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Bounded Oscillation of Second-Order Half-Linear Neutral Delay Dynamic Equations

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Abstract. In this paper, we deal with the oscillation of bounded solutions of a class of second-order half-linear neutral delay dynamic equations with an oscillating coefficient on a time scale. We establish several oscillation criteria for all bounded solutions of the equations by employing a generalized Riccati technique and an integral averaging technique. The results obtained here extend and complement some known results concerning the equations in which the coefficients are of one sign. Examples are given to illustrate our main results.

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1. Introduction

The goal of this paper is to establish some oscillation criteria for all bounded solutions of the following second-order half-linear neutral delay dynamic equation with an oscillating coefficient

$$\left(r(t)|x^{\Delta}(t)|^{\beta-1}x^{\Delta}(t)\right)^{\Delta}+f(t,y(\delta(t)))=0 \tag{1.1}$$

on a time scale $\mathbb{T}$, where

$$x(t):=y(t)+p(t)y(\tau(t)),\quad t\in\mathbb{T}. \tag{1.2}$$

Throughout this paper, we assume that $\sup\mathbb{T}=\infty$ since we are interested in the oscillation of solutions near infinity. Furthermore, in this paper we will use the following hypotheses:

(A_1) $p\in C_{rd}(\mathbb{T},\mathbb{R})$, p is an oscillating function, and $\lim_{t\to\infty}p(t)=0$;
(A_2) $\beta>0$ is a constant;
(A_3) $t_0\in\mathbb{T}$, $\mathbb{I}:=[t_0,\infty)$ is a time scale interval in $\mathbb{T}$, i.e., $\mathbb{I}:=\{t:t\in\mathbb{T},t\geq t_0\}$, $r\in C_{rd}(\mathbb{I},(0,\infty))$, and $\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{1/\beta}\Delta t=\infty$;
(A_4) $\tau\in C_{rd}(\mathbb{T},\mathbb{T})$ and $\lim_{t\to\infty}\tau(t)=\infty$;
(A_5) $\delta\in C_{rd}(\mathbb{T},\mathbb{T}),\delta(t)\leq t$ for $t\in\mathbb{I}$ and $\lim_{t\to\infty}\delta(t)=\infty$;

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(A_6) $f : \mathbb{I}\times\mathbb{R}\to\mathbb{R}$ is continuous function such that $uf(t,u) > 0$ for all $t\in\mathbb{I}$ and for all $u\neq 0$ and there exists a positive rd-continuous function q defined on $\mathbb{I}$ such that $|f(t,u)| \geq q(t)|u^{\beta}|$ for all $t\in\mathbb{I}$ and for all $u\in\mathbb{R}$.

By a solution of (1.1) we mean a nontrivial real function y such that $y(t)+p(t)y(\tau(t)) \in C^1_{rd}[t_y,\infty)$ and $r(t)|[y(t)+p(t)y(\tau(t))]^{\Delta}|^{\beta-1}[y(t)+p(t)y(\tau(t))]^{\Delta} \in C^1_{rd}[t_y,\infty)$ for a certain $t_y \geq t_0$ and satisfying (1.1) for $t\geq t_y$. Our attention is restricted to those solutions of (1.1) which exist on the half-line $[t_y,\infty)$ and satisfy $\sup\{|y(t)| : t > t_*\} > 0$ for any $t_* \geq t_y$. A solution y of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Since Stefan Hilger [21] introduced the theory of time scales, many authors have expounded on various aspects of this new theory; see the books [3,4] by Bohner and Peterson and the papers [1,2,5,8–18,20,22,23,28–31,35,37,40,41] and the references cited therein. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals $\mathbb{R}$ (see [3,4]), and the cases when this time scale is equal to the reals $\mathbb{R}$ or to the integers $\mathbb{Z}$ represent the classical theories of differential equations and of difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice—once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale. In this way results not only related to the set of real numbers $\mathbb{R}$ or the set of integers $\mathbb{Z}$ but those pertaining to more general time scales are obtained. Therefore, not only can the theory of dynamic equations unify the theories of differential equations and of difference equations, but it is also able to extend these classical cases to cases "in between," e.g., to the so-called q-difference equations.

Dynamic equations on time scales have many applications in biology, engineering, economics, physics, neural networks, social sciences and so on (see [3,33]). For instance, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [3]). A book on the subject of time scale, by Bohner and Peterson [3], summarizes and organizes much of time scale calculus. For advances of dynamic equations on time scales we refer the reader to the book [4].

During the last years, much interest has focused on obtaining oscillation criteria of solutions of different classes of dynamic equations on time scales, and we refer the reader to the papers [1,5,8–15,17,18,23,25,28–31,35,37,40,41] and the references cited therein.

Note that if $\beta > 0$ is a quotient of odd positive integers then (1.1) reduces to the equation

$$\text{(1.3)}\qquad \left(r(t)\left(\left[y(t)+p(t)y(\tau(t))\right]^{\Delta}\right)^{\beta}\right)^{\Delta} + f(t,y(\delta(t))) = 0, \quad t\in\mathbb{T}.$$

In 2006, for the case when $\beta \geq 1$ is a quotient of odd positive integers, Wu *et al.* [35] considered (1.3) where the conditions (A_3)–(A_6) and the following conditions are satisfied:

(A_7) $p\in C_{rd}(\mathbb{T},\mathbb{R})$ and $0\leq p(t) < 1$ for $t\in\mathbb{T}$;
(A_8) $\tau(t)\leq t$ for $t\in\mathbb{T}$;
(A_9) $\delta : \mathbb{R}\to\mathbb{R}$ is continuous, $\delta : \mathbb{T}\to\mathbb{R}$ is strictly increasing, and $\tilde{\mathbb{T}} := \delta(\mathbb{T}) = \{\delta(t) : t\in\mathbb{T}\}\subset\mathbb{T}$ is a time scale;

(A_{10}) $(\delta \circ \sigma)(t) = (\sigma \circ \delta)(t)$ for all $t \in \mathbb{T}$, here $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is the forward jump operator on $\mathbb{T}$.

Wu *et al.* [35] got several oscillation theorems for (1.3).

In 2007, Saker *et al.* [30] considered (1.3) where $\beta \geq 1$ is an odd positive integer and the conditions (A_3)–(A_8) and the following condition are satisfied:

(A_{11}) $\int_{t_0}^{\infty} \delta^{\beta}(s) q(s)[1 - p(\delta(s))]^{\beta} \Delta t = \infty.$

Saker *et al.* [30] removed the conditions (A_9) and (A_{10}) used in [35], and established some new oscillation criteria that can be applied on any time scale $\mathbb{T}$.

In 2010, Zhang and Wang [40] studied the oscillation of (1.3) where $\beta > 0$ is a quotient of odd positive integers and the conditions (A_3)–(A_8) and $r^{\Delta}(t) \geq 0$ for $t \in \mathbb{T}$ are assumed to hold. Their results for $\beta \geq 1$ extended and complemented the results in [30,35] and those for $0 < \beta < 1$ are new.

Very recently, Saker and O'Regan [31] were concerned with the oscillatory behavior of (1.3), where $\beta \geq 1$ is a quotient of odd positive integers, $\delta \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\lim_{t\to\infty} \delta(t) = \infty$ and the conditions (A_3), (A_4) and (A_6)–(A_8) are assumed to hold. Saker and O'Regan [31] didn't require the condition (A_{11}) used in [30] and the condition $r^{\Delta}(t) \geq 0$ for $t \in \mathbb{T}$ used in [40]. The results in [31] covered both the case when $\delta(t) > t$ for $t \in \mathbb{T}$ and the case when $\delta(t) \leq t$ for $t \in \mathbb{T}$ and improved some of those in [30,40].

For recent contributions on oscillatory and asymptotic properties of different classes differential equations and difference equations with an oscillating coefficient, we refer the reader to the papers [6,7,24,26,27,34,36,38,39,42,43].

In [27], Luo and Shen introduce a new technique to obtain some new oscillation criteria for the oscillating coefficient delay differential equation with piecewise constant argument of the form

$$x'(t) + a(t)x(t) + b(t)x([t-k]) = 0,$$

where $a(t)$ and $b(t)$ are right continuous functions on $[-k, \infty)$, k is a positive integer, $b(t)$ is oscillatory, and [.] denotes the greatest integer function.

Bolat and Akin [7] and Zhou and Yu [43] considered the higher-order neutral type nonlinear forced differential equation with an oscillating coefficient of the form

$$[y(t) + p(t)y(\tau(t))]^{(n)} + \sum_{i=1}^{m} q_i(t) f_i(y(\delta_i(t))) = s(t),$$

where $n \geq 2$ and the following conditions are always assumed to hold: (i) $p(t), q_i(t), \tau(t), s(t) \in C[t_0, \infty)$ for $i = 1, 2, \ldots, m$; (ii) $p(t)$ and $s(t)$ are oscillating functions; (iii) $q_i(t) \geq 0$ for $i = 1, 2, \ldots, m$; (iv) $\delta_i(t) \in C'[t_0, \infty), \delta_i'(t) > 0, \delta_i(t) \leq t, \lim_{t\to\infty} \delta_i(t) = \infty$ for $i = 1, 2, \ldots, m$, and $\lim_{t\to\infty} \tau(t) = \infty$; (v) $f_i(u) \in C(\mathbb{R})$ is nondecreasing function, $uf_i(u) > 0$ for $u \neq 0$ and $i = 1, 2, \ldots, m$. Bolat and Akin [7] established some oscillation criteria for the equation. By using Krasnoselskii's fixed point theorem and some new techniques, Zhou and Yu [43] obtained a necessary and sufficient criterion for every bounded solution of the equation to be oscillatory or to tend to zero and a sufficient condition for the existence of bounded positive solutions of the equation for general $p(t)$ and $s(t)$. In particular, Zhou and Yu [43] improved the results of Bolat and Akin [7] by removing certain conditions and relaxing some hypotheses used in [7].

Zein and Abu-Kaff [39] presented several sufficient conditions for the oscillation of bounded solutions of n-th order neutral type nonlinear differential equation with an oscillating coefficient of the form

$$[y(t)+p(t)y(\tau(t))]^{(n)}+f(t,y(t),y(\delta(t)))=s(t),$$

where $n\geq 2$, $p(t)\in C(\mathbb{R}_+,\mathbb{R})$ is an oscillatory function with $\lim_{t\to\infty}p(t)=0$, here $\mathbb{R}_+=[0,+\infty), \tau(t),\delta(t)\in C(\mathbb{R}_+,\mathbb{R}),\delta(t)\leq t,\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty, f:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, and $yf(t,x,y)>0$ for $xy>0$, and there exists an oscillatory function $r(t)\in C^n(\mathbb{R}_+,\mathbb{R})$ such that $r^{(n)}(t)=s(t)$ and $\lim_{t\to\infty}r(t)=0$. Zafer in [38] established some sufficient conditions for the oscillation of the equation when $0\leq p(t)<1$. Zein and Abu-Kaff [39] extended the results of Zafer in [38].

Yu and Tang [36] and Tang and Cheng [34] studied the oscillation of all solutions of the delay difference equation

$$y_{n+1}-y_n+p_ny_{n-k}=0,\quad n=0,1,2,\cdots,$$

where k is a positive integer and $\{p_n\}$ is an oscillatory real sequence. Yu and Tang [36] gave an interesting result on the oscillation of the equation by using an effective kind of method to evade those n values p_n taking on negative values. Tang and Cheng [34] obtained an oscillation criterion for the equation by making use of the convexity property of the function $x\ln x$. The results in [36] and [34] improved and extended some of the existing results.

Li [26] established some sufficient conditions for the oscillation of the second-order nonlinear difference equation

$$\Delta[a_ng(\Delta y_n)]+p_{n+1}f(y_{n+1})=0,\quad n\geq 0,$$

where Δ is the forward difference operator defined by $\Delta y_n=y_{n+1}-y_n$, $\{a_n\}$ is an eventually positive real sequence, $\{p_n\}$ is an oscillatory real sequence, and f and g are continuous real-valued functions on $\mathbb{R}$ and satisfy: $uf(u)>0,ug(u)>0,f'(u)\geq 0$ and $g'(u)>0$ for $u\neq 0$.

Bolat and Akin [6] were concerned with the oscillation of solutions of the higher-order nonlinear difference equation of the form

$$\Delta^n[y(k)+p(k)y(k-\tau)]+q(k)f(y(\delta(k)))=0,\quad n\geq 2\in\mathbb{N}_0,\quad k\in\mathbb{N}_0,$$

where $\mathbb{N}_0=\{0,1,2,\cdots\},p(k):\mathbb{N}_0\to\mathbb{R}$ is an oscillating function, $q(k):\mathbb{N}_0\to[0,\infty)$, τ is a positive integer, $\delta(k):\mathbb{N}_0\to\mathbb{Z}$, $\delta(k)\leq k,\lim_{k\to\infty}\delta(k)=\infty,f(u)\in C(\mathbb{R},\mathbb{R})$ is a nondecreasing function and $uf(u)>0$ for $u\neq 0$. They presented two sufficient conditions which ensure that every solution of the equation oscillates or converges to zero.

Zhou [42] dealt with the oscillatory and asymptotic properties of the higher-order nonlinear neutral difference equation of the form

$$\Delta(a_n(\Delta^m(y_n-p_ny_{n-\tau}))^\alpha)+f(n,y_{\delta(n)})=0,$$

where $n\in\mathbb{N}=\{1,2,\cdots\},m,\tau\in\mathbb{N},\alpha$ is a quotient of odd positive integers, $a_n:\mathbb{N}\to(0,\infty),\delta(n)\leq n$ and $\delta(n)\to\infty$ as $n\to\infty,p_n:\mathbb{N}\to\mathbb{R},f(n,u):\mathbb{N}\times\mathbb{R}\to\mathbb{R},uf(n,u)>0$, and $f(n,u)$ is continuous with respect to u, and $f(n,u)\geq f(n,v)$ for $u\geq v$ and for $n\in\mathbb{N}$. Zhou [42] obtained several necessary and sufficient conditions for every bounded solution of the equation to be oscillatory or to tend to zero for general p_n. Zhou [42] weakened some conditions of the results in [6,24].

The motivation of this paper principally comes from the papers [6, 7, 24, 26, 27, 30, 31, 34–36, 38–40, 42, 43]. Obviously, the condition (A_7) is indispensable for all the results in [30, 31, 35, 40]. Also, these results require the condition that β is a quotient of odd positive integers with $\beta \geq 1$ or $\beta > 0$. Therefore, we raise naturally the question whether it is possible to find some new oscillation criteria for (1.1) when $p \in C_{rd}(\mathbb{T},\mathbb{R})$ is an oscillating function and $\beta > 0$ is a constant. To the best of our knowledge, nothing is known regarding this question up to now. Our aim in this paper is to give an affirmative answer to this question. By using a generalized Riccati technique and an integral averaging technique, we obtain some sufficient conditions for the oscillation of all bounded solutions of (1.1) when $p \in C_{rd}(\mathbb{T},\mathbb{R})$ is an oscillating function and $\beta > 0$ is a constant. Our results extend and complement the results established in [30, 31, 35, 40]. We also illustrate the main results with several examples.

In what follows, for convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

2. Preliminaries on time scales and lemmas

For completeness, we recall the following concepts related to the notion of time scales. More details can be found in [3, 4].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Some examples of time scales are as follows: the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the positive integers $\mathbb{N}$, the nonnegative integers $\mathbb{N}_0$, $[0,1]\cup[2,3]$, $[0,1]\cup\mathbb{N}$, $h\mathbb{Z} := \{hk : k \in \mathbb{Z}, h > 0\}$ and $\overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}, q > 1\} \cup \{0\}$. But the rational numbers $\mathbb{Q}$, the complex numbers $\mathbb{C}$ and the open interval $(0,1)$ are not time scales. Many other interesting time scales exist, and they give rise to plenty of applications (see [3]).

For $t \in \mathbb{T}$, the forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{2.1}$$

where $\inf\emptyset = \sup\mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup\emptyset = \inf\mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), here $\emptyset$ denotes the empty set.

Let $t \in \mathbb{T}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by

$$\mu(t) := \sigma(t) - t. \tag{2.2}$$

We also need below the set $\mathbb{T}^\kappa$: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$, then we define the function $f^\sigma : \mathbb{T}^\kappa \to \mathbb{R}$ by

$$f^\sigma(t) := f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}^\kappa,$$

i.e., $f^\sigma := f \circ \sigma$.

For $a, b \in \mathbb{T}$ with $a < b$, we define the interval $[a,b]$ in $\mathbb{T}$ by

$$[a,b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-open intervals, etc. are defined accordingly.

Fix $t \in \mathbb{T}^{\kappa}$ and let $f : \mathbb{T} \to \mathbb{R}$. Define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighbourhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, we say that $f^{\Delta}(t)$ is the (delta) derivative of f at t and that f is (delta) differentiable at t.

Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^{\kappa}$. If f is (delta) differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t). \tag{2.3}$$

If $\mu(t) \neq 0$, then from (2.3) we have

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{2.4}$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided it is continuous at each right-dense point in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of all such rd-continuous functions is denoted by

$$C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are (delta) differentiable and whose (delta) derivative is rd-continuous is denoted by

$$C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

We will make use of the following product and quotient rules for the (delta) derivatives of the product fg and the quotient f/g of two (delta) differentiable functions f and g:

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma} \tag{2.5}$$

and

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}, \tag{2.6}$$

where $g^{\sigma} = g \circ \sigma$ and $gg^{\sigma} \neq 0$.

For $a, b \in \mathbb{T}$ and a (delta) differentiable function f, the Cauchy (delta) integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t \tag{2.7}$$

or

$$\int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t. \tag{2.8}$$

The infinite integral is defined as

$$\int_a^{\infty} f(s)\Delta s = \lim_{t \to \infty} \int_a^t f(s)\Delta s.$$

Next we present some lemmas which we will need in the proof of our main results.

Lemma 2.1 (Bohner and Peterson [3], p. 32, Theorem 1.87). *Let $f:\mathbb{R}\to\mathbb{R}$ be continuously differentiable and suppose $g:\mathbb{T}\to\mathbb{R}$ is delta differentiable. Then $f\circ g:\mathbb{T}\to\mathbb{R}$ is delta differentiable and satisfies*

$$(f\circ g)^{\Delta}(t)=\left\{\int_0^1 f'(g(t)+h\mu(t)g^{\Delta}(t))dh\right\}g^{\Delta}(t).$$

Lemma 2.2 (Hardy *et al.* [19]). *If X and Y are nonnegative, then*

$$\lambda XY^{\lambda-1}-X^{\lambda}\le(\lambda-1)Y^{\lambda}\quad \text{when } \lambda>1,$$

where the equality holds if and only if $X = Y$.

Lemma 2.3 (Bohner and Peterson [3], p. 29, Theorem 1.76 (ii)). *Assume $a,b\in\mathbb{T}$ and $f:\mathbb{T}\to\mathbb{R}$ is rd-continuous. If $[a,b]$ consists of finitely many isolated points, then*

$$\int_a^b f(t)\Delta t=\sum_{t\in[a,b)}\mu(t)f(t).$$

3. Main results

Theorem 3.1. *Suppose that (A_1)–(A_6) hold. Furthermore, assume that there exist a constant $M\in(0,1)$ and a positive function $\varphi\in C^1_{rd}(\mathbb{I},\mathbb{R})$ such that for all sufficiently large $T\ge t_0$,*

$$\limsup_{t\to\infty}\int_{T_1}^{t}\left[M^{\beta}\psi^{\beta}(s,T)q(s)\varphi(s)-\frac{r(s)(\varphi_+^{\Delta}(s))^{\beta+1}}{(\beta+1)^{\beta+1}\varphi^{\beta}(s)}\right]\Delta s=\infty, \tag{3.1}$$

where $T_1>T$ satisfies that $\delta(t)>T$ for $t\in[T_1,\infty)$,

$$\psi(s,T):=\Big(\int_T^s\frac{1}{r^{1/\beta}(u)}\Delta u\Big)^{-1}\int_T^{\delta(s)}\frac{1}{r^{1/\beta}(u)}\Delta u$$

for $T<\delta(s)$ and $\varphi_+^{\Delta}(s):=\max\{\varphi^{\Delta}(s),0\}$. Then every bounded solution of (1.1) is oscillatory.

Proof. Suppose that y is a bounded nonoscillatory solution of (1.1). Without loss of generality, we may assume that y is a bounded eventually positive solution of (1.1). Then from (1.2) and (A_1) we get that x is bounded. From (A_5) we obtain

$$\delta(t)>0\quad\text{and}\quad y(\delta(t))>0. \tag{3.2}$$

It follows from (A_6) and (3.2) that

$$f(t,y(\delta(t)))\ge q(t)[y(\delta(t))]^{\beta}>0. \tag{3.3}$$

From (1.1) and (3.3) we conclude

$$[r(t)|x^{\Delta}(t)|^{\beta-1}x^{\Delta}(t)]^{\Delta}=-f(t,y(\delta(t)))\le-q(t)[y(\delta(t))]^{\beta}<0. \tag{3.4}$$

Thus, there exists $t_1\in[t_0,\infty)$ such that $r(t)|x^{\Delta}(t)|^{\beta-1}x^{\Delta}(t)$ is strictly decreasing on $[t_1,\infty)$ and is eventually of one sign. Therefore, $x^{\Delta}(t)$ is eventually of one sign, too. We claim

$$x^{\Delta}(t)>0,\quad t\in[t_1,\infty). \tag{3.5}$$

Assume on the contrary, then there exists $t_2\in[t_1,\infty)$ such that $x^{\Delta}(t_2)\le0$. Hence, we have $r(t_2)|x^{\Delta}(t_2)|^{\beta-1}x^{\Delta}(t_2)\le0$. Take $t_3>t_2$. Since $r(t)|x^{\Delta}(t)|^{\beta-1}x^{\Delta}(t)$ is strictly decreasing on $[t_1,\infty)$, it is clear that $r(t_3)|x^{\Delta}(t_3)|^{\beta-1}x^{\Delta}(t_3)<r(t_2)|x^{\Delta}(t_2)|^{\beta-1}x^{\Delta}(t_2)$. Therefore, for

$t \in [t_3,\infty)$ we have $r(t)|x^{\Delta}(t)|^{\beta-1}x^{\Delta}(t) \le r(t_3)|x^{\Delta}(t_3)|^{\beta-1}x^{\Delta}(t_3) := c < 0$. Thus, we obtain $x^{\Delta}(t) \le -(-c)^{\frac{1}{\beta}}\left(\frac{1}{r(t)}\right)^{1/\beta}$ for $t \in [t_3,\infty)$. By integrating both sides of the last inequality from t_3 to t, we get

$$x(t)-x(t_3) \le -(-c)^{\frac{1}{\beta}}\int_{t_3}^{t}\Big(\frac{1}{r(s)}\Big)^{1/\beta}\Delta s, \quad t \in [t_3,\infty).$$

Letting $t \to \infty$ and using (A_3), we see $\lim_{t\to\infty} x(t) = -\infty$. This contradicts the fact that x is bounded. Hence, (3.5) holds. From (3.5) we find that $x(t)$ is strictly increasing on $[t_1,\infty)$ and is eventually of one sign. We now claim that $x(t)$ is eventually positive, i.e., there exists $t_4 \in [t_1,\infty)$ such that

$$x(t) > 0, \quad t \in [t_4,\infty). \tag{3.6}$$

Assume on the contrary that $x(t)$ is eventually nonpositive, then there exists $t_5 \in [t_1,\infty)$ such that $x(t) \le 0$ for $t \in [t_5,\infty)$. Therefore, from (1.2) we conclude

$$p(t)y(\tau(t)) = x(t)-y(t) < 0. \tag{3.7}$$

Since p is an oscillating function on $\mathbb{T}$ and $y(\tau(t)) > 0$, we find a contradiction to (3.7). Thus, (3.6) holds. From (3.5), (3.6) and the property that x is bounded, we get $\lim_{t\to\infty} x(t) := L > 0$. From (1.2) we have $y(t) = x(t)-p(t)y(\tau(t))$. Hence, for $M \in (0,1)$, from (A_1) we have

$$\lim_{t\to\infty}[y(t)-Mx(t)] = \lim_{t\to\infty}[(1-M)x(t)-p(t)y(\tau(t))] = (1-M)L > 0.$$

Then by the locally sign-preserving property of limit we conclude $y(t)-Mx(t) > 0$, i.e., $y(t) > Mx(t)$ for all sufficiently large t. Therefore, from (A_5) we obtain $y(\delta(t)) > Mx(\delta(t))$. In view of (3.4) and (3.5), there exists $T \in [t_4,\infty)$ such that

$$[r(t)(x^{\Delta}(t))^{\beta}]^{\Delta} \le -M^{\beta}q(t)[x(\delta(t))]^{\beta}, \quad t \in [T,\infty). \tag{3.8}$$

Define the function w by the generalized Riccati substitution

$$w(t) = \varphi(t)\frac{r(t)(x^{\Delta}(t))^{\beta}}{x^{\beta}(t)}, \quad t \in [T,\infty). \tag{3.9}$$

It is easy to see that $w(t) > 0$ for $t \in [T,\infty)$. Using (2.5) and (2.6), we get

$$\begin{aligned} w^{\Delta} &= [r(x^{\Delta})^{\beta}]^{\Delta}\frac{\varphi}{x^{\beta}} + [r(x^{\Delta})^{\beta}]^{\sigma}\Big(\frac{\varphi}{x^{\beta}}\Big)^{\Delta} \\ &= [r(x^{\Delta})^{\beta}]^{\Delta}\frac{\varphi}{x^{\beta}} + [r(x^{\Delta})^{\beta}]^{\sigma}\Big[\frac{\varphi^{\Delta}}{(x^{\beta})^{\sigma}} - \varphi\frac{(x^{\beta})^{\Delta}}{x^{\beta}(x^{\beta})^{\sigma}}\Big], \quad t \in [T,\infty). \end{aligned} \tag{3.10}$$

Hence, from (3.8)–(3.10) we have

$$w^{\Delta} \le -M^{\beta}q(x\circ\delta)^{\beta}\frac{\varphi}{x^{\beta}} + \frac{\varphi^{\Delta}}{\varphi^{\sigma}}w^{\sigma} - \varphi\frac{w^{\sigma}}{\varphi^{\sigma}}\frac{(x^{\beta})^{\Delta}}{x^{\beta}}, \quad t \in [T,\infty). \tag{3.11}$$

Since $r(t)(x^{\Delta}(t))^{\beta}$ is strictly decreasing on $[t_1,\infty)$, for $t \in [T,\infty)$ we obtain

$$x(t)-(x\circ\delta)(t) = \int_{\delta(t)}^{t}\frac{[r(u)(x^{\Delta}(u))^{\beta}]^{1/\beta}}{r^{1/\beta}(u)}\Delta u \le [(r\circ\delta)(t)(x^{\Delta}\circ\delta)^{\beta}(t)]^{1/\beta}\int_{\delta(t)}^{t}\frac{1}{r^{1/\beta}(u)}\Delta u$$

and

$$\frac{x(t)}{(x\circ\delta)(t)} \le 1 + \frac{[(r\circ\delta)(t)(x^{\Delta}\circ\delta)^{\beta}(t)]^{1/\beta}}{(x\circ\delta)(t)}\int_{\delta(t)}^{t}\frac{1}{r^{1/\beta}(u)}\Delta u. \tag{3.12}$$

Take $T_1 \in [T,\infty)$ such that $\delta(t) > T$ for $t \in [T_1,\infty)$. Then, for $t \in [T_1,\infty)$ we get

$$(x\circ\delta)(t) > (x\circ\delta)(t) - x(T) = \int_T^{\delta(t)} \frac{[r(u)(x^\Delta(u))^\beta]^{1/\beta}}{r^{1/\beta}(u)}\Delta u$$
$$\geq [(r\circ\delta)(t)(x^\Delta\circ\delta)^\beta(t)]^{1/\beta}\int_T^{\delta(t)} \frac{1}{r^{1/\beta}(u)}\Delta u$$

and

$$\frac{[(r\circ\delta)(t)(x^\Delta\circ\delta)^\beta(t)]^{1/\beta}}{(x\circ\delta)(t)} < \Big(\int_T^{\delta(t)} \frac{1}{r^{1/\beta}(u)}\Delta u\Big)^{-1}. \tag{3.13}$$

Therefore, (3.12) and (3.13) imply

$$\frac{x(t)}{(x\circ\delta)(t)} \leq 1 + \Big(\int_T^{\delta(t)} \frac{1}{r^{1/\beta}(u)}\Delta u\Big)^{-1}\int_{\delta(t)}^t \frac{1}{r^{1/\beta}(u)}\Delta u$$
$$= \int_T^t \frac{1}{r^{1/\beta}(u)}\Delta u\Big(\int_T^{\delta(t)} \frac{1}{r^{1/\beta}(u)}\Delta u\Big)^{-1}, \quad t \in [T_1,\infty).$$

Hence, from (3.11) we obtain

$$w^\Delta \leq -M^\beta\psi^\beta(t,T)q\varphi + \frac{\varphi^\Delta}{\varphi^\sigma}w^\sigma - \varphi\frac{w^\sigma}{\varphi^\sigma}\frac{(x^\beta)^\Delta}{x^\beta}, \quad t \in [T_1,\infty). \tag{3.14}$$

where ψ is defined as in Theorem 3.1. By (2.3) and Lemma 2.1, for $t \in [T_1,\infty)$ we have

$$(x^\beta(t))^\Delta = \beta\Big\{\int_0^1 [x(t) + h\mu(t)x^\Delta(t)]^{\beta-1}dh\Big\}x^\Delta(t)$$
$$= \beta\Big\{\int_0^1 [(1-h)x(t) + hx^\sigma(t)]^{\beta-1}dh\Big\}x^\Delta(t)$$
$$\geq \begin{cases} \beta(x^\sigma(t))^{\beta-1}x^\Delta(t), & 0 < \beta \leq 1, \\ \beta(x(t))^{\beta-1}x^\Delta(t), & \beta > 1. \end{cases} \tag{3.15}$$

From (3.14) and (3.15), if $0 < \beta \leq 1$, we get

$$w^\Delta \leq -M^\beta\psi^\beta(t,T)q\varphi + \frac{\varphi^\Delta}{\varphi^\sigma}w^\sigma - \beta\varphi\frac{w^\sigma}{\varphi^\sigma}\frac{x^\Delta}{x^\sigma}\Big(\frac{x^\sigma}{x}\Big)^\beta, \quad t \in [T_1,\infty), \tag{3.16}$$

whereas if $\beta > 1$, we find

$$w^\Delta \leq -M^\beta\psi^\beta(t,T)q\varphi + \frac{\varphi^\Delta}{\varphi^\sigma}w^\sigma - \beta\varphi\frac{w^\sigma}{\varphi^\sigma}\frac{x^\Delta}{x^\sigma}\frac{x^\sigma}{x}, \quad t \in [T_1,\infty). \tag{3.17}$$

Using the fact that $x(t)$ is strictly increasing, $r(t)(x^\Delta(t))^\beta$ is strictly decreasing and $\sigma(t) \geq t$, we conclude

$$x^\sigma(t) \geq x(t), \quad x^\Delta(t) \geq \Big(\frac{r^\sigma(t)}{r(t)}\Big)^{1/\beta}(x^\Delta(t))^\sigma, \quad t \in [T_1,\infty). \tag{3.18}$$

From (3.16)–(3.18), for $\beta > 0$ we obtain

$$w^\Delta \leq -M^\beta\psi^\beta(t,T)q\varphi + \frac{\varphi^\Delta}{\varphi^\sigma}w^\sigma - \beta\varphi\frac{w^\sigma}{\varphi^\sigma}\Big(\frac{r^\sigma}{r}\Big)^{1/\beta}\frac{(x^\Delta)^\sigma}{x^\sigma}, \quad t \in [T_1,\infty).$$

In view of (3.9), we get

$$w^{\Delta}(t) \le -M^{\beta}\psi^{\beta}(t,T)q(t)\varphi(t) + \frac{\varphi_{+}^{\Delta}(t)}{\varphi^{\sigma}(t)}w^{\sigma}(t) - \frac{\beta\varphi(t)(w^{\sigma}(t))^{\lambda}}{(\varphi^{\sigma}(t))^{\lambda}r^{1/\beta}(t)}, \quad t \in [T_1,\infty), \tag{3.19}$$

where φ_{+}^{Δ} is defined as in Theorem 3.1 and $\lambda := 1 + \frac{1}{\beta}$. Taking

$$X = \frac{(\beta\varphi(t))^{1/\lambda}w^{\sigma}(t)}{\varphi^{\sigma}(t)r^{1/(\beta+1)}(t)} \quad \text{and} \quad Y = \frac{(\beta r(t))^{1/\lambda}(\varphi_{+}^{\Delta}(t))^{\beta}}{(\beta+1)^{\beta}\varphi^{\beta/\lambda}(t)},$$

by Lemma 2.2 and (3.19) we have

$$w^{\Delta}(t) \le \frac{r(t)(\varphi_{+}^{\Delta}(t))^{\beta+1}}{(\beta+1)^{\beta+1}\varphi^{\beta}(t)} - M^{\beta}\psi^{\beta}(t,T)q(t)\varphi(t), \quad t \in [T_1,\infty).$$

Integrating from T_1 to t, we obtain

$$\int_{T_1}^{t}\left[M^{\beta}\psi^{\beta}(s,T)q(s)\varphi(s) - \frac{r(s)(\varphi_{+}^{\Delta}(s))^{\beta+1}}{(\beta+1)^{\beta+1}\varphi^{\beta}(s)}\right]\Delta s \le w(T_1) - w(t) < w(T_1), \quad t \in [T_1,\infty),$$

which implies a contradiction to (3.1). The proof is complete. ∎

The following theorem gives a Philos-type oscillation criterion for all bounded solutions of (1.1).

Theorem 3.2. *Suppose that (A_1)–(A_6) hold. Furthermore, suppose that there exist a constant $M \in (0,1)$, a positive function $\varphi \in C_{rd}^{1}(\mathbb{I},\mathbb{R})$ and a function $H \in C_{rd}(\mathbb{D},\mathbb{R})$, where $\mathbb{D} := \{(t,s) \in \mathbb{T}\times\mathbb{T} : t \ge s \ge t_0\}$, such that*

$$H(t,t) = 0 \quad for\, t \ge t_0, \quad H(t,s) > 0 \quad for\ (t,s) \in \mathbb{D}_0,$$

where $\mathbb{D}_0 := \{(t,s) \in \mathbb{T}\times\mathbb{T} : t > s \ge t_0\}$, and H has a nonpositive rd-continuous delta partial derivative $H^{\Delta_s}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable and satisfies, for all sufficiently large $T \ge t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T_1)}\int_{T_1}^{t}\left[M^{\beta}H(t,s)\psi^{\beta}(s,T)q(s)\varphi(s) - \frac{r(s)(h_{+}(t,s)\varphi^{\sigma}(s))^{\beta+1}}{(\beta+1)^{\beta+1}(H(t,s)\varphi(s))^{\beta}}\right]\Delta s = \infty, \tag{3.20}$$

where T_1 is defined as in Theorem 3.1 and $h_{+}(t,s) := \max\{0, H^{\Delta_s}(t,s) + H(t,s)\frac{\varphi_{+}^{\Delta}(s)}{\varphi^{\sigma}(s)}\}$, here φ_{+}^{Δ} is defined as in Theorem 3.1. Then all bounded solutions of (1.1) are oscillatory.

Proof. Suppose that y is a bounded nonoscillatory solution of (1.1). Without loss of generality, we may assume that y is a bounded eventually positive solution of (1.1). We proceed as in the proof of Theorem 3.1 to get that (3.19) holds. Multiplying (3.19) by $H(t,s)$ and integrating from T_1 to t, we find

$$\begin{aligned}
&\int_{T_1}^{t}M^{\beta}H(t,s)\psi^{\beta}(s,T)q(s)\varphi(s)\Delta s \\
&\le -\int_{T_1}^{t}H(t,s)w^{\Delta}(s)\Delta s + \int_{T_1}^{t}H(t,s)\frac{\varphi_{+}^{\Delta}(s)}{\varphi^{\sigma}(s)}w^{\sigma}(s)\Delta s \\
&\quad -\int_{T_1}^{t}H(t,s)\frac{\beta\varphi(s)(w^{\sigma}(s))^{\lambda}}{(\varphi^{\sigma}(s))^{\lambda}r^{1/\beta}(s)}\Delta s, \quad t \in [T_1,\infty).
\end{aligned} \tag{3.21}$$

Applying (2.7), for $t \in [T_1,\infty)$ we get

$$-\int_{T_1}^{t} H(t,s)w^{\Delta}(s)\Delta s = \Big[-H(t,s)w(s)\Big]_{s=T_1}^{s=t} + \int_{T_1}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s$$

$$= H(t,T_1)w(T_1) + \int_{T_1}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s. \tag{3.22}$$

Substituting (3.22) in (3.21), for $t \in [T_1,\infty)$ we obtain

$$\begin{aligned}
&\int_{T_1}^{t} M^{\beta} H(t,s)\psi^{\beta}(s,T)q(s)\varphi(s)\Delta s \\
&\le H(t,T_1)w(T_1) \\
&\quad + \int_{T_1}^{t} \left\{ \left[H^{\Delta_s}(t,s) + H(t,s)\frac{\varphi_+^{\Delta}(s)}{\varphi^{\sigma}(s)}\right] w^{\sigma}(s) - H(t,s)\frac{\beta\varphi(s)(w^{\sigma}(s))^{\lambda}}{(\varphi^{\sigma}(s))^{\lambda} r^{1/\beta}(s)} \right\}\Delta s \\
&\le H(t,T_1)w(T_1) + \int_{T_1}^{t} \left[h_+(t,s)w^{\sigma}(s) - H(t,s)\frac{\beta\varphi(s)(w^{\sigma}(s))^{\lambda}}{(\varphi^{\sigma}(s))^{\lambda} r^{1/\beta}(s)}\right]\Delta s,
\end{aligned} \tag{3.23}$$

where h_+ is defined as in Theorem 3.2. Therefore by using Lemma 2.2 in (3.23) with

$$X = \frac{(H(t,s)\beta\varphi(s))^{1/\lambda} w^{\sigma}(s)}{\varphi^{\sigma}(s) r^{1/(\beta+1)}(s)} \quad \text{and} \quad Y = \frac{r^{1/\lambda}(s)(h_+(t,s)\varphi^{\sigma}(s))^{\beta}}{\lambda^{\beta}(H(t,s)\beta\varphi(s))^{\beta/\lambda}},$$

we have for $t \in [T_1,\infty)$,

$$\begin{aligned}
&\int_{T_1}^{t} M^{\beta} H(t,s)\psi^{\beta}(s,T)q(s)\varphi(s)\Delta s \\
&\le H(t,T_1)w(T_1) + \int_{T_1}^{t} \frac{r(s)(h_+(t,s)\varphi^{\sigma}(s))^{\beta+1}}{(\beta+1)^{\beta+1}(H(t,s)\varphi(s))^{\beta}}\Delta s.
\end{aligned}$$

Therefore, we obtain for $t \in [T_1,\infty)$,

$$\frac{1}{H(t,T_1)}\int_{T_1}^{t} \left[M^{\beta} H(t,s)\psi^{\beta}(s,T)q(s)\varphi(s) - \frac{r(s)(h_+(t,s)\varphi^{\sigma}(s))^{\beta+1}}{(\beta+1)^{\beta+1}(H(t,s)\varphi(s))^{\beta}}\right]\Delta s \le w(T_1),$$

which contradicts (3.20). Thus, this completes the proof. ∎

Let $H(t,s) = (t-s)^m, (t,s) \in \mathbb{D}$, where $m \ge 1$ is a constant, then $H^{\Delta_s}(t,s) \le -m(t-\sigma(s))^{m-1}$ for $(t,s) \in \mathbb{D}_0$ (see Saker [32]). Therefore, from (3.22) we obtain for $t \in [T_1,\infty)$,

$$-\int_{T_1}^{t} H(t,s)w^{\Delta}(s)\Delta s \le H(t,T_1)w(T_1) + \int_{T_1}^{t} [-m(t-\sigma(s))^{m-1}]w^{\sigma}(s)\Delta s. \tag{3.24}$$

By replacing (3.22) with (3.24) and using methods similar to those of the proof of Theorem 3.2, we obtain the following Kamenev-type oscillation criterion for all bounded solutions of (1.1).

Theorem 3.3. *Assume that (A_1)–(A_6) hold. Furthermore, suppose that there exist constants $M \in (0,1)$, $m \ge 1$ and a positive function $\varphi \in C_{rd}^1(\mathbb{I},\mathbb{R})$ such that for all sufficiently large $T \ge t_0$,*

(3.25)

$$\limsup_{t\to\infty} \frac{1}{t^m}\int_{T_1}^{t} \left[M^{\beta}(t-s)^m\psi^{\beta}(s,T)q(s)\varphi(s) - \frac{r(s)\big(K_+(t,s)\varphi^{\sigma}(s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\big((t-s)^m\varphi(s)\big)^{\beta}}\right]\Delta s = \infty,$$

where T_1 is defined as in Theorem 3.1 and

$$K_+(t,s) := \max\{(t-s)^m \frac{\varphi_+^\Delta(s)}{\varphi(\sigma(s))} - m(t-\sigma(s))^{m-1}, 0\},$$

here φ_+^Δ is defined as in Theorem 3.1. Then all bounded solutions of (1.1) are oscillatory.

Remark 3.1. From Theorems 3.1–3.3, we can obtain many different sufficient conditions for the oscillation of all bounded solutions of (1.1) with different choices of the functions φ and H and the constant m.

For instance, let $\varphi(s) = s$, then Theorem 3.1 implies the following results.

Corollary 3.1. *Suppose that (A_1)–(A_6) hold and that there exists a constant $M \in (0,1)$ such that for all sufficiently large $T \geq t_0$,*

$$\limsup_{t\to\infty} \int_{T_1}^{t} \left[M^\beta \psi^\beta(s,T) s q(s) - \frac{r(s)}{(\beta+1)^{\beta+1} s^\beta} \right] \Delta s = \infty, \tag{3.26}$$

where T_1 and ψ are defined as in Theorem 3.1. Then every bounded solution of (1.1) is oscillatory.

Let $\varphi(s) = 1$, then from Theorem 3.1 we have the following.

Corollary 3.2. *Suppose that (A_1)–(A_6) hold and that for all sufficiently large $T \geq t_0$,*

$$\int_{T_1}^{\infty} \psi^\beta(s,T) q(s) \Delta s = \infty, \tag{3.27}$$

where T_1 and ψ are defined as in Theorem 3.1. Then all bounded solutions of (1.1) are oscillatory.

Next, we illustrate our main results with two examples, to which the results in [30, 31, 35, 40] fail to be applied.

Example 3.1. Consider the following second-order half-linear neutral delay dynamic equation with an oscillating coefficient

(3.28)
$$\left\{ \frac{1}{t^2} \left| \left[y(t) + \left(-\frac{1}{2}\right)^t y(t-h) \right]^\Delta \right| \left[y(t) + \left(-\frac{1}{2}\right)^t y(t-h) \right]^\Delta \right\}^\Delta + \frac{1}{(t+h)^2 h^3} \Big\{ 8 + (-1)^t V(t+h)$$
$$- (-1)^t V(t) + \frac{2th+h^2}{t^2} \left[4 - (-1)^t V(t) \right] \Big\} |y(t-3h)| y(t-3h) = 0, \quad t \in \mathbb{T}, \quad t > 0,$$

where h is an arbitrary odd positive integer,

$$\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\} = \{\cdots, -4h, -3h, -2h, -h, 0, h, 2h, 3h, 4h, \cdots\} \tag{3.29}$$

and

$$V(t) := -(-1)^t \left(\frac{1}{2}\right)^{2t+2h} - (-1)^t \left(\frac{1}{2}\right)^{2t} - 4\left(\frac{1}{2}\right)^{t+h} + 4\left(\frac{1}{2}\right)^t + 2(-1)^t \left(\frac{1}{2}\right)^{2t+h}. \tag{3.30}$$

It is easy to see that $\sigma(t) = t+h, \mu(t) = \sigma(t) - t = h$ (see (2.1) and (2.2)) and

$$\lim_{t\to\infty} V(t) = 0. \tag{3.31}$$

In (3.28), $r(t)=\frac{1}{t^2}, p(t)=\big(-\frac{1}{2}\big)^t, \tau(t)=t-h, \beta=2, \delta(t)=t-3h$ and

$$f(t,u)=\frac{1}{(t+h)^2h^3}\Big\{8+(-1)^tV(t+h)-(-1)^tV(t)+\frac{2th+h^2}{t^2}\big[4-(-1)^tV(t)\big]\Big\}|u|u.$$

Let

$$q(t)=\frac{1}{(t+h)^2h^3}\Big\{8+(-1)^tV(t+h)-(-1)^tV(t)+\frac{2th+h^2}{t^2}\big[4-(-1)^tV(t)\big]\Big\}. \tag{3.32}$$

From (3.31) we obtain

$$\lim_{t\to\infty}\big\{8+(-1)^tV(t+h)-(-1)^tV(t)+\frac{2th+h^2}{t^2}\big[4-(-1)^tV(t)\big]\big\}=8>0.$$

Thus, there exists $t_0\in\mathbb{T}$ such that $t_0\geq h$ and $8+(-1)^tV(t+h)-(-1)^tV(t)+\frac{2th+h^2}{t^2}\big[4-(-1)^tV(t)\big]>0$ for $t\geq t_0$. Therefore, we have $q(t)>0$ on the time scale interval $\mathbb{I}:=[t_0,\infty)=\{t_0,t_0+h,t_0+2h,\cdots\}$ and we find that (A_1)–(A_6) are satisfied. We will apply Corollary 3.1 and it remains to prove that (3.26) holds.

For every sufficiently large $T\geq t_0$ ($T\in\mathbb{T}$), take $T_1=T+4h$, then we have $\delta(t)>T$ for $t\in[T_1,\infty)$. For $s\in[T_1,\infty)$, by Lemma 2.3 we get

$$\begin{aligned}\psi(s,T)&:=\Big(\int_T^s\frac{1}{r^{1/\beta}(u)}\Delta u\Big)^{-1}\int_T^{\delta(s)}\frac{1}{r^{1/\beta}(u)}\Delta u=\Big(\int_T^s u\Delta u\Big)^{-1}\int_T^{s-3h}u\Delta u\\&=\Big(\sum_{u\in[T,s)}hu\Big)^{-1}\Big(\sum_{u\in[T,s-3h)}hu\Big)\\&=\Big[hT+h(T+h)+\cdots+h(s-h)\Big]^{-1}\Big[hT+h(T+h)+\cdots+h(s-4h)\Big]\\&=\Big[h(T+s-h)\big(\frac{s-T}{h}\big)/2\Big]^{-1}\Big[h(T+s-4h)\big(\frac{s-3h-T}{h}\big)/2\Big]\\&=\Big[(T+s-h)\big(\frac{s-T}{h}\big)\Big]^{-1}\Big[(T+s-4h)\big(\frac{s-T}{h}-3\big)\Big].\end{aligned}$$

Therefore, we have

$$\lim_{s\to\infty}\psi(s,T)=1. \tag{3.33}$$

Take an arbitrary constant $M\in(0,1)$. It follows from (3.31)–(3.33) that

$$\begin{aligned}&\limsup_{t\to\infty}\int_{T_1}^t\Big[M^\beta\psi^\beta(s,T)sq(s)-\frac{r(s)}{(\beta+1)^{\beta+1}s^\beta}\Big]\Big/\Big(\frac{1}{s}\Big)\Delta s\\&=\limsup_{t\to\infty}\int_{T_1}^t\Big[M^2\psi^2(s,T)s^2q(s)-\frac{1}{9s^3}\Big]\Delta s=\frac{8M^2}{h^3}>0.\end{aligned}$$

Since $\limsup_{t\to\infty}\int_{T_1}^t\frac{1}{s}\Delta s=\infty$, we get

$$\limsup_{t\to\infty}\int_{T_1}^t\Big[M^\beta\psi^\beta(s,T)sq(s)-\frac{r(s)}{(\beta+1)^{\beta+1}s^\beta}\Big]\Delta s=\infty,$$

which implies that (3.26) holds. Thus by Corollary 3.1, every bounded solution of (3.28) is oscillatory. In fact, $y(t)=(-1)^t$ is such a solution of (3.28). The verification is as follows:

Let $y(t)=(-1)^t$. In view of the fact that h is an odd positive integer, form (2.4) we get

$$\begin{aligned}\big[y(t)+\big(-\frac{1}{2}\big)^t y(t-h)\big]^\Delta &= [(-1)^t+\big(-\frac{1}{2}\big)^t(-1)^{t-h}\big]^\Delta\\ &= [(-1)^t-\big(\frac{1}{2}\big)^t\big]^\Delta\\ &= \Big\{\big[(-1)^{t+h}-\big(\frac{1}{2}\big)^{t+h}\big]-\big[(-1)^t-\big(\frac{1}{2}\big)^t\big]\Big\}/h\\ &= \Big[-2(-1)^t-\big(\frac{1}{2}\big)^{t+h}+\big(\frac{1}{2}\big)^t\Big]/h.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\Big|\big[y(t)+\big(-\frac{1}{2}\big)^t y(t-h)\big]^\Delta\Big|\big[y(t)+\big(-\frac{1}{2}\big)^t y(t-h)\big]^\Delta\\ &=\Big\{\Big[2+(-1)^t\big(\frac{1}{2}\big)^{t+h}-(-1)^t\big(\frac{1}{2}\big)^t\Big]/h\Big\}\\ &\qquad\cdot\Big\{-(-1)^t\Big[2+(-1)^t\big(\frac{1}{2}\big)^{t+h}-(-1)^t\big(\frac{1}{2}\big)^t\Big]/h\Big\}\\ &=\frac{1}{h^2}\Big[-4(-1)^t-(-1)^t\big(\frac{1}{2}\big)^{2t+2h}-(-1)^t\big(\frac{1}{2}\big)^{2t}\\ &\qquad -4\big(\frac{1}{2}\big)^{t+h}+4\big(\frac{1}{2}\big)^t+2(-1)^t\big(\frac{1}{2}\big)^{2t+h}\Big]\\ &=\frac{1}{h^2}\big[-4(-1)^t+V(t)\big]\quad\text{for}\quad t\ge t_0\ge h,\end{aligned}\tag{3.34}$$

where $V(t)$ is defined as in (3.30). Therefore, from (3.34) and (2.4) we have

$$\begin{aligned}&\Big\{\frac{1}{t^2}\Big|\big[y(t)+\big(-\frac{1}{2}\big)^t y(t-h)\big]^\Delta\Big|\big[y(t)+\big(-\frac{1}{2}\big)^t y(t-h)\big]^\Delta\Big\}^\Delta\\ &=\Big\{\frac{1}{t^2h^2}\big[-4(-1)^t+V(t)\big]\Big\}^\Delta\\ &=\Big\{\frac{1}{(t+h)^2h^2}\big[-4(-1)^{t+h}+V(t+h)\big]-\frac{1}{t^2h^2}\big[-4(-1)^t+V(t)\big]\Big\}/h\\ &=\frac{(-1)^t}{(t+h)^2h^3}\Big\{8+(-1)^tV(t+h)-(-1)^tV(t)+\frac{2th+h^2}{t^2}\big[4-(-1)^tV(t)\big]\Big\}\\ &=(-1)^tq(t)\quad\text{for}\quad t\ge t_0\ge h,\end{aligned}\tag{3.35}$$

where $q(t)$ is defined as in (3.32). Also, we have

$$|y(t-3h)|y(t-3h)=|(-1)^{t-3h}|(-1)^{t-3h}=-(-1)^t.\tag{3.36}$$

Thus, from (3.35), (3.32) and (3.36) we obtain

$$\begin{aligned}&\Big\{\frac{1}{t^2}\Big|\big[y(t)+\big(-\frac{1}{2}\big)^t y(t-h)\big]^\Delta\Big|\big[y(t)+\big(-\frac{1}{2}\big)^t y(t-h)\big]^\Delta\Big\}^\Delta\\ &\quad+\frac{1}{(t+h)^2h^3}\Big\{8+(-1)^tV(t+h)-(-1)^tV(t)+\frac{2th+h^2}{t^2}\big[4-(-1)^tV(t)\big]\Big\}\\ &\qquad\cdot|y(t-3h)|y(t-3h)=(-1)^tq(t)+q(t)[-(-1)^t]=0\quad\text{for}\quad t\ge t_0\ge h,\end{aligned}$$

which implies that $y(t) = (-1)^t$ is a bounded solution of (3.28). Clearly, $y(t) = (-1)^t$ is oscillatory on $\mathbb{T}$.

Example 3.2. Consider the following second-order half-linear neutral delay dynamic equation with an oscillating coefficient

$$\left((t+1)^\beta |x^\Delta(t)|^{\beta-1} x^\Delta(t)\right)^\Delta + \frac{1}{\psi^\beta(t,t^*)t} |y(\delta(t))|^{\beta-1} y(\delta(t)) = 0, \quad t \in \mathbb{T}, \tag{3.37}$$

where $x(t) := y(t) + p(t)y(\tau(t))$, p, β, τ and δ satisfy (A_1), (A_2), (A_4) and (A_5), respectly, $t^* \in \mathbb{T}$ and $t^* > 0$. In (3.37), $r(t) = (t+1)^\beta$ and $f(t,y) = \frac{1}{\psi^\beta(t,t^*)t}|y|^{\beta-1}y$. Take $t_0 > t^*$ such that $\delta(t) > t^*$ for $t \in [t_0,\infty)$. Let $q(t) = \frac{1}{\psi^\beta(t,t^*)t}$. Then we find that ($A_1$)–($A_6$) are satisfied. We will apply Corollary 3.2 and it remains to prove that (3.27) holds. Since $\lim_{t\to\infty} \frac{\psi(t,T)}{\psi(t,t^*)} = 1$, we have

$$\int_{T_1}^\infty \psi^\beta(s,T)q(s)\Delta s = \int_{T_1}^\infty \left(\frac{\psi(s,T)}{\psi(s,t^*)}\right)^\beta \frac{1}{s}\Delta s = \int_{T_1}^\infty \frac{1}{s}\Delta s = \infty,$$

which yields that (3.27) holds. Thus, all bounded solutions of (3.37) are oscillatory by Corollary 3.2.

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References

[1] R. P. Agarwal, M. Bohner, D. O'Regan and A. Peterson, Dynamic equations on time scales: a survey, *J. Comput. Appl. Math.* **141** (2002), no. 1–2, 1–26.

[2] M. Benchohra, S. K. Ntouyas and A. Ouahab, Existence results for second order boundary value problem of impulsive dynamic equations on time scales, *J. Math. Anal. Appl.* **296** (2004), no. 1, 65–73.

[3] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2001.

[4] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[5] M. Bohner and S. H. Saker, Oscillation criteria for perturbed nonlinear dynamic equations, *Math. Comput. Modelling* **40** (2004), no. 3–4, 249–260.

[6] Y. Bolat and Ö. Akin, Oscillatory behaviour of a higher-order nonlinear neutral type functional difference equation with oscillating coefficients, *Appl. Math. Lett.* **17** (2004), no. 9, 1073–1078.

[7] Y. Bolat and O. Akin, Oscillatory behaviour of higher order neutral type nonlinear forced differential equation with oscillating coefficients, *J. Math. Anal. Appl.* **290** (2004), no. 1, 302–309.

[8] D.-X. Chen, Oscillation and asymptotic behavior for *n*th-order nonlinear neutral delay dynamic equations on time scales, *Acta Appl. Math.* **109** (2010), no. 3, 703–719.

[9] D. X. Chen and J. C. Liu, Oscillation theorems for second-order nonlinear neutral dynamic equations on time scales with distributed delay, *J. Systems Sci. Math. Sci.* **30** (2010), no. 9, 1191–1205.

[10] D.-X. Chen and J.-C. Liu, Asymptotic behavior and oscillation of solutions of third-order nonlinear neutral delay dynamic equations on time scales, *Can. Appl. Math. Q.* **16** (2008), no. 1, 19–43.

[11] A. Del Medico and Q. Kong, Kamenev-type and interval oscillation criteria for second-order linear differential equations on a measure chain, *J. Math. Anal. Appl.* **294** (2004), no. 2, 621–643.

[12] O. Došlý and S. Hilger, A necessary and sufficient condition for oscillation of the Sturm-Liouville dynamic equation on time scales, *J. Comput. Appl. Math.* **141** (2002), no. 1–2, 147–158.

[13] L. Erbe, T. S. Hassan and A. Peterson, Oscillation criteria for nonlinear damped dynamic equations on time scales, *Appl. Math. Comput.* **203** (2008), no. 1, 343–357.

[14] L. Erbe, A. Peterson and S. H. Saker, Hille and Nehari type criteria for third-order dynamic equations, *J. Math. Anal. Appl.* **329** (2007), no. 1, 112–131.

[15] L. Erbe, A. Peterson and S. H. Saker, Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales, *J. Comput. Appl. Math.* **181** (2005), no. 1, 92–102.

[16] G. Sh. Guseinov, Integration on time scales, *J. Math. Anal. Appl.* **285** (2003), no. 1, 107–127.

[17] T. S. Hassan, Oscillation of third order nonlinear delay dynamic equations on time scales, *Math. Comput. Modelling* **49** (2009), no. 7–8, 1573–1586.

[18] T. S. Hassan, Oscillation criteria for half-linear dynamic equations on time scales, *J. Math. Anal. Appl.* **345** (2008), no. 1, 176–185.

[19] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, reprint of the 1952 edition, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 1988.

[20] Z. He, Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales, *J. Comput. Appl. Math.* **182** (2005), no. 2, 304–315.

[21] S. Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. thesis, Universität Würzburg, Würzburg, 1988.

[22] J. Hoffacker and C. C. Tisdell, Stability and instability for dynamic equations on time scales, *Comput. Math. Appl.* **49** (2005), no. 9–10, 1327–1334.

[23] B. Karpuz, Asymptotic behaviour of bounded solutions of a class of higher-order neutral dynamic equations, *Appl. Math. Comput.* **215** (2009), no. 6, 2174–2183.

[24] W.-T. Li, Oscillation of higher-order neutral nonlinear difference equations, *Appl. Math. Lett.* **11** (1998), no. 4, 1–8.

[25] T. Li, Z. Han, S. Sun and Y. Zhao, Oscillation results for third order nonlinear delay dynamic equations on time scales, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 3, 639–648.

[26] W.-T. Li, Oscillation theorems for second-order nonlinear difference equations, *Math. Comput. Modelling* **31** (2000), no. 6–7, 71–79.

[27] Z. Luo and J. Shen, New results on oscillation for delay differential equations with piecewise constant argument, *Comput. Math. Appl.* **45** (2003), no. 12, 1841–1848.

[28] L. Ou, Atkinson's super-linear oscillation theorem for matrix dynamic equations on a time scale, *J. Math. Anal. Appl.* **299** (2004), no. 2, 615–629.

[29] Y. Şahiner, Oscillation of second-order delay differential equations on time scales, *Nonlinear Anal.* **63** (2005), no. 5–7, 1073–1080.

[30] S. H. Saker, R. P. Agarwal and D. O'Regan, Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales, *Appl. Anal.* **86** (2007), no. 1, 1–17.

[31] S. H. Saker and D. O'Regan, New oscillation criteria for second-order neutral functional dynamic equations via the generalized Riccati substitution, *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011), no. 1, 423–434.

[32] S. H. Saker, Oscillation of second-order nonlinear neutral delay dynamic equations on time scales, *J. Comput. Appl. Math.* **187** (2006), no. 2, 123–141.

[33] V. Spedding, Taming Nature's Numbers, New Scientist, 179 (2003), no. 2402, 28–31.

[34] X.-H. Tang and S. S. Cheng, An oscillation criterion for linear difference equations with oscillating coefficients, *J. Comput. Appl. Math.* **132** (2001), no. 2, 319–329.

[35] H.-W. Wu, R.-K. Zhuang and R. M. Mathsen, Oscillation criteria for second-order nonlinear neutral variable delay dynamic equations, *Appl. Math. Comput.* **178** (2006), no. 2, 321–331.

[36] J. S. Yu and X. H. Tang, Sufficient conditions for the oscillation of linear delay difference equations with oscillating coefficients, *J. Math. Anal. Appl.* **250** (2000), no. 2, 735–742.

[37] A. Zafer, On oscillation and nonoscillation of second-order dynamic equations, *Appl. Math. Lett.* **22** (2009), no. 1, 136–141.

[38] A. Zafer, Oscillation criteria for even order neutral differential equations, *Appl. Math. Lett.* **11** (1998), no. 3, 21–25.

[39] A. Zein and T. Abu-Kaff, Bounded oscillation of higher order neutral differential equations with oscillating coefficients, *Appl. Math. E-Notes* **6** (2006), 126–131 (electronic).

[40] S.-Y. Zhang and Q.-R. Wang, Oscillation of second-order nonlinear neutral dynamic equations on time scales, *Appl. Math. Comput.* **216** (2010), no. 10, 2837–2848.

[41] B. G. Zhang and Z. Shanliang, Oscillation of second-order nonlinear delay dynamic equations on time scales, *Comput. Math. Appl.* **49** (2005), no. 4, 599–609.

[42] X. Zhou, Oscillatory and asymptotic properties of higher order nonlinear neutral difference equations with oscillating coefficients, *Appl. Math. Lett.* **21** (2008), no. 11, 1142–1148.

[43] X. Zhou and R. Yu, Oscillatory behavior of higher order nonlinear neutral forced differential equations with oscillating coefficients, *Comput. Math. Appl.* **56** (2008), no. 6, 1562–1568.