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Existence and Multiplicity of Solutions for $p(x)$-Kirchhoff-Type Problem in $\mathbf{R}^N$

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Abstract. In this paper we study the $p(x)$-Kirchhoff-type problem

$$\begin{cases} -m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=f(x,u) & \text{in } \mathbf{R}^N \\ u\in W^{1,p(x)}(\mathbf{R}^N). \end{cases}$$

We first establish the compact imbedding $W^{1,p(x)}(\mathbf{R}^N)\hookrightarrow L^{q(x)}_{b(x)}(\mathbf{R}^N)$, where $L^{p(x)}_{b(x)}(\mathbf{R}^N)=\{u$ is measurable on $\mathbf{R}^N$: $\int_{\mathbf{R}^N} b(x)|u|^{p(x)}dx<\infty\}$. Based on it, the existence and multiplicity of solutions for the problem are obtained by variational methods.

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1. Introduction and main results

In this paper, we aim to discuss the existence and multiplicity of solutions for the following $p(x)$-Kirchhoff-type problem in $\mathbf{R}^N$:

$$\text{(1.1)}\qquad \begin{cases} -m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=f(x,u) & \text{in } \mathbf{R}^N \\ u\in W^{1,p(x)}(\mathbf{R}^N) \end{cases}$$

where $N\geq 2$, p is a function defined on $\mathbf{R}^N$, $m:\mathbf{R}\to\mathbf{R}$ is continuous, $f:\mathbf{R}^N\times\mathbf{R}\to\mathbf{R}$ satisfies Caratheodory conditions, i.e., $f(x,t)$ is continuous in x for almost every t and measurable in t for all x.

When $p(x)\equiv p$ (a constant), problem (1.1) is the p-Kirchhoff-type problem in $\mathbf{R}^N$. There have been many studies on the existence of solutions for p-Kirchhoff-type problem, $p(x)$-Kirchhoff-type problem and nonlocal $p(x)$-Laplacian Dirichlet problems on a bounded domain (refer to [1–6, 16]). In [8, 9], the Kirchhoff-type problems in $\mathbf{R}^N$ have been studied. The study on the existence of solutions for p-Kirchhoff-type problem and $p(x)$-Kirchhoff-type problem in $\mathbf{R}^N$ is a new topic. We know that in the study of equations in $\mathbf{R}^N$, a

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main difficulty arises from that the imbedding $W^{1,p(x)}(\mathbf{R}^N) \hookrightarrow L^{q(x)}(\mathbf{R}^N)$ is not compact any more. In [8], the radially symmetric case is studied. In [9], the study is based on the compact imbedding $E \hookrightarrow L^s(\mathbf{R}^N)$ $(E = \{u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N}(|\nabla u|^2 + V(x)u^2)dx < \infty\})$. In this paper, we establish the compact imbedding $W^{1,p(x)}(\mathbf{R}^N) \hookrightarrow L^{q(x)}_{b(x)}(\mathbf{R}^N)$ $\big(L^{p(x)}_{b(x)}(\mathbf{R}^N) = \{u$ is measurable on $\mathbf{R}^N : \int_{\mathbf{R}^N} b(x)|u|^{p(x)}dx < \infty\}\big)$ to overcome the difficulty, which is a new method for this problem.

Denote by $\mathscr{M}(\mathbf{R}^N)$ the set of all measurable real functions defined on $\mathbf{R}^N$, elements in $\mathscr{M}(\mathbf{R}^N)$ which are equal to each other almost everywhere are considered as one element. For a function $p(x)$ defined on $\mathbf{R}^N$, let

$$p_\sharp := \inf_{\mathbf{R}^N} p(x) \quad \text{and} \quad p^\sharp := \sup_{\mathbf{R}^N} p(x).$$

The elementary assumptions on p and m are as follows:

(p) p is Lipschitz continuous, $p \in L^\infty(\mathbf{R}^N)$, $1 < p_\sharp \le p^\sharp < N$;
(m_0) there exists $m_0 > 0$ such that $m(t) \ge m_0$ for all $t \ge 0$;
(m_1) there is $0 < \mu < 1$ such that $M(t) \ge \mu t m(t)$ for all $t \ge 0$, where $M(t) = \int_0^t m(s)ds$.

For the functions $b(x)$ and $q(x)$, we suppose that they satisfy the following conditions:

(b_0) $b(x) \ge 0$, $b \ne 0$ and $b \in C(\mathbf{R}^N, \mathbf{R})$;
(b_1) $b(x) \ge 0$, and $b \in L^{r(x)}(\mathbf{R}^N)$, where $r \in L^\infty(\mathbf{R}^N)$, $r_\sharp \ge 1$;
(q) $q \in L^\infty(\mathbf{R}^N)$ and $1 \le q_\sharp \le q^\sharp < (p^*)_\sharp$, where $p^*(x) := (Np(x))/(N - p(x))$.

For $p \in L^\infty(\mathbf{R}^N)$ and $p_\sharp \ge 1$, define

$$L^{p(x)}(\mathbf{R}^N) = \left\{ u \in \mathscr{M}(\mathbf{R}^N) : \int_{\mathbf{R}^N} |u|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{L^{p(x)}(\mathbf{R}^N)} = |u|_{p(x)} = \inf\left\{ \lambda > 0 : \int_{\mathbf{R}^N} \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1 \right\},$$

and

$$W^{1,p(x)}(\mathbf{R}^N) = \{u \in L^{p(x)}(\mathbf{R}^N) \,|\, |\nabla u| \in L^{p(x)}(\mathbf{R}^N)\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\mathbf{R}^N)} = |u|_{L^{p(x)}(\mathbf{R}^N)} + |\nabla u|_{L^{p(x)}(\mathbf{R}^N)}.$$

In this paper we will use the following equivalent norm on $W^{1,p(x)}(\mathbf{R}^N)$:

$$\|u\| := \inf\left\{ \lambda > 0 : \int_{\mathbf{R}^N} \left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + \left|\frac{u}{\lambda}\right|^{p(x)} \right) dx \le 1 \right\}.$$

For $p \in L^\infty(\mathbf{R}^N)$ with $p_\sharp \ge 1$ and b satisfying (b_0) we define

$$L^{p(x)}_{b(x)}(\mathbf{R}^N) = \left\{ u \in \mathscr{M}(\mathbf{R}^N) : \int_{\mathbf{R}^N} b(x)|u|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{L^{p(x)}_{b(x)}(\mathbf{R}^N)} = \inf\left\{ \lambda > 0 : \int_{\mathbf{R}^N} b(x) \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1 \right\}.$$

It is easily verified that $\rho(u) := \int_{\mathbf{R}^N} b(x)|u|^{p(x)}dx$ is semimodular (see [20, Definition 2.1.1]). Therefore, according to [20, Theorem 2.3.13], the space $L^{p(x)}_{b(x)}(\mathbf{R}^N)$ is a Banach space. As $1 \le p_\sharp \le p^\sharp < N$, we can obtain that the space is separable and reflexive. The proof is similar to the proof of Lemma 3.4.4 and Theorem 3.4.7 in [20].

Before we give the main results, we first give a embedding result which is new and critical in the paper.

Lemma 1.1. *Suppose that conditions (p) and (q) hold. Assume that (b_1) holds with r satisfying $p(x) \le s(x) := r(x)q(x)/(r(x)-1) \le p^*$ for a.e. $x \in \mathbf{R}^N$. Then the embedding $W^{1,p(x)}(\mathbf{R}^N) \hookrightarrow L^{q(x)}_{b(x)}(\mathbf{R}^N)$ is continuous.*

Moreover if $s^\sharp < (p^*)_\sharp$, the embedding $W^{1,p(x)}(\mathbf{R}^N) \hookrightarrow L^{q(x)}_{b(x)}(\mathbf{R}^N)$ is compact.

Remark 1.1. This result is new. As we all know, the embedding $W^{1,p(x)}(\mathbf{R}^N) \hookrightarrow L^{q(x)}(\mathbf{R}^N)$ is no longer compact. It leads to the difficulty in proving the (PS) condition. And this is the main difficulty in studying problems in $\mathbf{R}^N$. This result provides a new tool to overcome the difficulty.

Next, we give our main results.

Theorem 1.1. *Suppose that m satisfies (m_0), (m_1) and f satisfies the following conditions:*

(f1) $|f(x,t)| \le b(x)|t|^{q(x)-1}$, $\forall (x,t) \in \mathbf{R}^N \times \mathbf{R}$, *where b and q satisfy the conditions of Lemma 1.1;*

(f2) *there exists $\delta > 0$ such that $f(x,t) \ge b_0(x)t^{q_0(x)-1}$ for $x \in \mathbf{R}^N$ and $0 < t \le \delta$, where b_0 satisfies condition (b_0) and q_0 satisfies (q) with $q_0^\sharp < p_\sharp$.*

Let $q^\sharp < p_\sharp$ in $(f1)$, then problem (1.1) *has a nontrivial solution.*

Remark 1.2. Let $F(x,t) = \int_0^t f(x,s)ds$. On the one hand, $(f1)$ guarantees that $\int_{\mathbf{R}^N} F(x,u)dx$ is well defined. On the other hand, from $(f1)$ we can obtain $f(x,0) = 0$, that is, 0 is a solution of problem (1.1), combining with condition $(f2)$, a nontrivial solution of problem (1.1) can be obtained.

If $q_\sharp > p^\sharp$ in $(f1)$, we can obtain the following result.

Theorem 1.2. *Suppose that m satisfies (m_0), (m_1), f satisfies $(f1)$ with $q_\sharp > p^\sharp$, and*
(f3) there is a positive constant $\alpha > p^\sharp/\mu$ such that

$$0 < \alpha F(x,t) \le t f(x,t), \quad \forall\, x \in \mathbf{R}^N,\ t \ne 0.$$

Then problem (1.1) *has a nontrivial solution.*

Remark 1.3. When $q_\sharp > p^\sharp$, the energy functional related to problem (1.1) is not coercive any more. $(f4)$ guarantees that the (PS) sequence is bounded. We will prove this theorem through the Mountain Pass Theorem.

Besides, we can obtain the existence of infinitely many solutions for the problem (1.1) by the Fountain Theorem.

Theorem 1.3. *Suppose that m satisfies (m_0), (m_1), f satisfies $(f1)$ with $q_\sharp > p^\sharp$, $(f3)$ and*
(f4) $f(x,-t) = -f(x,t)$, for a.e. $x \in \mathbf{R}^N$ and $t \in \mathbf{R}$.
Then problem (1.1) *has a sequence of nontrivial solutions $\{\pm u_k\}$ such that*

$$M\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla(\pm u_k)|^{p(x)}dx\right) + \int_{\mathbf{R}^N} \frac{1}{p(x)}|\pm u_k|^{p(x)}dx - \int_{\mathbf{R}^N} F(x,\pm u_k)dx \to +\infty$$

as $k \to \infty$.

We also can get the existence of infinitely many solutions for the problem (1.1) without the restriction to the relationship between p and q.

Theorem 1.4. *Suppose that m satisfies (m_0), (m_1), f satisfies $(f1)$, $(f2)$ and $(f4)$. Then problem* (1.1) *has a sequence of nontrivial negative energy solutions $\{u_k\}$ with $u_k \to 0$ in $W^{1,p(x)}(\mathbf{R}^N)$ as $k \to \infty$.*

Remark 1.4. In this result, there is not any restriction to the relationship between p and q.

Remark 1.5. If $m(t) = a + bt$, problem (1.1) reduces to

$$\begin{cases} -\left(a + b\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = f(x,u) \quad \text{in } \mathbf{R}^N \\ u \in W^{1,p(x)}(\mathbf{R}^N). \end{cases}$$

It is clear that

$$m(t) \geq a > 0, \quad \text{for all} \quad t \geq 0.$$

Taking $\mu = 1/2$, we have

$$M(t) = at + \frac{1}{2}bt^2 \geq \frac{1}{2}(a+bt)t = \mu m(t)t, \quad \text{for all} \quad t \geq 0.$$

That is, conditions (m_0) and (m_1) are satisfied. So the results corresponding to Theorem 1.1–1.4 can be obtained. The problem and results are all new.

Remark 1.6. There are some studies of p-Kirchhoff-type problem like

$$-\left[m\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1} \triangle_p u = f(x,u) \quad \text{in } \Omega,$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$, we refer to [1–3]. Motivated by these studies, we can consider the following problem

$$(1.2) \qquad \begin{cases} -\left(a + b\int_{\mathbf{R}^N} |\nabla u|^p dx\right)^{p-1} \triangle_p u + |u|^{p-2}u = f(x,u) \quad \text{in } \mathbf{R}^N, \\ u \in W^{1,p}(\mathbf{R}^N). \end{cases}$$

where a, b are two positive constants. It is clear that problem (1.2) is a special case of problem (1.1) with $p(x) \equiv p$ and $m(t) = (a+bpt)^{p-1}$. We know that

$$m(t) \geq a^{p-1} > 0, \quad \text{for all} \quad t \geq 0.$$

Let $\mu = 1/p$, we have

$$M(t) = \frac{1}{bp^2}(a+bpt)^p - \frac{a^p}{bp^2} \geq \frac{1}{p}(a+bpt)^{p-1}t = \mu m(t)t, \quad \text{for all} \quad t \geq 0.$$

That is, conditions (m_0) and (m_1) are satisfied. Therefore, the results corresponding to Theorem 1.1–1.4 can be obtained. The problem and results are also all new.

2. Preliminary

$u \in W^{1,p(x)}(\mathbf{R}^N)$ is called a weak solution of problem (1.1) if

$$m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_{\mathbf{R}^N} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v dx + \int_{\mathbf{R}^N} |u|^{p(x)-2}uvdx = \int_{\mathbf{R}^N} f(x,u)vdx$$

for all $v \in W^{1,p(x)}(\mathbf{R}^N)$. Set

$$F(x,t) = \int_0^t f(x,s)ds, \quad \forall (x,t) \in \mathbf{R}^N \times \mathbf{R},$$

$$\varphi(u) = M\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) + \int_{\mathbf{R}^N} \frac{1}{p(x)}|u|^{p(x)}dx - \int_{\mathbf{R}^N} F(x,u)dx, \forall u \in W^{1,p(x)}(\mathbf{R}^N).$$

Note that m is continuous, so $M \in C^1(\mathbf{R},\mathbf{R})$. Refer to [7], we know that $\int_{\mathbf{R}^N}(1/p(x))|\nabla u|^{p(x)}dx$, $\int_{\mathbf{R}^N}(1/p(x))|u|^{p(x)}dx$, $\int_{\mathbf{R}^N} F(x,u)dx$ are all in $C^1(W^{1,p(x)}(\mathbf{R}^N),\mathbf{R})$. Therefore, $\varphi \in C^1(W^{1,p(x)}(\mathbf{R}^N),\mathbf{R})$ and

$$\begin{aligned}\langle \varphi'(u), v\rangle = m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_{\mathbf{R}^N} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v dx \\ + \int_{\mathbf{R}^N} |u|^{p(x)-2}uvdx - \int_{\mathbf{R}^N} f(x,u)vdx\end{aligned}$$

for all $u,v \in W^{1,p(x)}(\mathbf{R}^N)$. That is, the critical points of φ are just the weak solutions of problem (1.1).

Let Ω be an open subset of $\mathbf{R}^N$, on the basic properties of space $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ we refer to [11–15, 17, 18]. As we all know, $\mathbf{R}^N$ is a special open subset. So those properties also hold for $L^{p(x)}(\mathbf{R}^N)$ and $W^{1,p(x)}(\mathbf{R}^N)$. In the following, we display some facts (refer to [7, 10]) which we will use later.

Proposition 2.1. [7] *The spaces $L^{p(x)}(\mathbf{R}^N)$ and $W^{1,p(x)}(\mathbf{R}^N)$ are separable and reflexive Banach spaces.*

Proposition 2.2. [7] *The conjugate space of $L^{p(x)}(\mathbf{R}^N)$ is $L^{q(x)}(\mathbf{R}^N)$, where $1/p(x)+1/q(x) = 1$. For any $u \in L^{p(x)}(\mathbf{R}^N)$ and $v \in L^{q(x)}(\mathbf{R}^N)$, one has*

$$\int_{\mathbf{R}^N} |uv|dx \le \left(\frac{1}{p_\sharp}+\frac{1}{q_\sharp}\right)|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

Proposition 2.3. [7] *Assume that $p:\mathbf{R}^N \to \mathbf{R}$ is Lipschitz continuous and $p^\sharp < N$. Then for $q \in L^\infty(\mathbf{R}^N)$ with $q_\sharp \ge 1$, $p(x) \le q(x) \le p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\mathbf{R}^N) \hookrightarrow L^{q(x)}(\mathbf{R}^N)$.*

Proposition 2.4. [7] *Let Ω be a bounded domain in $\mathbf{R}^N$, $p \in C(\overline{\Omega})$, $p^\sharp < N$. Then for any $q \in L^\infty(\mathbf{R}^N)$ with $1 \le q_\sharp \le q^\sharp < (p^*)_\sharp$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Proposition 2.5. [7] *Suppose that $|u|^{q(x)} \in L^{s(x)/q(x)}(\mathbf{R}^N)$, where $q,s \in L^\infty(\mathbf{R}^N)$ with $q_\sharp \ge 1$, $s_\sharp \ge 1$, $q(x) \le s(x)$. Then $u \in L^{s(x)}(\mathbf{R}^N)$ and there is a number $\bar{q} \in [q_\sharp, q^\sharp]$ such that $||u|^{q(x)}|_{s(x)/q(x)} = (|u|_{s(x)})^{\bar{q}}$.*

Proposition 2.6. [7] *Let $K(u)=\int_{\mathbf{R}^N}(1/p(x))(|\nabla u|^{p(x)}+|u|^{p(x)})dx$, then $K\in C^1(W^{1,p(x)}(\mathbf{R}^N),\mathbf{R})$ and K is a convex functional, $K':W^{1,p(x)}(\mathbf{R}^N)\to W^{1,p(x)}(\mathbf{R}^N)^*$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type, namely $u_n\rightharpoonup u$ and $\overline{\lim}_{n\to\infty}\langle K'(u_n),u_n-u\rangle\le 0$ implies $u_n\to u$.*

Definition 2.1. *Let $\varphi\in C^1(X,\mathbf{R})$. We say that φ satisfies the (PS) condition if any sequence $\{u_k\}$ in X such that $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k)\to 0$ in X^* as $k\to\infty$ has a convergent subsequence in X.*

Lemma 2.1. $\lim_{k\to\infty}|u_k|_{L^{p(x)}_{b(x)}(\mathbf{R}^N)}=0\Longleftrightarrow \lim_{k\to\infty}\int_{\mathbf{R}^N}b(x)|u_k|^{p(x)}dx=0.$

Proof. ($\Rightarrow$) From $\lim_{k\to\infty}|u_k|_{L^{p(x)}_{b(x)}(\mathbf{R}^N)}=0$ we know that for every $0<\varepsilon<1$, there exists $k_1>0$ such that for all $k>k_1$,

$$\inf\left\{\lambda:\int_{\mathbf{R}^N}b(x)\left|\frac{u_k}{\lambda}\right|^{p(x)}dx\le 1\right\}<\varepsilon^{\frac{1}{p_\sharp}},$$

so there is $\lambda_\varepsilon\le\varepsilon^{1/p_\sharp}<1$ such that

$$\int_{\mathbf{R}^N}b(x)\left|\frac{u_k}{\lambda_\varepsilon}\right|^{p(x)}dx\le 1,$$

therefore

$$\int_{\mathbf{R}^N}b(x)|u_k|^{p(x)}dx\le\lambda_\varepsilon^{p_\sharp}\le\varepsilon,$$

that is, $\lim_{k\to\infty}\int_{\mathbf{R}^N}b(x)|u_k|^{p(x)}dx=0$.

($\Leftarrow$) From $\lim_{k\to\infty}\int_{\mathbf{R}^N}b(x)|u_k|^{p(x)}dx=0$ we know that for every $0<\varepsilon<1$, there exists $k_2>0$ such that for all $k>k_2$,

$$\int_{\mathbf{R}^N}b(x)|u_k|^{p(x)}dx\le\varepsilon^{p^\sharp},$$

so

$$\int_{\mathbf{R}^N}b(x)\left|\frac{u_k}{\varepsilon}\right|^{p(x)}dx\le\frac{1}{\varepsilon^{p^\sharp}}\int_{\mathbf{R}^N}b(x)|u_k|^{p(x)}dx\le 1,$$

therefore

$$\inf\left\{\lambda:\int_{\mathbf{R}^N}b(x)\left|\frac{u_k}{\lambda}\right|^{p(x)}dx\le 1\right\}\le\varepsilon,$$

that is, $\lim_{k\to\infty}|u_k|_{L^{p(x)}_{b(x)}(\mathbf{R}^N)}=0$. ∎

Lemma 2.2. *Suppose that b and q satisfy the conditions in Lemma 1.1. Then for $u\in W^{1,p(x)}(\mathbf{R}^N)$, there is a number $\bar q\in[q_\sharp,q^\sharp]$ and a constant $c>0$, such that $\int_{\mathbf{R}^N}b(x)|u|^{q(x)}dx\le c\|u\|^{\bar q}$.*

Proof. From the the continuous imbedding $W^{1,p(x)}(\mathbf{R}^N)\hookrightarrow L^{q(x)}_{b(x)}(\mathbf{R}^N)$, there is a constant $c_1>0$ such that $|u|_{L^{q(x)}_{b(x)}}\le c_1\|u\|$. So there exists $\lambda\le c_1\|u\|$ such that

$$\int_{\mathbf{R}^N}b(x)\left|\frac{u}{\lambda}\right|^{q(x)}dx\le 1. \tag{2.1}$$

When $c_1\|u\| \geq 1$, we have

$$\int_{\mathbf{R}^N} b(x)\left|\frac{u}{\lambda}\right|^{q(x)} dx \geq \int_{\mathbf{R}^N} \frac{1}{(c_1\|u\|)^{q(x)}} b(x)|u|^{q(x)}dx \geq \frac{1}{(c_1\|u\|)^{q^\sharp}} \int_{\mathbf{R}^N} b(x)|u|^{q(x)}dx.$$

When $c_1\|u\| \leq 1$, one has

$$\int_{\mathbf{R}^N} b(x)\left|\frac{u}{\lambda}\right|^{q(x)} dx \geq \int_{\mathbf{R}^N} \frac{1}{(c_1\|u\|)^{q(x)}} b(x)|u|^{q(x)}dx \geq \frac{1}{(c_1\|u\|)^{q_\sharp}} \int_{\mathbf{R}^N} b(x)|u|^{q(x)}dx.$$

Hence there exists $\bar{q} \in [q_\sharp, q^\sharp]$ such that

$$\int_{\mathbf{R}^N} b(x)\left|\frac{u}{\lambda}\right|^{q(x)} dx \geq \frac{1}{(c_1\|u\|)^{\bar{q}}} \int_{\mathbf{R}^N} b(x)|u|^{q(x)}dx. \tag{2.2}$$

It follows from (2.1) and (2.2), we obtain $\int_{\mathbf{R}^N} b(x)|u|^{q(x)}dx \leq c_1^{\bar{q}}\|u\|^{\bar{q}} \leq c\|u\|^{\bar{q}}$. ∎

Lemma 2.3. *Suppose that m satisfies (m_0), (m_1). Assume that f satisfies $(f1)$ and $(f3)$. Then φ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be an arbitrary sequence satisfying $|\varphi(u_n)| \leq c$ for some $c > 0$ and $\varphi'(u_n) \to 0$. In [7], we know that

$$\|u\| \geq 1 \Leftrightarrow \|u\|^{p_\sharp} \leq \int_{\mathbf{R}^N} (|\nabla u|^{p(x)} + |u|^{p(x)})dx \leq \|u\|^{p^\sharp}. \tag{2.3}$$

When $\|u_n\| \geq 1$, from (m_0), (m_1), $(f3)$ and (2.3), we obtain

$$\begin{aligned}
c + \|u_n\| &\geq \varphi(u_n) - \frac{1}{\alpha}\langle \varphi'(u_n), u_n\rangle \\
&= M\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right) + \int_{\mathbf{R}^N} \frac{1}{p(x)}|u_n|^{p(x)}dx - \int_{\mathbf{R}^N} F(x,u_n)dx \\
&\quad - \frac{1}{\alpha}m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_{\mathbf{R}^N} |\nabla u_n|^{p(x)}dx - \frac{1}{\alpha}\int_{\mathbf{R}^N} |u_n|^{p(x)}dx \\
&\quad + \int_{\mathbf{R}^N} \frac{1}{\alpha} f(x,u_n)u_n dx \\
&\geq \mu m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \\
&\quad - \frac{1}{\alpha}m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_{\mathbf{R}^N} |\nabla u_n|^{p(x)}dx \\
&\quad + \left(\frac{1}{p^\sharp} - \frac{1}{\alpha}\right)\int_{\mathbf{R}^N} |u_n|^{p(x)}dx + \int_{\mathbf{R}^N} \left[\frac{1}{\alpha}f(x,u_n)u_n - F(x,u_n)\right]dx \\
&\geq \left(\frac{\mu}{p^\sharp} - \frac{1}{\alpha}\right) m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_{\mathbf{R}^N} |\nabla u_n|^{p(x)}dx \\
&\quad + \left(\frac{1}{p^\sharp} - \frac{1}{\alpha}\right)\int_{\mathbf{R}^N} |u_n|^{p(x)}dx \\
&\geq \min\left\{\left(\frac{\mu}{p^\sharp} - \frac{1}{\alpha}\right)m_0, \left(\frac{1}{p^\sharp} - \frac{1}{\alpha}\right)\right\}\int_{\mathbf{R}^N} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx \\
&\geq \min\left\{\left(\frac{\mu}{p^\sharp} - \frac{1}{\alpha}\right)m_0, \left(\frac{1}{p^\sharp} - \frac{1}{\alpha}\right)\right\}\|u_n\|^{p_\sharp}.
\end{aligned}$$

As $\alpha > p^\sharp/\mu > p^\sharp$ and $p_\sharp > 1$, we get $\{u_n\}$ is bounded.

Without loss of generality, we assume that $u_n \rightharpoonup u$, then $\langle \varphi'(u_n), u_n - u\rangle \to 0$ as $n \to \infty$. Thus, we have

$$\begin{aligned}\langle \varphi'(u_n), u_n - u\rangle = m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_{\mathbf{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(u_n - u)dx \\ + \int_{\mathbf{R}^N} |u_n|^{p(x)-2}u_n(u_n-u)dx - \int_{\mathbf{R}^N} f(x,u_n)(u_n-u)dx \\ \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

From $(f1)$ we obtain

$$\begin{aligned}\left|\int_{\mathbf{R}^N} f(x,u_n)(u_n-u)dx\right| &\le \int_{\mathbf{R}^N} |f(x,u_n)||u_n-u|dx \le \int_{\mathbf{R}^N} b(x)|u_n|^{q(x)-1}|u_n-u|dx \\ &\le 2^{q^\sharp-1}\left(\int_{\mathbf{R}^N} b(x)|u_n-u|^{q(x)}dx + \int_{\mathbf{R}^N} b(x)|u|^{q(x)-1}|u_n-u|dx\right)\end{aligned}$$

As $u_n \rightharpoonup u$, we have $\int_{\mathbf{R}^N} b(x)|u|^{q(x)-1}|u_n-u|dx \to 0$. According to Lemma 1.1, we have $u_n \to u$ strongly in $L^{q(x)}_{b(x)}(\mathbf{R}^N)$, and from Lemma 2.1 we obtain $\int_{\mathbf{R}^N} b(x)|u_n-u|^{q(x)}dx \to 0$. Consequently, $\int_{\mathbf{R}^N} f(x,u_n)(u_n-u)dx \to 0$. Therefore, we have

$$\begin{aligned}m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_{\mathbf{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(u_n-u)dx + \int_{\mathbf{R}^N} |u_n|^{p(x)-2}u_n(u_n-u)dx \\ \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

In view of (m_0), one has

$$\begin{aligned}m\left(\int_{\mathbf{R}^N} \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_{\mathbf{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(u_n-u)dx + \int_{\mathbf{R}^N} |u_n|^{p(x)-2}u(u_n-u)dx \\ \ge \min\{m_0, 1\}\langle K'(u_n), u_n - u\rangle.\end{aligned}$$

So, $\overline{\lim}_{n\to\infty}\langle K'(u_n), u_n-u\rangle \le 0$. By Proposition 2.6, one has $u_n \to u$. According to Definition 2.1, we conclude that φ satisfies the (PS) condition. ∎

Proposition 2.7. (Fountain Theorem) *Let X be a Banach space with the norm $\|\cdot\|$ and let X_j be a sequence of subspace of X with $dimX_j < \infty$ for each $j \in \mathbf{N}$. Further, $X = \overline{\bigoplus_{j\in\mathbf{N}} X_j}$, the closure of the direct sum of all X_j. Set $Y_k = \bigoplus_{j=0}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$. Assume that $\varphi \in C^1(X, \mathbf{R})$ satisfies the (PS) condition, $\varphi(-u) = \varphi(u)$. Suppose that for every $k \in \mathbf{N}$, there exist $\rho_k > r_k > 0$ such that*

(A1) $\inf_{u\in Z_k, \|u\|=r_k} \varphi(u) \to +\infty$ *as* $k \to \infty$,
(A2) $\max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) \le 0$.

Then φ has an unbounded sequence of critical values.

As $W^{1,p(x)}(\mathbf{R}^N)$ is a separable and reflexive Banach space, there exist $\{e_n\}_{n=1}^\infty \subset X$ and $\{f_n\}_{n=1}^\infty \subset X^*$ such that

$$f_n(e_m) = \delta_{n,m},$$

$$W^{1,p(x)}(\mathbf{R}^N) = \overline{\text{span}}\{e_n : n = 1, 2, \cdots,\}, \quad (W^{1,p(x)}(\mathbf{R}^N))^* = \overline{\text{span}}\{f_n : n = 1, 2, \cdots,\}.$$

For $k = 1, 2, \cdots$, denote

$$X_k = \text{span}\{e_k\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{2.4}$$

Lemma 2.4. *For $k = 1, 2, \cdots$, write*

$$\theta_k = \sup_{u \in Z_k, \|u\| \leq 1} \int_{\mathbf{R}^N} b(x)|u|^{q(x)} dx,$$

where b, q satisfy conditions in $(f1)$. Then $\theta_k > 0$ and $\theta_k \to 0$ as $k \to \infty$.

Proof. Obviously, $0 \leq \theta_{k+1} \leq \theta_k$, so there is $\theta \geq 0$ such that $\theta_k \to \theta$ as $k \to \infty$. For each $k = 1, 2, \cdots$, taking $u_k \in Z_k$, $\|u_k\| \leq 1$ such that

$$0 \leq \theta_k - \int_{\mathbf{R}^N} b(x)|u_k|^{q(x)} dx < \frac{1}{k}.$$

From Proposition 2.1, we know that $W^{1,p(x)}(\mathbf{R}^N)$ is reflexive. So $\{u_k\}$ has a weakly convergent subsequence, without loss of generality we suppose that $u_k \rightharpoonup u$. We claim that $u = 0$. In fact, for any f_n, $n = 1, 2, \cdots$, we have $f_n(u_k) = 0$ when $k > n$, so $f_n(u_k) \to 0$ as $k \to \infty$, this conclude that for any f_n, $n = 1, 2, \cdots$, $f_n(u) = 0$, therefore, $u = 0$. That is, $u_k \rightharpoonup 0$ weakly in $W^{1,p(x)}(\mathbf{R}^N)$, as $k \to \infty$. By Lemma 1.1 and Lemma 2.1, one has

$$\int_{\mathbf{R}^N} b(x)|u_k|^{q(x)} dx \to 0 \quad \text{as} \quad k \to \infty.$$

So, $\theta_k \to 0$ as $k \to \infty$. ∎

Next, we will state the Symmetric Mountain Pass Lemma. For this purpose, we should first introduce the definition of genus.

Definition 2.2. *Let X be a real Banach space and A a subset of X. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbf{R}^k \setminus \{0\}$. If there does not exist such a k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denotes the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$.*

For the convenience of the readers, we summarize the property of a genus. We refer the readers to [19] for the proof of the next Proposition.

Proposition 2.8. [19] *Let A and B be closed symmetric subsets of X which do not contain the origin. Then (i)–(v) below hold.*

(i) *If there is an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$;*
(ii) *If there is an odd homeomorphism from A onto B, then $\gamma(A) = \gamma(B)$;*
(iii) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;*
(iv) *If A is compact, then $\gamma(A) < \infty$ and $\gamma(N_\delta(A)) = \gamma(A)$ for $\delta > 0$ small enough;*
(v) *The n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk-Ulam theorem.*

Now, we recall the Symmetric Mountain Pass Lemma, which can be found in [21].

Proposition 2.9. [21] *Let X be an infinite demensional Banach space and $\varphi \in C^1(X, \mathbf{R})$ satisfies (B1) and (B2):*

(B1) *$\varphi(u)$ is even, bounded from below, $\varphi(0) = 0$ and $\varphi(u)$ satisfies the (PS) condition.*

(B2) *For each $k \in N$, there exists an $A_k \in \Gamma_k$ such that* $\sup_{u \in A_k} \varphi(u) < 0$.

Then $\varphi(u)$ admits a sequence of critical points $\{u_k\}$ such that $\varphi(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

3. Proof of main results

Proof of Lemma 1.1. For the proof of the continuous imbedding, we just need to prove that $W^{1,p(x)}(\mathbf{R}^N) \subset L^{q(x)}_{b(x)}(\mathbf{R}^N)$. As $p(x) \leq s(x) \leq p^*(x)$, from Proposition 2.3, the embedding $W^{1,p(x)}(\mathbf{R}^N) \hookrightarrow L^{s(x)}(\mathbf{R}^N)$ is continuous. Hence, for any $u \in W^{1,p(x)}(\mathbf{R}^N)$, we have $u \in L^{s(x)}(\mathbf{R}^N)$. Then one has

$$\int_{\mathbf{R}^N} (|u|^{q(x)})^{\frac{s(x)}{q(x)}} dx = \int_{\mathbf{R}^N} |u|^{s(x)} dx < \infty,$$

hence, $|u|^{q(x)} \in L^{s(x)/q(x)}(\mathbf{R}^N)$. From $b \in L^{r(x)}(\mathbf{R}^N)$, $1/r(x) + q(x)/s(x) = 1$ and Proposition 2.2, we obtain

$$\int_{\mathbf{R}^N} b(x)|u|^{q(x)} dx \leq 2|b|_{L^{r(x)}} ||u|^{q(x)}|_{L^{s(x)/q(x)}} < \infty,$$

hence, $u \in L^{q(x)}_{b(x)}(\mathbf{R}^N)$. That is, the imbedding $W^{1,p(x)}(\mathbf{R}^N) \hookrightarrow L^{q(x)}_{b(x)}(\mathbf{R}^N)$ is continuous.

Next, we prove that when $s^\sharp < (p^*)_\sharp$ the imbedding is compact. Suppose that $u_k \rightharpoonup u$ weakly in $W^{1,p(x)}(\mathbf{R}^N)$, then $\{\|u_k\|\}$ is bounded, further $\{|u_k|_{s(x)}\}$ is bounded. So there is a positive constant C such that

$$\max\{||u_k|^{q(x)}|_{s(x)/q(x)}, ||u|^{q(x)}|_{s(x)/q(x)}\} \leq C. \tag{3.1}$$

Set $B_n = \{x \in \mathbf{R}^N : |x| < n\}$, $b \in L^{r(x)}(\mathbf{R}^N)$ implies that

$$|b|_{L^{r(x)}(\mathbf{R}^N \setminus B_n)} \to 0 \quad \text{as} \quad n \to \infty.$$

For any $\varepsilon > 0$, we can find $n_1 > 0$ big enough such that

$$|b|_{L^{r(x)}(\mathbf{R}^N \setminus B_{n_1})} \leq \frac{\varepsilon}{2^{q^\sharp+3}C}. \tag{3.2}$$

Proposition 2.4 and $s^\sharp < (p^*)_\sharp$ conclude that there is a compact embedding $W^{1,p(x)}(B_{n_1}) \hookrightarrow L^{s(x)}(B_{n_1})$, so $u_k \rightharpoonup u$ implies that $|u_k - u|_{L^{s(x)}(B_{n_1})} \to 0$ as $k \to \infty$. According to Proposition 2.2 we have

$$\int_{B_{n_1}} b(x)|u_k - u|^{q(x)} dx \leq |b(x)|_{L^{r(x)}(B_{n_1})} ||u_k - u|^{q(x)}|_{L^{s(x)/q(x)}(B_{n_1})}.$$

It follows from Proposition 2.5 that $||u_k - u|^{q(x)}|_{L^{s(x)/q(x)}(B_{n_1})} = |u_k - u|^{\bar{q}}_{L^{s(x)}(B_{n_1})} \to 0$ as $k \to \infty$. Since $|b(x)|_{L^{r(x)}(B_{n_1})}$ is bounded, we have

$$\int_{B_{n_1}} b(x)|u_k - u|^{q(x)} dx \to 0 \quad \text{as} \quad k \to \infty.$$

Thus, there exists $k_1 > 0$ such that for any $k \geq k_1$ one has

$$\int_{B_{n_1}} b(x)|u_k - u|^{q(x)} dx \leq \frac{\varepsilon}{2}. \tag{3.3}$$

According to Proposition 2.2 and combining with (3.1), (3.2), (3.3) we have

$$\begin{aligned}&\int_{\mathbf{R}^N} b(x)|u_k-u|^{q(x)}dx\\ &\le \int_{B_{n_1}} b(x)|u_k-u|^{q(x)}dx+2^{q^\sharp}\int_{\mathbf{R}^N\backslash B_{n_1}}|b(x)|(|u_k|^{q(x)}+|u|^{q(x)})dx\\ &\le \frac{\varepsilon}{2}+2^{q^\sharp+1}|b|_{L^{r(x)}(\mathbf{R}^N\backslash B_{n_1})}(||u_k|^{q(x)}|_{L^{s(x)/q(x)}(\mathbf{R}^N\backslash B_{n_1})}+||u|^{q(x)}|_{L^{s(x)/q(x)}(\mathbf{R}^N\backslash B_{n_1})})\\ &\le \frac{\varepsilon}{2}+2^{q^\sharp+1}\times\frac{\varepsilon}{2^{q^\sharp+3}C}\times 2C\le\varepsilon,\end{aligned}$$

this concludes that $\int_{\mathbf{R}^N} b(x)|u_k-u|^{q(x)}dx\to 0$ as $k\to\infty$. Then by Lemma 2.1, we have $u_k\to u$ strongly in $L^{q(x)}_{b(x)}(\mathbf{R}^N)$, that is, the embedding $W^{1,p(x)}(\mathbf{R}^N)\hookrightarrow L^{q(x)}_{b(x)}(\mathbf{R}^N)$ is compact. ∎

Proof of Theorem 1.1. From Proposition 2.6 and $(f1)$, we know that φ is weakly lower semi-continuous. Next, we prove that φ is coercive on $W^{1,p(x)}(\mathbf{R}^N)$, that is, $\varphi(u)\to+\infty$ as $\|u\|\to\infty$. According to Lemma 2.2, (m_0), $(f1)$ and (2.3), when $\|u\|>1$ we have

$$\begin{aligned}\varphi(u)&\ge m_0\int_{\mathbf{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx+\int_{\mathbf{R}^N}\frac{1}{p(x)}|u|^{p(x)}dx-\int_{\mathbf{R}^N}F(x,u)dx\\ &\ge\frac{\min\{m_0,1\}}{p^\sharp}\left(\int_{\mathbf{R}^N}|\nabla u|^{p(x)}dx+\int_{\mathbf{R}^N}|u|^{p(x)}dx\right)-\int_{\mathbf{R}^N}\frac{1}{q_\sharp}b(x)|u|^{q(x)}dx \qquad (3.4)\\ &\ge\frac{\min\{m_0,1\}}{p^\sharp}\|u\|^{p_\sharp}-\frac{c}{q_\sharp}\|u\|^{\bar q}\ge\frac{\min\{m_0,1\}}{p^\sharp}\|u\|^{p_\sharp}-\frac{c}{q_\sharp}\|u\|^{q^\sharp},\end{aligned}$$

where $\bar q\in[q_\sharp,q^\sharp]$. Since $q^\sharp<p_\sharp$, $\varphi(u)\to+\infty$ as $\|u\|\to\infty$. According to the least action principle, there exists a critical point which minimizes φ on $W^{1,p(x)}(\mathbf{R}^N)$.

Besides, we can obtain $\inf_{u\in W^{1,p(x)}(\mathbf{R}^N)}\varphi(u)<0$. In fact, from (m_1), we know that the function $g(t)=M(t)/t^{1/\mu}$ is decreasing, so for any $t_0>0$ when $t>t_0$, we can obtain

$$M(t)\le\frac{M(t_0)}{t_0^{1/\mu}}t^{1/\mu}\le ct^{1/\mu}. \qquad (3.5)$$

Fix a $\bar u\in W^{1,p(x)}(\mathbf{R}^N)$ with $\|\bar u\|=1$, let $\int_{\mathbf{R}^N}(b_0(x)/q_0(x))|\bar u|^{q_0(x)}dx=s$. For $t\in(0,1)$, according to (3.5) we have

$$\begin{aligned}\varphi(t\bar u)&=M\left(\int_{\mathbf{R}^N}\frac{1}{p(x)}|\nabla t\bar u|^{p(x)}dx\right)+\int_{\mathbf{R}^N}\frac{1}{p(x)}|t\bar u|^{p(x)}dx-\int_{\mathbf{R}^N}F(x,t\bar u)dx\\ &\le c\left(\int_{\mathbf{R}^N}\frac{1}{p(x)}|\nabla t\bar u|^{p(x)}dx\right)^{1/\mu}+\frac{t^{p_\sharp}}{p_\sharp}\int_{\mathbf{R}^N}|\bar u|^{p(x)}dx-\int_{\mathbf{R}^N}\frac{b_0(x)t^{q_0(x)}|\bar u|^{q_0(x)}}{q_0(x)}dx\\ &\le\frac{c}{(p_\sharp)^{1/\mu}}t^{p_\sharp/\mu}\left(\int_{\mathbf{R}^N}|\nabla\bar u|^{p(x)}dx\right)^{1/\mu}+\frac{t^{p_\sharp}}{p_\sharp}\int_{\mathbf{R}^N}|\bar u|^{p(x)}dx-t^{q_0^\sharp}s\end{aligned}$$

As $q_0^\sharp<p_\sharp<p_\sharp/\mu$, we can find t small enough such that

$$\varphi(t\bar u)<0.$$

Hence, $\inf_{u\in W^{1,p(x)}(\mathbf{R}^N)}\varphi(u)<0$. That is, the critical value of φ which we have obtained before is not zero. Then the critical point is also not zero. Therefore, problem (1.1) has a nontrivial solution. ∎

Proof of Theorem 1.2. We will prove this theorem by the Mountain Pass Theorem. Firstly, according to Lemma 2.3, the (PS) condition holds. Secondly, we verify that there exist $\rho > 0$ and $\beta > 0$ such that when $\|u\| = \rho$, $\varphi(u) \geq \beta > 0$. In [7], we know that

$$\|u\| \leq 1 \Leftrightarrow \|u\|^{p^\sharp} \leq \int_{\mathbf{R}^N} (|\nabla u|^{p(x)} + |u|^{p(x)})dx \leq \|u\|^{p_\sharp}. \tag{3.6}$$

According to Lemma 2.2, (m_0), $(f1)$ and (3.6), when $\|u\| \leq 1$ we have

$$\begin{aligned}\varphi(u) &\geq \frac{\min\{m_0,1\}}{p^\sharp}\int_{\mathbf{R}^N}(|\nabla u|^{p(x)}+|u|^{p(x)})dx - \int_{\mathbf{R}^N}\frac{1}{q_\sharp}b(x)|u|^{q(x)}dx\\ &\geq \frac{\min\{m_0,1\}}{p^\sharp}\int_{\mathbf{R}^N}(|\nabla u|^{p(x)}+|u|^{p(x)})dx - \frac{c}{q_\sharp}\|u\|^{\bar q}\\ &\geq \frac{\min\{m_0,1\}}{p^\sharp}\|u\|^{p^\sharp} - \frac{c}{q_\sharp}\|u\|^{q_\sharp},\end{aligned}$$

where $\bar q \in [q_\sharp, q^\sharp]$. As $q_\sharp > p^\sharp$, when $\|u\|$ small enough $\varphi(u) \geq 0$, so there are two constants $0 < \rho < 1$ and $\beta > 0$ such that when $\|u\| = \rho$, $\varphi(u) \geq \beta > 0$.

Lastly, we prove that there exists $e \in W^{1,p(x)}(\mathbf{R}^N)$ with $\|e\| > \rho$ such that $\varphi(e) < 0$. Condition $(f3)$ implies that

$$F(x,tu) \geq t^\alpha F(x,u), \quad \forall t \geq 1. \tag{3.7}$$

Choose $\bar u \in W^{1,p(x)}(\mathbf{R}^N)$ with $\|\bar u\| = 1$. Then from (3.5) and (3.7), for any $t > 1$ we have

$$\begin{aligned}\varphi(t\bar u) &= M\left(\int_{\mathbf{R}^N}\frac{1}{p(x)}|t\nabla \bar u|^{p(x)}dx\right) + \int_{\mathbf{R}^N}\frac{1}{p(x)}|t\bar u|^{p(x)}dx - \int_{\mathbf{R}^N}F(x,t\bar u)dx\\ &\leq c\left(\int_{\mathbf{R}^N}\frac{1}{p(x)}|t\nabla\bar u|^{p(x)}dx\right)^{1/\mu} + \frac{1}{p_\sharp}\int_{\mathbf{R}^N}|t\bar u|^{p(x)}dx - t^\alpha\int_{\mathbf{R}^N}F(x,\bar u)dx\\ &\leq \frac{c}{(p_\sharp)^{1/\mu}}t^{\frac{p^\sharp}{\mu}}\left(\int_{\mathbf{R}^N}|\nabla\bar u|^{p(x)}dx\right)^{1/\mu} + \frac{1}{p_\sharp}t^{p^\sharp}\int_{\mathbf{R}^N}|\bar u|^{p(x)}dx - t^\alpha\int_{\mathbf{R}^N}F(x,\bar u)dx\\ &\to -\infty \quad \text{as} \quad t \to +\infty,\end{aligned}$$

due to $\alpha > p^\sharp/\mu > p^\sharp$. Then $e = t\bar u$ ($t > \max\{1,\rho\}$) is what we need.

Consequently, the result is proved. ∎

Proof of Theorem 1.3. Let us verify for φ the conditions in the Fountain Theorem item by item. From condition $(f4)$, $\varphi(-u) = \varphi(u)$. We know that $\varphi(u)$ satisfies the (PS) condition by Lemma 2.3. Next we verify that $(A1)$ and $(A2)$ in Proposition 2.7 are satisfied. $(A1)$ When $u \in Z_k$ with $\|u\| \geq 1$, from $(f1)$, there exists $c > 0$ such that $F(x,t) \leq cb(x)|t|^{q(x)}$. Then from Lemma 2.4, we have

$$\begin{aligned}\varphi(u) &\geq m_0\int_{\mathbf{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\mathbf{R}^N}\frac{1}{p(x)}|u|^{p(x)}dx - c\int_{\mathbf{R}^N}b(x)|u|^{q(x)}dx\\ &\geq \frac{\min\{m_0,1\}}{p^\sharp}\int_{\mathbf{R}^N}(|\nabla u|^{p(x)}+|u|^{p(x)})dx - c\theta_k\|u\|^{q^\sharp}\\ &\geq \frac{\min\{m_0,1\}}{p^\sharp}\|u\|^{p_\sharp} - c\theta_k\|u\|^{q^\sharp}.\end{aligned}$$

Let

$$r_k = \left(\frac{\min\{m_0, 1\} p_\sharp}{p^\sharp q^\sharp c \theta_k} \right)^{1/(q^\sharp - p_\sharp)},$$

when $u \in Z_k$ and $\|u\| = r_k$, for sufficiently large k,

$$\varphi(u) \geq \left(\frac{\min\{m_0, 1\} p_\sharp}{p^\sharp q^\sharp} \right)^{q^\sharp/(q^\sharp - p_\sharp)} \frac{q^\sharp - p_\sharp}{p_\sharp} \left(\frac{1}{c\theta_k} \right)^{p_\sharp/(q^\sharp - p_\sharp)}.$$

Now $\theta_k \to 0$ and $q^\sharp > p_\sharp$ implies that

$$\inf_{u \in Z_k, \|u\| = r_k} \varphi(u) \to +\infty \quad \text{as} \quad k \to \infty,$$

hence $(A1)$ is satisfied.

$(A2)$ From (3.5) and (3.7), for any $v \in Y_k$ with $\|v\| = 1$ and $t > 1$, we have

$$\begin{aligned}
\varphi(tv) &= M\left(\int_{\mathbf{R}^N} \frac{1}{p(x)} |t\nabla v|^{p(x)} dx \right) + \int_{\mathbf{R}^N} \frac{1}{p(x)} |tv|^{p(x)} dx - \int_{\mathbf{R}^N} F(x, tv) dx \\
&\leq c \left(\int_{\mathbf{R}^N} |t\nabla v|^{p(x)} dx \right)^{1/\mu} + \frac{1}{p_\sharp} \int_{\mathbf{R}^N} |tv|^{p(x)} dx - t^\alpha \int_{\mathbf{R}^N} F(x, v) dx \\
&\leq c t^{\frac{p^\sharp}{\mu}} \left(\int_{\mathbf{R}^N} |\nabla v|^{p(x)} dx \right)^{1/\mu} + \frac{1}{p_\sharp} t^{p^\sharp} \int_{\mathbf{R}^N} |v|^{p(x)} dx - t^\alpha \int_{\mathbf{R}^N} F(x, v) dx \\
&\to -\infty \quad \text{as} \quad t \to +\infty,
\end{aligned}$$

due to $\alpha > p^\sharp/\mu > p^\sharp$. So there exists $\rho_k > r_k$ such that $t = \rho_k$ concludes $\varphi(tv) \leq 0$, and then

$$\max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \leq 0,$$

hence $(A2)$ is satisfied. ∎

Proof of Theorem 1.4. Choose $h \in C^\infty([0,\infty), \mathbf{R})$ such that $0 \leq h(t) \leq 1$ for $t \in [0,\infty)$, and for every $\varepsilon > 0$, $h(t) = 1$ for $0 \leq t \leq \varepsilon/2$, $h(t) = 0$ for $t \geq \varepsilon$. Let $\psi(u) = h(\|u\|)$. We consider the truncated functional

$$\Phi(u) := M\left(\int_{\mathbf{R}^N} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) + \int_{\mathbf{R}^N} \frac{1}{p(x)} |u|^{p(x)} dx - \psi(u) \int_{\mathbf{R}^N} F(x, u) dx.$$

We know that $\Phi \in C^1(W^{1,p(x)}(\mathbf{R}^N), \mathbf{R})$. If we can prove that Φ admits a sequence of nontrivial weak solutions $\{u_n\}$ with $u_n \to 0$ as $n \to \infty$ in $W^{1,p(x)}(\mathbf{R}^N)$, Theorem 1.4 is proved. In fact, for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n > N$, $\|u_n\| < \varepsilon/2$, $\varphi(u_n) = \Phi(u_n)$, that is, $\{u_n\}_{n>N}$ are just weak solutions of problem (1.1). By applying Proposition 2.9 we show that Φ admits a sequence of nontrivial weak solutions converges to zero in $W^{1,p(x)}(\mathbf{R}^N)$.

For $\|u\| \geq 1$, we have $\Phi(u) \geq (\min\{m_0, 1\}/p^\sharp)\|u\|^{p_\sharp}$, which implies that $\Phi(u) \to \infty$ as $\|u\| \to \infty$. Hence Φ is coercive on $W^{1,p(x)}(\mathbf{R}^N)$. Thus $\Phi(u)$ is bounded from below and the (PS) sequence is bounded. As the proof of Lemma 2.3, the (PS) condition is satisfied. By $(f4)$, it is easy to see that $\Phi(-u) = \Phi(u)$ and $\Phi(0) = 0$. This shows that $(B1)$ holds.

As $b_0(x) \not\equiv 0$ and $b_0(x) \geq 0$, we can find a bounded open set $\Omega \subset \mathbf{R}^N$ such that $b_0(x) > 0$ for all $x \in \Omega$. There exists a map $\phi : W_0^{1,p(x)}(\Omega) \to W^{1,p(x)}(\mathbf{R}^N)$, such that for every $u \in W_0^{1,p(x)}(\Omega)$, $\phi(u) = u$, if $x \in \Omega$; $\phi(u) = 0$, if $x \in \mathbf{R}^N \setminus \Omega$. Under this map, $W_0^{1,p(x)}(\Omega)$

is isomorphic to a subspace of $W^{1,p(x)}(\mathbf{R}^N)$. So in the isomorphic meaning, the space $W_0^{1,p(x)}(\Omega)$ is a subspace of $W^{1,p(x)}(\mathbf{R}^N)$. For any k, we can choose a k-dimensional linear subspace E_k of $W_0^{1,p(x)}(\Omega)$ such that $E_k \subset C_0^\infty(\Omega)$. As the norms on E_k are equivalent each other, there exists $\rho_k \le \min\{1, \varepsilon/2\}$ such that $u \in E_k$ with $\|u\| \le \rho_k$ implies $|u|_{L^\infty} \le \delta$. Set

$$S_{\rho_k}^{(k)} = \{u \in E_k | \|u\| = \rho_k\},$$

the compactness of $S_{\rho_k}^{(k)}$ and condition $(f2)$ conclude that there exists a constant $d_k > 0$ such that

$$\int_\Omega \frac{b_0(x)|u|^{q_0(x)}}{q_0(x)} dx \ge d_k, \quad \forall u \in S_{\rho_k}^{(k)}. \tag{3.8}$$

If not, for every n, there exists $u_n \in S_{\rho_k}^{(k)}$ such that

$$\int_\Omega \frac{b_0(x)|u_n|^{q_0(x)}}{q_0(x)} dx < \frac{1}{n}.$$

Let $n \to \infty$, we have

$$\int_\Omega \frac{b_0(x)|u_n|^{q_0(x)}}{q_0(x)} dx \to 0. \tag{3.9}$$

Since $S_{\rho_k}^{(k)}$ is compact, $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$ and $u \in S_{\rho_k}^{(k)}$. From (3.9), we obtain

$$\int_\Omega \frac{b_0(x)|u|^{q_0(x)}}{q_0(x)} dx = 0.$$

For $b_0 > 0$ on Ω, and $(q_0)_\sharp \ge 1$, we obtain $u = 0$. It is contrary to $u \in S_{\rho_k}^{(k)}$. So (3.8) holds. For $u \in S_{\rho_k}^{(k)}$ and $t \in (0,1)$, from (3.5) we have

$$\begin{aligned}
\Phi(tu) &= M\left(\int_\Omega \frac{1}{p(x)} |\nabla tu|^{p(x)} dx\right) + \int_\Omega \frac{1}{p(x)} |tu|^{p(x)} dx - \psi(u) \int_\Omega F(x,tu) dx \\
&\le c\left(\int_\Omega \frac{1}{p(x)} |\nabla tu|^{p(x)} dx\right)^{1/\mu} + \frac{t^{p_\sharp}}{p_\sharp} \int_\Omega |u|^{p(x)} dx - \int_\Omega \frac{b_0(x) t^{q_0(x)} |u|^{q_0(x)}}{q_0(x)} dx \\
&\le \frac{c t^{p_\sharp/\mu}}{(p_\sharp)^{1/\mu}} \rho_k^{\frac{p_\sharp}{\mu}} + \frac{t^{p_\sharp}}{p_\sharp} \rho_k^{p_\sharp} - t^{q_0^\sharp} d_k.
\end{aligned}$$

As $q_0^\sharp < p_\sharp < p_\sharp/\mu$, we can find $t_k \in (0,1)$ such that

$$\Phi(t_k u) < 0, \quad \forall u \in S_{\rho_k}^{(k)},$$

that is,

$$\Phi(u) < 0, \quad \forall u \in S_{t_k\rho_k}^{(k)}.$$

Therefore, $S_{t_k\rho_k}^{(k)} \subset \{u \in W^{1,p(x)}(\mathbf{R}^N) | \Phi(u) < 0\}$. Since $S_{t_k\rho_k}^{(k)}$ is a sphere with radius $t_k\rho_k$ in E_k, a k-dimensional subspace of E_k, so $\gamma(S_{t_k\rho_k}^{(k)}) = k+1$ by Proposition 2.8. Then $\gamma(\{u \in W^{1,p(x)}(\mathbf{R}^N) | \Phi(u) < 0\}) \ge \gamma(S_{t_k\rho_k}^{(k)}) = k+1$. Let $A_k = \{u \in W^{1,p(x)}(\mathbf{R}^N) | \Phi(u) < 0\}$, we have

$A_k \in \Gamma_k$, and $\sup_{u \in A_k} \Phi(u) < 0$. This shows that $(B2)$ holds. Hence, by Proposition 2.9 we complete the proof of Theorem 1.4. ∎

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