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Two-Point Boundary Value Problems for Fractional Differential Equations at Resonance

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Abstract. In this paper, by using the coincidence degree theory, we consider the following two-point boundary value problem for fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t,x(t),x'(t)), & t \in [0,1], \\ x(0) = 0,\ x'(0) = x'(1), \end{cases}$$

where D_{0+}^{α} denotes the Caputo fractional differential operator of order α, $1 < \alpha \leq 2$. A new result on the existence of solutions for above fractional boundary value problem is obtained.

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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be noninteger. This subject, as old as the problem of ordinary differential calculus, can go back to the times when Leibniz and Newton invented differential calculus. As is known to all, the problem for fractional derivative was originally raised by Leibniz in a letter, dated September 30, 1695. In recent years, the fractional differential equations have received more and more attention. The fractional derivative has been occurring in many physical applications such as a non-Markovian diffusion process with memory [17], charge transport in amorphous semiconductors [20], propagations of mechanical waves in viscoelastic media [15], etc. Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equations of fractional order (see [4, 6, 7, 14, 18, 19]).

Recently boundary value problems (BVPs for short) for fractional differential equations at nonresonance have been studied in many papers (see [1–3, 8, 9, 12, 13, 21]). Moreover, the BVPs for differential equations at resonance have also been studied in some papers (see [5, 10]). Motivated by the work above, in this paper, we consider the following BVP of

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fractional equation at resonance

$$(1.1)\qquad \begin{cases} D_{0^+}^{\alpha}x(t) = f(t,x(t),x'(t)), & t\in[0,1],\\ x(0)=0,\ x'(0)=x'(1), \end{cases}$$

where $D_{0^+}^{\alpha}$ denotes the Caputo fractional differential operator of order α, $1<\alpha\le 2$. $f:[0,1]\times\mathbb{R}^2\to\times\mathbb{R}$ is continuous.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, we establish a theorem on existence of solutions for BVP (1.1) under nonlinear growth restriction of f, basing on the coincidence degree theory due to Mawhin (see [16]). Finally, in Section 4, an example is given to illustrate the main result.

2. Preliminaries

In this section, we will introduce notations, definitions and preliminary facts which are used throughout this paper. Let X and Y be real Banach spaces and let $L:\operatorname{dom}L\subset X\to Y$ be a Fredholm operator with index zero, and $P:X\to X$, $Q:Y\to Y$ be projectors such that

$$\operatorname{Im}P=\operatorname{Ker}L,\quad \operatorname{Ker}Q=\operatorname{Im}L,\quad X=\operatorname{Ker}L\oplus\operatorname{Ker}P,\quad Y=\operatorname{Im}L\oplus\operatorname{Im}Q.$$

It follows that

$$L|_{\operatorname{dom}L\cap\operatorname{Ker}P}:\operatorname{dom}L\cap\operatorname{Ker}P\to\operatorname{Im}L$$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X, and $\operatorname{dom}L\cap\overline{\Omega}\neq\varnothing$, the map $N:X\to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N:\overline{\Omega}\to X$ is compact.

Lemma 2.1. [16] *Let $L:\operatorname{dom}L\subset X\to Y$ be a Fredholm operator of index zero and $N:X\to Y$ $L-$compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*

(1) $Lx\neq\lambda Nx$ *for every* $(x,\lambda)\in[(\operatorname{dom}L\setminus\operatorname{Ker}L)]\cap\partial\Omega\times(0,1)$;
(2) $Nx\notin\operatorname{Im}L$ *for every* $x\in\operatorname{Ker}L\cap\partial\Omega$;
(3) $\deg(QN|_{\operatorname{Ker}L},\operatorname{Ker}L\cap\Omega,0)\neq 0$, *where $Q:Y\to Y$ is a projection such that* $\operatorname{Im}L=\operatorname{Ker}Q$.

Then the equation $Lx=Nx$ has at least one solution in $\operatorname{dom}L\cap\overline{\Omega}$.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha>0$ of a function $x:(0,+\infty)\to\mathbb{R}$ is given by*

$$I_{0^+}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)ds,$$

provided that the right side integral is pointwise defined on $(0,+\infty)$.

Definition 2.2. *The Caputo fractional derivative of order $\alpha>0$ of a continuous function $x:(0,+\infty)\to\mathbb{R}$ is given by*

$$D_{0^+}^{\alpha}x(t)=I_{0^+}^{n-\alpha}\frac{d^nx(t)}{dt^n}=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}x^{(n)}(s)ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0,+\infty)$.

Lemma 2.2. [11] *For $\alpha > 0$, the general solution of the Caputo fractional differential equation*

$$D_{0+}^{\alpha}x(t) = 0$$

is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, here n is the smallest integer greater than or equal to α.

Lemma 2.3. [11] *Assume that $x \in C(0,1) \cap L(0,1)$ with a Caputo fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$, here n is the smallest integer greater than or equal to α.

In this paper, we denote $X = C^1[0,1]$ with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$ and $Y = C[0,1]$ with the norm $\|y\|_Y = \|y\|_\infty$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Obviously, both X and Y are Banach spaces. Define the operator $L : \mathrm{dom} L \subset X \to Y$ by

$$Lx = D_{0+}^{\alpha} x, \tag{2.1}$$

where

$$\mathrm{dom} L = \{x \in X | D_{0+}^{\alpha} x(t) \in Y,\ x(0) = 0, x'(0) = x'(1)\}.$$

Let $N : X \to Y$ be the Nemytskii operator

$$Nx(t) = f(t, x(t), x'(t)), \quad \forall t \in [0,1].$$

Then BVP (1.1) is equivalent to the operator equation

$$Lx = Nx, \quad x \in \mathrm{dom} L.$$

3. Main result

In this section, a theorem on existence of solutions for BVP (1.1) will be given.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that*

(H_1) *there exist nonnegative functions $p, q, r \in C[0,1]$ with $\Gamma(\alpha) - 2q_1 - 2r_1 >$ such that*

$$|f(t,u,v)| \le p(t) + q(t)|u| + r(t)|v|, \ \forall\, t \in [0,1],\ (u,v) \in \mathbb{R}^2,$$

where $p_1 = \|p\|_\infty$, $q_1 = \|q\|_\infty$, $r_1 = \|r\|_\infty$.

(H_2) *there exists a constant $B > 0$ such that for all $v \in \mathbb{R}$ with $|v| > B$ either*

$$vf(t,u,v) > 0, \quad \forall\, t \in [0,1],\ u \in \mathbb{R}$$

or

$$vf(t,u,v) < 0, \quad \forall\, t \in [0,1],\ u \in \mathbb{R}.$$

Then BVP (1.1) has at leat one solution in X.

Now, we begin with some lemmas below.

Lemma 3.1. *Let L be defined by (2.1), then*

$$\mathrm{Ker} L = \{x \in X | x(t) = c_1 t,\ c_1 \in \mathbb{R},\ \forall t \in [0,1]\}, \tag{3.1}$$

$$\mathrm{Im} L = \left\{ y \in Y | \int_0^1 (1-s)^{\alpha-2} y(s) ds = 0 \right\}. \tag{3.2}$$

Proof. By Lemma 2.2, $D^{\alpha}_{0+}x(t) = 0$ has solution

$$x(t) = c_0 + c_1 t, \quad c_0, c_1 \in \mathbb{R}.$$

Combining with the boundary value condition of BVP (1.1), one has (3.1) hold.

For $y \in \mathrm{Im}L$, there exists $x \in \mathrm{dom}L$ such that $y = Lx \in Y$. By Lemma 2.3, we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + c_0 + c_1 t.$$

Then, we have

$$x'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s)ds + c_1.$$

By conditions of BVP (1.1), we can get that y satisfies

$$\int_0^1 (1-s)^{\alpha-2} y(s)ds = 0.$$

Thus we get (3.2). On the other hand, suppose $y \in Y$ and satisfies $\int_0^1 (1-s)^{\alpha-2} y(s)ds = 0$. Let $x(t) = I^{\alpha}_{0+}y(t)$, then $x \in \mathrm{dom}\,L$ and $D^{\alpha}_{0+}x(t) = y(t)$. So that, $y \in \mathrm{Im}\,L$. ∎

Lemma 3.2. *Let L be defined by* (2.1)*, then L is a Fredholm operator of index zero, and the linear continuous projector operators $P : X \to X$ and $Q : Y \to Y$ can be defined as*

$$Px(t) = x'(0)t, \quad \forall t \in [0,1],$$

$$Qy(t) = (\alpha-1) \int_0^1 (1-s)^{\alpha-2} y(s)ds, \quad \forall t \in [0,1].$$

Furthermore, the operator $K_P : \mathrm{Im}\,L \to \mathrm{dom}\,L \cap \mathrm{Ker}\,P$ can be written by

$$K_P y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds, \quad \forall t \in [0,1].$$

Proof. Obviously, $\mathrm{Im}P = \mathrm{Ker}L$ and $P^2x = Px$. It follows from $x = (x - Px) + Px$ that $X = \mathrm{Ker}P + \mathrm{Ker}L$. By simple calculation, we can get that $\mathrm{Ker}L \cap \mathrm{Ker}P = \{0\}$. Then we get

$$X = \mathrm{Ker}L \oplus \mathrm{Ker}P.$$

For $y \in Y$, we have

$$Q^2 y = Q(Qy) = Qy(\alpha-1) \int_0^1 (1-s)^{\alpha-2} ds = Qy.$$

Let $y = (y - Qy) + Qy$, where $y - Qy \in \mathrm{Ker}Q = \mathrm{Im}L$, $Qy \in \mathrm{Im}Q$. It follows from $\mathrm{Ker}Q = \mathrm{Im}L$ and $Q^2y = Qy$ that $\mathrm{Im}Q \cap \mathrm{Im}L = \{0\}$. Then, we have

$$Y = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Thus

$$\dim \mathrm{Ker}L = \dim \mathrm{Im}Q = \mathrm{codim}\, \mathrm{Im}L = 1.$$

This means that L is a Fredholm operator of index zero.

From the definitions of P, K_P, it is easy to see that the generalized inverse of L is K_P. In fact, for $y \in \mathrm{Im}\,L$, we have

$$LK_P y = D^{\alpha}_{0+} I^{\alpha}_{0+} y = y. \tag{3.3}$$

Moreover, for $x \in \mathrm{dom}\,L \cap \mathrm{Ker}\,P$, we get $x'(0) = x(0) = 0$. By Lemma 2.3, we obtain that

$$I^{\alpha}_{0+} Lx(t) = I^{\alpha}_{0+} D^{\alpha}_{0+} x(t) = x(t) + c_0 + c_1 t, \quad c_0,\ c_1 \in \mathbb{R},$$

which together with $x'(0) = x(0) = 0$ yields that

$$K_P Lx = x. \tag{3.4}$$

Combining (3.3) with (3.4), we know that $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. ∎

Lemma 3.3. *Assume $\Omega \subset X$ is an open bounded subset such that $\operatorname{dom} L \cap \overline{\Omega} \neq \varnothing$, then N is L-compact on $\overline{\Omega}$.*

Proof. By the continuity of f, we can get that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. So, in view of the Arzelà-Ascoli theorem, we need only prove that $K_P(I-Q)N(\overline{\Omega}) \subset X$ is equicontinuous.

From the continuity of f, there exists constant $A > 0$ such that $|(I-Q)Nx| \le A$, $\forall x \in \overline{\Omega}, t \in [0,1]$. Furthermore, denote $K_{P,Q} = K_P(I-Q)N$ and for $0 \le t_1 < t_2 \le 1$, $x \in \overline{\Omega}$, we have

$$\begin{aligned}
&|(K_{P,Q}x)(t_2) - (K_{P,Q}x)(t_1)| \\
&\le \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2-s)^{\alpha-1}(I-Q)Nx(s)ds - \int_0^{t_1} (t_1-s)^{\alpha-1}(I-Q)Nx(s)ds \right| \\
&\le \frac{A}{\Gamma(\alpha)} \left[\int_0^{t_1} (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right] = \frac{A}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha)
\end{aligned}$$

and

$$\begin{aligned}
&|(K_{P,Q}x)'(t_2) - (K_{P,Q}x)'(t_1)| \\
&= \frac{\alpha-1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2-s)^{\alpha-2}(I-Q)Nx(s)ds - \int_0^{t_1} (t_1-s)^{\alpha-2}(I-Q)Nx(s)ds \right| \\
&\le \frac{A}{\Gamma(\alpha-1)} \left[\int_0^{t_1} (t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2} ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-2} ds \right] \\
&\le \frac{A}{\Gamma(\alpha)} [t_2^{\alpha-1} - t_1^{\alpha-1} + 2(t_2-t_1)^{\alpha-1}].
\end{aligned}$$

Since t^α and $t^{\alpha-1}$ are uniformly continuous on $[0,1]$, we can get that $K_{P,Q}(\overline{\Omega}) \subset C[0,1]$ and $(K_{P,Q})'(\overline{\Omega}) \subset C[0,1]$ are equicontinuous. Thus, we get that $K_{P,Q} : \overline{\Omega} \to X$ is compact. ∎

Lemma 3.4. *Suppose $(H_1), (H_2)$ hold, then the set*

$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L \mid Lx = \lambda Nx,\ \lambda \in (0,1)\}$$

is bounded.

Proof. Take $x \in \Omega_1$, then $Nx \in \operatorname{Im} L$. By (3.2), we have

$$\int_0^1 (1-s)^{\alpha-2} f(s, x(s), x'(s)) ds = 0.$$

Then, by the integral mean value theorem, there exists a constant $\xi \in (0,1)$ such that $f(\xi, x(\xi), x'(\xi)) = 0$. Then from (H_2), we have $|x'(\xi)| \le B$.

From $x \in \operatorname{dom} L$, we get $x(0) = 0$. Therefore

$$|x(t)| = \left| x(0) + \int_0^t x'(s) ds \right| \le \|x'\|_\infty.$$

That is

$$\|x\|_\infty \leq \|x'\|_\infty. \tag{3.5}$$

By $Lx = \lambda Nx$, we have

$$x(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s))ds + x(0) + x'(0)t, \quad t \in [0,1].$$

Then we get

$$x'(t) = \frac{\lambda}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, x(s), x'(s))ds + x'(0), \quad t \in [0,1].$$

Take $t = \xi$, we get

$$x'(\xi) = \frac{\lambda}{\Gamma(\alpha-1)} \int_0^\xi (\xi-s)^{\alpha-2} f(s, x(s), x'(s))ds + x'(0).$$

Together with $|x'(\xi)| \leq B$, (H_1) and (3.6), we have

$$\begin{aligned}
|x'(0)| &\leq |x'(\xi)| + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^\xi (\xi-s)^{\alpha-2} |f(s, x(s), x'(s))| ds \\
&\leq B + \frac{1}{\Gamma(\alpha-1)} \int_0^\eta (\xi-s)^{\alpha-2} [p(s) + q(s)|x(s)| + r(s)|x'(s)|] ds \\
&\leq B + \frac{1}{\Gamma(\alpha-1)} \int_0^\xi (\xi-s)^{\alpha-2} [p_1 + q_1 \|x\|_\infty + r_1 \|x'\|_\infty] ds \\
&\leq B + \frac{1}{\Gamma(\alpha-1)} \int_0^\xi (\xi-s)^{\alpha-2} [p_1 + (q_1 + r_1)\|x'\|_\infty] ds \\
&\leq B + \frac{1}{\Gamma(\alpha)} [p_1 + (q_1 + r_1)\|x'\|_\infty].
\end{aligned}$$

So, we have

$$\begin{aligned}
\|x'\|_\infty &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} |f(s, x(s), x'(s))| ds + |x'(0)| \\
&\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} [p(s) + q(s)|x(s)| + r(s)|x'(s)|] ds + |x'(0)| \\
&\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} [p_1 + q_1 \|x\|_\infty + r_1 \|x'\|_\infty] ds + |x'(0)| \\
&\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} [p_1 + (q_1 + r_1)\|x'\|_\infty] ds + |x'(0)| \\
&\leq B + \frac{2}{\Gamma(\alpha)} [p_1 + (q_1 + r_1)\|x'\|_\infty].
\end{aligned}$$

Thus, from $\Gamma(\alpha) - 2q_1 - 2r_1 > 0$, we obtain that

$$\|x'\|_\infty \leq \frac{B\Gamma(\alpha) + 2p_1}{\Gamma(\alpha) - 2q_1 - 2r_1} := M_1, \quad \text{and} \quad \|x\|_\infty \leq \|x'\|_\infty \leq M_1$$

Therefore, $\|x\|_X \leq M_1$. So Ω_1 is bounded. ∎

Lemma 3.5. *Suppose (H_2) holds, then the set*

$$\Omega_2 = \{x | x \in \operatorname{Ker} L,\ Nx \in \operatorname{Im} L\}$$

is bounded.

Proof. For $x \in \Omega_2$, we have $x(t) = ct,\ c \in \mathbb{R}$, and $Nx \in \operatorname{Im} L$. Then we get

$$\int_0^1 (1-s)^{\alpha-2} f(s, cs, c) ds = 0,$$

which together with (H_2) implies $|c| \le B$. Thus, we have $\|x\|_X \le B$. Hence, Ω_2 is bounded. ∎

Lemma 3.6. *Suppose the first part of (H_2) holds, then the set*

$$\Omega_3 = \{x | x \in \operatorname{Ker} L,\ \lambda x + (1-\lambda) QNx = 0,\ \lambda \in [0,1]\}$$

is bounded.

Proof. For $x \in \Omega_3$, we have $x(t) = ct,\ c \in \mathbb{R}$, and

$$\lambda ct + (1-\lambda)(\alpha-1) \int_0^1 (1-s)^{\alpha-2} f(s, cs, c) ds = 0. \tag{3.6}$$

If $\lambda = 0$, then $|c| \le B$ because of the first part of (H_2). If $\lambda \in (0,1]$, we can also obtain $|c| \le B$. Otherwise, if $|c| > B$, in view of the first part of (H_2), one has

$$\lambda c^2 t + (1-\lambda)(\alpha-1) \int_0^1 (1-s)^{\alpha-2} c f(s, cs, c) ds > 0,$$

which contradicts to (3.6). Therefore, Ω_3 is bounded. The proof is complete. ∎

Remark 3.1. Suppose the second part of (H_2) hold, then the set

$$\Omega_3' = \{x | x \in \operatorname{Ker} L,\ -\lambda x + (1-\lambda) QNx = 0,\ \lambda \in [0,1]\}$$

is bounded.

The proof of Theorem 3.1. Set $\Omega = \{x \in X | \|x\|_X < \max\{M_1, B\} + 1\}$. It follows from Lemma 3.2 and 3.3 that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By Lemma 3.4 and 3.5, we get that the following two conditions are satisfied

(1) $Lx \ne \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;
(2) $Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$.

Take

$$H(x, \lambda) = \pm\lambda x + (1-\lambda) QNx.$$

According to Lemma 3.6 (or Remark 3.1), we know that $H(x, \lambda) \ne 0$ for $x \in \operatorname{Ker} L \cap \partial\Omega$. Therefore

$$\begin{aligned}\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0) = \deg(H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0) \\ &= \deg(\pm I, \Omega \cap \operatorname{Ker} L, 0) \ne 0.\end{aligned}$$

So that, the condition (3) of Lemma 2.1 is satisfied. By Lemma 2.1, we can get that $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. Therefore the BVP (1.1) has at least one solution. ∎

4. An example

Example 4.1. Consider the following BVP

$$\begin{cases} D_{0+}^{\frac{3}{2}}x(t) = \frac{1}{4}(x'(t)-10) + \frac{t}{2}e^{-|x(t)|}, & t \in [0,1] \\ x(0)=0,\ x'(0)=x'(1) \end{cases} \tag{4.1}$$

Where

$$f(t,u,v) = \frac{1}{4}(v-10) + \frac{t}{2}e^{-|u|}.$$

Choose $p(t) = 11/4,\ q(t) = 0,\ r(t) = 1/4,\ B = 10$. We can get that $q_1 = 0,\ r_1 = 1/4$ and

$$\Gamma\left(\frac{3}{2}\right) - 2q_1 - 2r_1 > 0.$$

Then, all conditions of Theorem hold, so BVP (4.1) has at least one solution.

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