with perfect matchings. In this paper, we consider the bicyclic graphs with minimal number of matchings and maximal number of independent sets, respectively, in \( \mathcal{B}(2m, m) \).

In order to state our results, we introduce some notation and terminology. For other undefined notation we refer to Bollobás [1]. For a vertex \( v \) of \( G \), denote the degree of \( v \) by \( d_G(v) \). Set \( N_G(v) = \{ u | uv \in E(G) \} \), \( N_G[v] = N_G(v) \cup \{ v \} \). If \( W \subset V(G) \), we denote by \( G - W \) the subgraph of \( G \) obtained by deleting the vertices of \( W \) and the edges incident with them. Similarly, if \( E \subset E(G) \), we denote by \( G - E \) the subgraph of \( G \) obtained by deleting the edges of \( E \). If \( W = \{ v \} \) and \( E = \{ xy \} \), we write \( G - v \) and \( G - xy \) instead of \( G - \{ v \} \) and \( G - \{ xy \} \), respectively. Denote by \( F_n \) the \( n \)th Fibonacci number. Recall that \( F_n = F_{n-1} + F_{n-2}, n \geq 2 \) with initial conditions \( F_0 = F_1 = 1 \). Then \( i(P_n) = F_{n+1}, z(P_n) = F_n \).

Now we give some lemmas that will be used in the proof of our main results.

**Lemma 1.1.** [9] Let \( G = (V, E) \) be a graph.

(i) If \( uv \in E(G) \), then \( z(G) = z(G - uv) + z(G - \{ u, v \}) \);

(ii) If \( v \in V(G) \), then \( z(G) = z(G - v) + \sum_{u \in N_G(v)} z(G - \{ u, v \}) \);

(iii) If \( G_1, G_2, \ldots, G_t \) are the components of the graph \( G \), then \( z(G) = \prod_{j=1}^{t} z(G_j) \).

**Lemma 1.2.** [9] Let \( G = (V, E) \) be a graph.

(i) If \( uv \in E(G) \), then \( i(G) = i(G - uv) - i(G - N_G[u] \cup N_G[v]) \);

(ii) If \( v \in V(G) \), then \( i(G) = i(G - v) + i(G - N_G[v]) \);

(iii) If \( G_1, G_2, \ldots, G_t \) are the components of the graph \( G \), then \( i(G) = \prod_{j=1}^{t} i(G_j) \).

**Lemma 1.3.** [18] Let \( H, X, Y \) be three connected graphs disjoint in pair. Suppose that \( u, v \) are two vertices of \( H \), \( v' \) is a vertex of \( X \), \( u' \) is a vertex of \( Y \). Let \( G \) be the graph obtained from \( H, X, Y \) by identifying \( v \) with \( v' \) and \( u \) with \( u' \), respectively. Let \( G_1^* \) be the graph obtained from \( H, X, Y \) by identifying vertices \( v, v', u' \) and \( G_2^* \) be the graph obtained from \( H, X, Y \) by identifying vertices \( u, v', u' \). Then

(i) \( z(G_1^*) < z(G) \) or \( z(G_2^*) < z(G) \);

(ii) \( i(G_1^*) > i(G) \) or \( i(G_2^*) > i(G) \).

Let \( G \) consist of connected graph \( G_1 \) and a pendent tree \( T \), where \( G_1 \cap T = r \). Vertex \( r \) is called the root of \( T \) on \( G_1 \) and \( T \) is named the attaching tree to \( G_1 \) rooted at \( r \). Denote by \( |V(T)| \) the order of \( T \) not including the root \( r \) of \( T \).
Lemma 2.1. [2] Let $G$ be a connected graph with perfect matchings which consists of a connected subgraph $H$ and a tree $T$ such that $T$ is attached to a root-vertex $r$ of $H$. If $|V(T)| \geq 2$ and $v \in V(T)$ is a vertex furthest from the root $r$. Then $v$ is a pendent vertex and adjacent to a vertex $u$ of degree 2.

2. Preliminaries

Hoffman and Smith [13] define an internal path of $G$ as a walk $u_0u_1 \ldots u_s(s \geq 1)$ such that the vertices $u_0, u_1, \ldots, u_{s-1}$ are distinct, $d(u_0) > 2, d(u_s) > 2$, and $d(u_i) = 2$, whenever $0 < i < s$. An internal path is closed if $u_0 = u_s$.

**Transformation A** Let $G \in \mathcal{B}(2m,m)$, $P = v_0v_1 \ldots v_3$ be an internal path of $G$. If $s = 2$ and $v_0v_2 \notin E(G)$, joining $v_0$ and $v_2$ by an edge in $G - v_1$, the resulting graph is denoted by $H'$; then, attaching a pendent edge $v_0v_1$ to $v_0$ in $H'$ if $v_0v_1$ belongs to the perfect matchings of $G$, and a pendent edge $v_2v_1$ to $v_2$ if $v_1v_2$ belongs to the perfect matchings of $G$. The resulting graph is denoted by $H''$. If $s \geq 3$, $v_0 \neq v_3$ and $v_0v_3 \notin E(G)$, joining $v_0$ and $v_3$ by an edge in $G - \{v_1,v_2\}$, the resulting graph is denoted by $G''$; then, attaching a path of length 2 to $v_0$ in $G''$, denote the path by $v_0v_1v_2$. The resulting graph is denoted by $G'$.

**Lemma 2.1.** Let $G \in \mathcal{B}(2m,m)$, $P = v_0v_1 \ldots v_3$ be an internal path of $G$. $H''$, $G''$ be graphs as described in Transformation A.

(i) If $s = 2$ and $v_0v_2 \notin E(G)$, $z(G) > z(H'')$ and $i(G) < i(H'')$;

(ii) If $s \geq 3$, $v_0 \neq v_3$ and $v_0v_3 \notin E(G)$, $z(G) > z(G'')$ and $i(G) < i(G'')$.

**Proof.** (i) Without loss of generality, let $v_0v_1$ belong to the perfect matchings of $G$. By Lemma 1.1 and Lemma 1.2, we have

$$z(G) = z(G - v_1v_2) + z(G - \{v_1,v_2\}),$$

$$z(H'') = z(H'' - v_0v_2) + z(H'' - v_0v_2) = z(H'' - v_0v_2) + z(H'' - \{v_0,v_1,v_2\});$$

$$i(G) = i(G - v_0) + i(G - N_G[v_0]),$$

$$i(H'') = i(H'' - v_0) + i(H'' - N_{H''}[v_0]).$$

Note that

$$G - v_1v_2 \cong H'' - v_0v_2, \quad H'' - \{v_0,v_1,v_2\} \subset G - \{v_1,v_2\},$$

$$G - v_0 \cong H'' - v_0, \quad H'' - N_{H''}[v_0] \subset G - N_G[v_0].$$

Note that $H'' - N_{H''}[v_0]$ and $G - N_G[v_0]$ have the same order. Then

$$z(G - v_1v_2) = z(H'' - v_0v_2), \quad z(G - \{v_1,v_2\}) = z(H'' - \{v_0,v_1,v_2\});$$

$$i(G - v_0) = i(H'' - v_0), \quad i(G - N_G[v_0]) = i(H'' - N_{H''}[v_0]).$$

Hence $z(G) > z(H'')$ and $i(G) < i(H'')$.

(ii) By Lemma 1.1 and Lemma 1.2, we have

$$z(G) = z(G - v_2v_3) + z(G - \{v_2,v_3\}) = z(G - v_2v_3) + z(G - \{v_1,v_2,v_3\})$$

$$z(G'') = z(G'' - v_0v_3) + z(G'' - v_0v_3) = z(G'' - v_0v_3) + 2z(G'' - \{v_0,v_1,v_2,v_3\});$$

$$i(G) = i(G - v_3) + i(G - N_G[v_3]) = i(G - v_3) + i(G - N_G[v_3] \cup N_G[v_0]).$$
Hence so Lemma 1.1, we have

\[ i(G'') = i(G'' - v_3) + i(G'' - N_{G''}[v_3]) = i(G'' - v_3) + 3i(G'' - N_{G''}[v_3] \cup \{v_1, v_2\}). \]

Note that

\[ G - v_2v_3 \cong G'' - v_0v_3, G - \{v_0, v_1, v_2, v_3\} \cong G'' - \{v_0, v_1, v_2, v_3\}; \]

\[ G - v_3 \cong G'' - v_3, G - N_{G''}[v_3] \cup \{v_0, v_1\} \cong G'' - N_{G''}[v_3] \cup \{v_1, v_2\}; \]

\[ G - N_{G''}[v_3] \cup N_G[v_0] \subset G'' - N_{G''}[v_3] \cup \{v_1, v_2\} \]

Hence \( z(G) > z(G''), i(G) < i(G''). \)

Let \( \hat{G} \) be a graph on \( m \) vertices, attach a pendent edge at each vertex of \( \hat{G} \), denote the resulted graph by \( C(\hat{G}) \). Obviously, \( C(\hat{G}) \) has an unique perfect matchings which consists of all pendent edges. Contracting each edge of the matching in \( C(\hat{G}) \) yields the graph \( \hat{G} \) on \( m \) vertices. We call the graph \( \hat{G} \) the contracted graph of the graph \( C(\hat{G}) \).

**Lemma 2.2.** Let \( \hat{G} \) be the contracted graph of \( G \), if there exists either an internal path of length no less than 2 or a closed internal path of length no less than 4 in \( \hat{G} \), then there exists a connected graph \( G' \) with a path of length 2 attached such that \( G' = C(\hat{G}') \) for some \( \hat{G}' \) and \( z(G) > z(G''), i(G) < i(G'') \).

**Proof.** Let \( P = v_0v_1 \ldots v_s \) be an internal path of length no less than 2 or a closed internal path of length no less than 4 in \( \hat{G} \), and \( v'_0, v'_1, \ldots, v'_s \) the pendent vertices corresponding to \( v_0, v_1, \ldots, v_s \) in \( G \), respectively. Denote by \( H \) the graph obtained from \( G - v_1v_2 \) by joining \( v_0, v_2 \) with an edge.

**Case 1.** If \( P = v_0v_1 \ldots v_s \) is a closed internal path of length no less than 4 in \( \hat{G} \). By Lemma 1.1, we have

\[
z(G) = z(G - v_1v_2) + z(G - \{v_1, v_2\}) \\
  = z(G - v_1v_2) + z(G - \{v'_0, v_1, v_2\}) + z(G - \{v_0, v'_0, v_1, v_2\}) \\
  = z(G - v_1v_2) + z(G - \{v'_0, v_1, v'_1, v_2\}) + 2z(G - \{v_0, v'_0, v_1, v'_1, v_2\}) + z(G - \{v_0, v'_0, v_1, v'_1, v'_2\}) \\
  = z(G - v_1v_2) + z(G - \{v'_0, v_1, v'_1, v_2\}) + z(G - \{v_0, v'_0, v_1, v'_1, v'_2\}) \\
  = z(H - v_0v_2) + z(H - \{v_0, v_2\} \\
  = z(H - v_0v_2) + z(H - \{v_0, v_0, v_1, v'_1, v_2, v'_2\} \cup 2P_1 \cup P_2) \\
  = z(H - v_0v_2) + 2z(H - \{v_0, v'_0, v_1, v'_1, v'_2\}) \\
\]

and

\[
G - v_1v_2 \cong H - v_0v_2, G - \{v_0, v'_0, v_1, v'_1, v_2, v'_2\} \cong H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2\}, \\
H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2\} \subset G - \{v'_0, v_1, v'_1, v_2, v'_2\},
\]

so

\[
z(G - v_1v_2) = z(H - v_0v_2), z(G - \{v_0, v'_0, v_1, v'_1, v_2, v'_2\}) = z(H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2\}), \\
z(G - \{v_0, v'_0, v_1, v'_1, v'_2\}) > z(H - \{v_0, v'_0, v_1, v'_1, v'_2\}).
\]

Hence \( z(G) > z(H) \).
By Lemma 1.2, we have
\[ i(G) = i(G - v_2) + i(G - N_G[v_2]) \]
\[ = i(G - v_2) + i(G - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}) \]
\[ = i(G - v_2) + 4i(G - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}) \]
\[ = i(G - v_2) + 4i(G - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\} \cup N_G[v_0]) \]
\[ + 2i(G - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}), \]
\[ i(H) = i(H - v_2) + i(H - N_H[v_2]) \]
\[ = i(H - v_2) + i(H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}) \cup 2P_1 \cup P_2 \]
\[ = i(H - v_2) + 12i(H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}). \]

Note that
\[ G - v_2 \cong H - v_2, G - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\} \cong H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}, \]
\[ G - \{v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\} \cup N_G[v_0] \subset H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}, \]
then
\[ i(G - v_2) = i(H - v_2), i(G - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}) = i(H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}), \]
\[ i(G - \{v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\} \cup N_G[v_0]) < i(H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2, v_3, v'_3\}). \]

Hence \( i(G) < i(H) \).

**Case 2.** If \( P = v_0 v'_1 \ldots v_s \) is an internal path of length no less than 2 in \( \hat{G} \). By Lemma 1.1 and Lemma 1.2, we have
\[ z(G) = z(G - v_1 v_2) + z(G - \{v_1, v_2\}) = z(G - v_1 v_2) + z(G - \{v_1, v'_1, v_2, v'_2\}) \]
\[ = z(G - v_1 v_2) + z(G - \{v'_0, v_1, v'_1, v_2, v'_2\}) + z(G - \{v_0, v'_0, v_1, v'_1, v_2, v'_2\}), \]
\[ z(H) = z(H - v_0 v_2) + z(H - \{v_0, v_2\}) = z(H - v_0 v_2) + 2z(H - \{v_0, v_0, v_1, v'_1, v_2, v'_2\}); \]
\[ i(G) = i(G - v_2) + i(G - N_G[v_2]) = i(G - v_2) + 2i(G - N_G[v_2] \cup \{v'_1\}) \]
\[ = i(G - v_2) + 2i(G - N_G[v_2] \cup \{v_0, v'_0\}) + 2i(G - N_G[v_2] \cup \{v_0, v'_0, v'_1\}) \]
\[ = i(G - v_2) + 2i(G - N_G[v_2] \cup N_G[v_0] \cup \{v'_1\}) + 4i(G - N_G[v_2] \cup \{v_0, v'_0, v'_1\}) \]
\[ = i(H - v_2) + i(H - N_H[v_2]) = i(H - v_2) + 6i(H - N_H[v_2] \cup \{v_0, v_1, v'_1\}). \]

Note that
\[ G - v_1 v_2 \cong H - v_0 v_2, G - \{v'_0, v_1, v'_1, v_2, v'_2\} \cong H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2\}; \]
\[ H - \{v_0, v'_0, v_1, v'_1, v_2, v'_2\} \subset G - \{v'_0, v_1, v'_1, v_2, v'_2\}; \]
\[ G - v_2 \cong H - v_2, G - N_G[v_2] \cup \{v_0, v'_0, v'_1\} \cong H - N_H[v_2] \cup \{v_0, v_1, v'_1\}, \]
\[ G - N_G[v_2] \cup N_G[v_0] \cup \{v'_1\} \subset H - N_H[v_2] \cup \{v_0, v_1, v'_1\}. \]

Then \( z(G) > z(H), i(G) < i(H) \).

Select \( H = \hat{G} \), then we obtain our desirable results. \( \square \)
3. Main results

Let $G$ be a bicyclic graph. The base of $G$, denoted by $B(G)$, is the minimal bicyclic subgraph of $G$. Obviously, $B(G)$ is the unique bicyclic subgraph of $G$ containing no pendant vertex, and $G$ can be obtained from $B(G)$ by planting trees to some vertices of $B(G)$. It is well known that bicyclic graphs have the following two types of bases: $B(p, l, q)$ and $P(p, q, r)$, where $B(p, l, q)$ is the graph obtained by joining a new path $u_1u_2\ldots u_l$ between two cycles $C_p$ and $C_q$ with $u_1 \in V(C_p), u_l \in V(C_q)$, and $P(p, q, r)$ is the bicyclic graph consisting of three pairwise internal disjoint paths $P_{p+1}, P_{q+1}, P_{r+1}$ with common endpoints $u, v$. Let $B_1(2m) = \{ G \in \mathcal{B}(2m, m) | B(G) = B(p, l, q), p \leq q \}; B_2(2m) = \{ G \in \mathcal{B}(2m, m) | B(G) = P(p, q, r) \}$. Then $\mathcal{B}(2m) = B_1(2m) \cup B_2(2m)$.

![Figure 1. The graph $B_{2m}$.](image)

**Lemma 3.1.** Let $B_{2m}$ be graph of the form in Figure 1. Then $z(B_{2m}) = 4 \cdot 2^{m-1} + (m - 3) \cdot 2^{m-2}$ and $i(B_{2m}) = 2 \cdot 3^{m-1} + 2^{m-3}$.

**Proof.** By Lemma 1.1 and Lemma 1.2, we have

\[
z(B_{2m}) = z(B_{2m} - u) + \sum_{v \in NB_{2m}(u)} z(B_{2m} - \{u, v\}) = z((m - 1)P_2 \cup P_1) + z((m - 1)P_2 + 4z((m - 2)P_2 \cup 2P_1) + (m - 3)z((m - 2)P_2 \cup 2P_1) = 4 \cdot 2^{m-1} + (m - 3) \cdot 2^{m-2},
\]

\[
i(B_{2m}) = i(B_{2m} - u) + i(B_{2m} - NB_{2m}[u]) = i((m - 1)P_2 \cup P_1) + i((m - 3)P_1)
\]

\[= 2 \cdot 3^{m-1} + 2^{m-3}.
\]

Let $G_1, G_2, \ldots, G_{14}$ be graphs of the form in Figure 2, by direct calculation, we have

\[
z(G_1) = 20, z(G_2) = 16, z(G_3) = 38, z(G_4) = 52, z(G_5) = 20;
\]

\[
z(G_6) = 45, z(G_7) = 46, z(G_8) = 42, z(G_9) = 250, z(G_{10}) = 40;
\]

\[
z(G_{11}) = 94, z(G_{12}) = 99, z(G_{13}) = 142, z(G_{14}) = 143.
\]

(3.1)

And

\[
i(G_1) = 17, i(G_2) = 19, i(G_3) = 52, i(G_4) = 48, i(G_5) = 15;
\]

\[
i(G_6) = 45, i(G_7) = 44, i(G_8) = 47, i(G_9) = 384, i(G_{10}) = 48;
\]

(3.2)

\[
i(G_{11}) = 136, i(G_{12}) = 132, i(G_{13}) = 128, i(G_{14}) = 128.
\]
Theorem 3.1. Let \( G \) be a graph in \( B_1(2m) \), \( m \geq 3 \). Then \( z(G) \geq z(B_{2m}) \) and \( i(G) \leq i(B_{2m}) \), the equalities hold if and only if \( G \cong B_{2m} \).

Proof. When \( m = 3 \), \( B_1(2m) = \{ G_1, G_2, G_3, \tilde{B}(3, 1, 4) \} \). By direct calculation, \( z(\tilde{B}(3, 1, 4)) = 20, i(\tilde{B}(3, 1, 4)) = 17 \), combining (3.1) and (3.2), we have \( z(G) \geq z(G_2) = z(B_{2m}), i(G) \leq i(G_2) = i(B_{2m}) \).

Now we suppose \( m \geq 4 \). Let \( G \in B_1(2m) \).

Case 1. If \( G \) has a pendent vertex \( v' \) with its adjacent vertex \( u' \) of degree 2. Let \( N_G(u') = \{ v', r \} \). By Lemmas 1.1 and 1.2, we have

\[
z(G) = z(G - v') + z(G - \{ v', u' \}) = z(G - \{ v', u', r \}) + 2z(G - \{ v', u' \}),
\]

\[
z(B_{2m}) = z(B_{2m} - v') + z(B_{2m} - \{ v', u' \}) = z(B_{2m} - \{ v', u', u' \}) + 2z(B_{2m} - \{ v', u' \})
\]

\[
= z(K_1 \cup (m-2)K_2) + 2z(B_{2m} - \{ v', u' \});
\]

\[
i(G) = i(G - v') + i(G - \{ v', u' \}) = i(G - \{ v', u', r \}) + 2i(G - \{ v', u' \}),
\]

\[
i(B_{2m}) = i(B_{2m} - v') + i(B_{2m} - \{ v', u' \}) = i(B_{2m} - \{ v', u', u' \}) + 2i(B_{2m} - \{ v', u' \})
\]

\[
= i(K_1 \cup (m-2)K_2) + 2i(B_{2m} - \{ v', u' \}).
\]

Since \( G - \{ v', u', r \} \) is a graph on \( 2m - 3 \) vertices with \( (m-2) \)-matching, \( K_1 \cup (m-2)K_2 \) is a spanning subgraph of \( G - \{ v', u', r \} \) when \( G \not\cong B_{2m} \), then \( z(G - \{ v', u', r \}) > z(K_1 \cup (m-2)K_2) \), \( i(G - \{ v', u', r \}) < i(K_1 \cup (m-2)K_2) \). Since \( G - \{ v', u' \} \) is a graph on \( 2m - 2 \) vertices with perfect matching, by induction hypothesis, we have \( z(G - \{ v', u' \}) > z(B_{2m} - \{ v', u' \}) \), \( i(G - \{ v', u' \}) < i(B_{2m} - \{ v', u' \}) \).

Then \( z(G) \geq z(B_{2m}), i(G) \leq i(B_{2m}) \).
Case 2. If $G$ has not a pendent vertex $v'$ with its adjacent vertex $u'$ of degree 2. Then $G$ can be obtained from $B(p,l,q)$ by attaching some pendent edges at some vertices of $B(p,l,q)$. In fact, if there is a vertex $u \in V(B(p,l,q))$ attaching a tree $T$ with $|V(T)| \geq 2$, by Lemma 1.4, it is contradict to the choice of $G$. Let $P = v_0v_1 \ldots v_s$ be the longest internal path in $G$.

Subcase 2.1. $s = 1$.

When $d(u_1) \geq 3$, then $G \cong C(B(p,l,q))$. If there exists either an internal path of length no less than 2 or a closed internal path of length no less than 4 in $B(p,l,q)$, by Lemma 2.2 and Case 1, we have the desired results. Otherwise, $B(p,l,q) \cong B(3,1,3)$ or $B(3,2,3)$, then

$$G \in \{C(B(3,1,3)), C(B(3,2,3))\}.$$ 

By direct calculation, we have

$$z(C(B(3,1,3))) = 90, z(C(B(3,2,3))) = 221; i(C(B(3,1,3))) = 144, i(C(B(3,2,3))) = 224.$$ 

By Lemma 3.1,

$$z(B_{10}) = 52, z(B_{12}) = 76; i(B_{10}) = 166, i(B_{12}) = 494.$$ 

Hence, we also have the desired results.

When $d(u_1) = 3$, then $l = 2$ and $d(u) = 3$ for any $u \in V(B(G))$. Let $G'$ be the graph obtained form $G - u_1u_2$ by identifying $u_1$ with $u_2$ and adding a pendent edge at $u_1$, obviously, $G' \in B_1(2m)$ and $G' \cong C(B(p,l - 1, q))$. By Lemma 1.3, we have $z(G) > z(G')$ and $i(G) < i(G')$, as above discussion, we have $z(G') > z(B_{2m})$ and $i(G') < i(B_{2m})$. Then we obtain the desired results.

Subcase 2.2. $s = 2$, then at least one of $v_0, v_2$ must be in $\{u_1, u_l\}$. Otherwise, $v_1$ must be an unmatched vertex, a contradiction. Then there are at most two internal paths of length 2.

Subcase 2.2.1. $G$ has only one internal path of length 2.

When $v_0v_2 \notin E(G)$, by Lemma 2.1, we can obtain a connected graph $G'$ such that $G' \cong C(B(p,l',q))$, where $l' \leq l$. By Subcase 2.1., we have the desired results.

When $v_0v_2 \in E(G)$, then $B(G) \cong B(3,1,q)$. Without loss of generality, let $v_0 = u_1$. If $l \geq 3$, let $G'$ be the graph obtained from $G$ by deleting $u_2u_3$ and adding $u_1u_3$. Set $G - u_2u_3 = A \cup D$, where $u_2 \in V(A), u_3 \in V(D)$. By Lemmas 1.1 and 1.2, we have

$$z(G) = z(G - u_2u_3) + z(G - \{u_2,u_3\}) = z(A \cup D) + z((A - u_2) \cup (D - u_3))$$

$$= z(A \cup D) + 6z(D - u_3);$$

$$z(G') = z(G' - u_1u_3) + z(G' - \{u_1,u_3\}) = z(A \cup D) + z((A - u_1) \cup (D - u_3))$$

$$= z(A \cup D) + 3z(D - u_3);$$

$$i(G) = i(G - u_3) + i(G - N_G[u_3]) = i(A \cup (D - u_3)) + i((A - u_2) \cup (D - N_D[u_3]))$$

$$= i(A \cup (D - u_3)) + 14i(D - N_D[u_3]),$$

$$i(G') = i(A \cup (D - u_3)) + 15i(D - N_D[u_3]).$$

Then $z(G) > z(G')$ and $i(G) < i(G')$ and there is a pendent vertex $v'$ with its adjacent vertex $u_2$ of degree 2 in $G'$. Similarly, if $q \geq 4$, we also can find a graph $G'$ which satisfy $z(G) > z(G')$ and $i(G) < i(G')$ and there is a pendent vertex $v'$ with its adjacent vertex $u'$ of degree 2 in $G'$. By Case 1, $z(G') > z(B_{2m})$ and $i(G') < i(B_{2m})$. Then we obtain the desired results. If $l \leq 2$ and $q = 3$, then $G \in \{G_4, G_{12}\}$. By (3.1), (3.2) and Lemma 3.1, we have the desired results.
Subcase 2.2.2. G have two internal paths of length 2, let \( P = v_0v_1v_2 \) and \( P' = v_0'v_1'v_2' \) be the two paths.

When at least one of \( v_0v_2, v_0'v_2' \notin E(G) \), by Lemma 2.1, we can obtain a graph \( G' \) such that \( z(G) > z(G') \) and \( i(G) < i(G') \), and \( G' \) has one internal path of length 2, by Subcase 2.2.1., we have the desired results.

When all of \( v_0v_2, v_0'v_2' \in E(G) \), then \( B(G) \cong B(3, l, 3) \). If \( l \geq 3 \), similar to Subcase 2.2.1., we can obtain the desired results. If \( l \leq 2 \), then \( G \cong G_6 \). By (3.1), (3.2) and Lemma 3.1, we have the desired results.

Subcase 2.3. \( s = 3 \).

If there exists an internal path \( P = v_0v_1v_2v_3 \) with \( v_0v_3 \notin E(G), v_0 \neq v_3 \). By Transformation A and Lemma 2.1, we can find a graph \( G' \) such that \( z(G) > z(G') \), \( i(G) < i(G') \), and there is a pendant vertex \( v' \) with its adjacent vertex \( u' \) of degree 2 in \( G' \). By Case 1, we have \( z(G') \geq z(B_{2m}), i(G') \leq i(B_{2m}) \), as desired. Otherwise, any internal path \( P = v_0v_1v_2v_3 \in G \), it has \( v_0v_3 \in E(G) \) or \( v_0 = v_3 \).

Subcase 2.3.1. Any internal path \( P = v_0v_1v_2v_3 \) in \( G \), it has \( v_0v_3 \in E(G) \). Obviously, there are at most two such internal paths.

When there are two such internal paths in \( G \), then \( B(G) \cong B(4, l, 4) \). Further \( l \geq 2 \), otherwise \( G \notin B_1(2m) \). If \( d(u_1) = 3 \), then \( u_1u_2 \) must be an matching edge and \( d(u_2) = 2, d(u_3) \geq 3 \). Let \( G' \) be the graph obtained from \( G \) by deleting \( u_2u_3 \) and adding \( u_1u_3 \). Similar to the procedure of (3.4), we have \( z(G) > z(G') \) and \( i(G) < i(G') \). To find the extremal graph, we can set \( d(u_1), d(u_i) \geq 4 \). Then we have \( d(u_i) = 3 \) for \( i = 2, \ldots, l - 1 \), otherwise, it must have another internal path \( P' = v_0'v_1'v_2'v_3' \) with \( v_0'v_3' \notin E(G) \), a contradiction. If \( l \geq 3 \), similar to the discussion of Subcase 2.2.1., we have the desired results. For \( l = 2 \), \( G \cong G_9 \), by (3.1-3.3), we have the desired results.

When there is only one such internal path in \( G \), then \( B(G) \cong B(4, l, q) \). If \( l \geq 3 \) or \( q \geq 4 \), similar to the discussion of Subcase 2.2.1., we have the desired results. If \( l \leq 2 \) and \( q = 3, G \in \{G_{13}, G_{14}\} \), by (3.1-3.3), we have the desired results.

Subcase 2.3.2. Any internal path \( P = v_0v_1v_2v_3 \) in \( G \), it has \( v_0 = v_3 \). Obviously, there are at most two such internal paths.

When there are two such internal paths in \( G \), then \( B(G) \cong B(3, l, 3) \). If \( l \geq 3 \), similar to the discussion of Subcase 2.2.1., we have the desired results. If \( l \leq 2 \), \( G \in \{G_2, G_5, G_{10}\} \), by (3.1–3.3), we have the desired results.

When there is only one such internal path in \( G \), then \( B(G) \cong B(3, l, q) \). If \( l \geq 3 \) or \( q \geq 4 \), similar to the discussion of Subcase 2.2.1., we have the desired results. If \( l \leq 2 \) and \( q = 3, G \in \{G_1, G_3, G_7, G_8, G_{11}\} \), by (3.1–3.3), we have the desired results.

Subcase 2.4. \( s \geq 4 \). By Translation A, Lemma 2.1 and Case 1, we have the desired results.

This completes the proof.

Let \( W_1, W_2, \ldots, W_{12} \) be graphs of the form in Figure 3, by direct calculation, we have

\[
\begin{align*}
z(W_1) &= 22, z(W_2) = 20, z(W_3) = 20, z(W_4) = 19, z(W_5) = 24, z(W_6) = 18, \\
z(W_7) &= 19, z(W_8) = 26, z(W_9) = 21, z(W_{10}) = 46, z(W_{11}) = 108, z(W_{12}) = 44.
\end{align*}
\]

And

\[
\begin{align*}
i(W_1) &= 17, i(W_2) = 18, i(W_3) = 16, i(W_4) = 12, i(W_5) = 17, i(W_6) = 18, \\
i(W_7) &= 17, i(W_8) = 17, i(W_9) = 16, i(W_{10}) = 52, i(W_{11}) = 136, i(W_{12}) = 48.
\end{align*}
\]
**Theorem 3.2.** Let $G$ be a graph in $B_2(2m), m \geq 3$. Then $z(G) > z(B_{2m})$ and $i(G) < i(B_{2m})$.

**Proof.** When $m = 3$, $B_1(2m) = \{G_1, G_2, G_5, \hat{B}(3,1,4)\}$, $B_2(2m) = \{W_1, W_2, \ldots, W_9\}$. By (3.5), (3.6), Lemma 3.1 and Theorem 3.2, we have $z(G) > z(B_{2m}), i(G) < i(B_{2m})$.

We now suppose $m \geq 4$. For any graph $G \in B_2(2m)$, $B(G) \cong P(p, q, r)$. For convenience, let $q \leq r \leq p$.

**Case 1.** $G$ has a pendent vertex $v'$ with its adjacent vertex $u'$ of degree 2. Similar to the proof of Case 1 in Theorem 3.2, we have the desired results.

**Case 2.** $G$ hasn’t a pendent vertex $v'$ with its adjacent vertex $u'$ of degree 2 and $(p, q, r) \neq (2,1,2)$. Then $G$ can be obtained from $P(p, q, r)$ by attaching some pendent edges at some vertices of $P(p, q, r)$. Let $P_{p+1} = uu_1u_2 \ldots u_{p-1}v$, $u_i'$ be the pendent vertex which is adjacent to $u_i (i = 1, 2, \ldots, p-1)$, respectively, and $P = v_0v_1\ldots v_5$ be the longest internal path in $G$.

**Subcase 2.1.** $s = 1$. If $d_G(u) = 4, G \cong C(P(p, q, r))$. By Lemma 2.2 and Case 1, we have the desired results. If $d_G(u) = 3$, then $d_G(v) = 3, q = 1$ and $p \geq 3$.

If $p \geq 4$, let $G'$ be the graph obtained from $G$ by deleting $u_2u_3$ and adding $u_1u_3$. By Lemmas 1.1 and 1.2, we have

$$z(G) = z(G - u_2u_3) + z(G - \{u_2, u_3\}) = z(G - u_2u_3) + z(G - \{u_2, u'_2, u_3, u'_3\})$$

$$= z(G - u_2u_3) + z(G - \{u'_1, u_2, u'_2, u_3, u'_3\}) + z(G - \{u_1, u'_1, u_2, u'_2, u_3, u'_3\})$$

$$z(G') = z(G' - u_1u_3) + z(G' - \{u_1, u_3\}) = z(G' - u_1u_3) + 2z(G' - \{u_1, u'_1, u_2, u'_2, u_3, u'_3\}),$$

$$i(G) = i(G - u_3) + i(G - N_G[u_3]) = i(G - u_3) + 2i(G - N_G[u_3] \cup \{u'_2\})$$

$$= i(G - u_3) + 2[i(G - N_G[u_3] \cup \{u'_1, u'_2\}) + i(G - N_G[u_3] \cup \{u_1, u'_1, u'_2\})]$$

$$i(G') = i(G' - u_3) + i(G' - N_{G'}[u_3]) = i(G' - u_3) + 6i(G' - N_{G'}[u_3] \cup \{u'_1, u_2, u'_2\}).$$

Note that

$$G - u_2u_3 \cong G' - u_1u_3, G - \{u_1, u'_1, u_2, u'_2, u_3, u'_3\} \cong G' - \{u_1, u'_1, u_2, u'_2, u_3, u'_3\},$$

$$G' - \{u_1, u'_1, u_2, u'_2, u_3, u'_3\} \subset G - \{u'_1, u_2, u'_2, u_3, u'_3\};$$
Lemma 2.1 and Case 1, we have the desired results. Otherwise, any internal path

Then \( z(G) > z(G') \) and \( i(G) < i(G') \). By Case 1, we have the desired results.

If \( p = 3 \), then \( r \leq 3 \), and \( G \in \{W_1, W_2\} \). By (3.5), (3.6), Lemma 3.1 and Theorem 3.2, we have \( z(G) > z(B_{2m}), i(G) < i(B_{2m}) \).

Subcase 2.2. \( s = 2 \), then at least one of \( v_0, v_2 \) must be in \( \{ u, v \} \). Otherwise, \( v_1 \) must be an unmatched vertex, a contradiction. Then there are at most two internal paths of length 2.

Subcase 2.2.1. \( G \) has only one internal path of length 2.

When \( v_0v_2 \not\in E(G) \), by Lemma 2.1, we can obtain a connected graph \( G' \) such that \( G' \cong C(P(p', q', r')) \). By Subcase 2.1., we have the desired results.

When \( v_0v_2 \in E(G) \), then \( B(G) \cong P(p, 1, 2) \), where \( p \geq 3 \). Without loss of generality, let \( v_1v_2 \) be a matching edge and \( u = v_0, v = v_2 \), then \( d_G(u) = 4, d_G(v) = 3 \). Set \( u' \) be the pendant vertex which is adjacent to \( u \). Let \( G' \) be the graph obtained from \( G \) by deleting \( u_1u_2 \) and adding \( uu_2 \). Obviously, \( G' \) has a pendant vertex which is adjacent to a vertex of degree 2.

By Lemmas 1.1 and 1.2, we have

Then \( z(G) > z(G') \) and \( i(G) < i(G') \). By Case 1, we have the desired results.

Subcase 2.2.2. \( G \) has two internal paths of length 2, let \( P = v_0v_1v_2 \) and \( P' = v_0'v_1'v_2' \) be the two paths. Then at least one of \( v_0v_2, v_0'v_2' \) is not an edge of \( G \), by Lemma 2.1, we can obtain a graph \( G' \) such that \( z(G) > z(G') \) and \( i(G) < i(G') \), and \( G' \) has one internal path of length 2, by Subcase 2.2.1., we have the desired results.

Subcase 2.3. \( s = 3 \).

If there exists an internal path \( P = v_0v_1v_2v_3 \) with \( v_0v_3 \not\in E(G) \). By Transformation A, Lemma 2.1 and Case 1, we have the desired results. Otherwise, any internal path \( P = v_0v_1v_2v_3 \) in \( G \), it has \( v_0v_3 \in E(G) \). Obviously, there are at most two such internal paths.

When there are two such internal paths in \( G \), then \( B(G) \cong P(3, 1, 3) \). Then \( G \in \{W_8, W_{10}\} \). By (3.5), (3.6), Lemma 3.1 and Theorem 3.2, we have \( z(G) > z(B_{2m}), i(G) < i(B_{2m}) \).
When there is only one such internal path in \( G \), then \( B(G) \cong P(3, 1, q) \). Let \( G' \) be the graph obtained from \( G \) by deleting \( v_1 v_2 \) and adding \( uv_2 \). By Lemmas 1.1 and 1.2, we have

\[
\begin{align*}
z(G) &= z(G - v_1 v_2) + z(G - \{v_1, v_2\}), \\
z(G') &= z(G' - uv_2) + z(G' - \{u, v_2\}) = z(G' - uv_2) + z(G' - \{u, v_1, v_2\}), \\
i(G) &= i(G - v_2) + i(G - N_G[v_2]) = i(G - v_2) + i(G - \{v_1, v_2\}) \\
&= i(G - v_2) + i(G - \{u, v, v_1, v_2\}) + i(G - \{v, v_1, v_2\} \cup N_G[u]), \\
i(G') &= i(G' - v_2) + i(G' - N_G[v_2]) = i(G' - v_2) + 2i(G' - \{u, v, v_1, v_2\}).
\end{align*}
\]

Note that

\[
\begin{align*}
G - v_1 v_2 &\cong G' - uv_2, G' - \{u, v_1, v_2\} \subset G - \{v_1, v_2\}; \\
G - v_2 &\cong G' - v_2, G - \{u, v, v_1, v_2\} \cong G - \{u, v_1, v_2\}, \\
G - \{v, v_1, v_2\} \cup N_G[u] &\subset G' - \{u, v, v_1, v_2\}.
\end{align*}
\]

Then \( z(G) > z(G') \) and \( i(G) < i(G') \). Hence \( s = 2 \) in \( G' \), by Subcase 2.2., we have the desired results.

**Subcase 2.4.** \( s \geq 4 \). By Translation A, Lemma 2.1 and Case 1, we have the desired results.

**Case 3.** \( G \) hasn’t a pendent vertex \( v' \) with its adjacent vertex \( u' \) of degree 2 and \((p, q, r) = (2, 1, 2)\). Then \( G \in \{W_1, W_2, C(P(2, 1, 2))\} \). Note that \( z(C(P(2, 1, 2))) = 38, i(C(P(2, 1, 2))) = 52 \). By (3.5), (3.6), Lemma 3.1 and Theorem 3.2, we have \( z(G) > z(B_{2m}), i(G) < i(B_{2m}) \).

This completes the proof.

By Lemma 3.1, Theorem 3.2 and 3.3, we obtain our main results.

**Theorem 3.3.** Let \( G \) be a graph in \( \mathcal{B}(2m, m), m \geq 2 \).

(i) If \( m = 2 \), \( G \cong P(2, 1, 2), z(G) = 8, i(G) = 6 \);

(ii) If \( m \geq 3, z(G) \geq 4 \cdot 2^{m-1} + (m - 3) \cdot 2^{m-2} \) and \( i(G) \leq 2 \cdot 3^{m-1} + 2^{m-3} \), the equalities hold if and only if \( G \cong B_{2m} \).

References


