On Lockett Pairs and Lockett Conjecture for \(\pi\)-Soluble Fitting Classes

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Abstract. In this paper, we construct a new and wide family of Lockett pairs in the class of all finite \(\pi\)-soluble groups and give a new characteristic of the validity of Lockett conjecture. As application, some known results are followed.

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1. Introduction

In the theory of finite soluble groups, many well known results related to research of structures of Fitting classes and canonical subgroups are closed connected with the operators \(\ast\ast\) and \(\ast\) defined by Lockett [13] (see also [8, chapter X]). In fact, every nonempty Fitting class \(\mathfrak{F}\) has the associated Fitting classes \(\mathfrak{F}\ast\) and \(\mathfrak{F}\ast\ast\), where \(\mathfrak{F}\ast\) is the smallest Fitting class containing \(\mathfrak{F}\) such that the \(\mathfrak{F}\ast\) -radical of the direct product \(G \times H\) of any two groups \(G\) and \(H\) is equal to the direct product of the \(\mathfrak{F}\ast\) -radical of \(G\) and the \(\mathfrak{F}\ast\) -radical of \(H\); \(\mathfrak{F}\ast\ast\) is the intersection of all Fitting classes \(\mathfrak{X}\) such that \(\mathfrak{X}\ast=\mathfrak{F}\ast\ast\) (see [13] or [8, Chapter X]). If \(\mathfrak{F}\ast\ast=\mathfrak{F}\ast\), then the Fitting class is called a Lockett class. The interest for the research of \(\mathfrak{F}\ast\ast\) and \(\mathfrak{F}\ast\) is determined mainly by the following circumstances. Firstly, the family of Fitting classes satisfying \(\mathfrak{F}\ast\ast=\mathfrak{F}\ast\) is vast. In fact, by [8, Theorem X.1.25], every Fitting class closed about homomorphic images or closed about finite subdirect products, and every Fischer class (see [8, IX.3.3]) are all Lockett classes. Secondly, Lockett [13] formulated the conjecture that for every Fitting class \(\mathfrak{F}\), there exists a normal Fitting class \(\mathfrak{X}\) such that \(\mathfrak{F}=\mathfrak{F}\ast\ast\cap\mathfrak{X}\). Later, in this case, we say that \(\mathfrak{F}\) satisfies Lockett conjecture in \(\mathfrak{S}\), where \(\mathfrak{S}\) is the class of all soluble groups.

About the Lockett conjecture, Bryce and Cossey [6] proved that Lockett conjecture holds for all soluble \(\mathfrak{S}\)-closed local Fitting classes and that every soluble Fitting class \(\mathfrak{F}\) satisfies Lockett conjecture if and only if \(\mathfrak{F}\ast\ast=\mathfrak{S}^\ast\cap\mathfrak{S}\ast\), where \(\mathfrak{S}\ast\) is the smallest normal Fitting class. In the paper [6], Bryce and Cossey also gave the concept of Lockett pair \((\mathfrak{F},\mathfrak{S})\) (see

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They call an ordered pair \((\mathfrak{F}, \mathfrak{H})\) of two Fitting classes \(\mathfrak{F}\) and \(\mathfrak{H}\) a Lockett pair (or in brevity, an \(\mathcal{L}\)-pair) if \(\mathfrak{F} \cap \mathfrak{H} = (\mathfrak{F} \cap \mathfrak{H})^*\).

It is easy to see that if \(\mathfrak{F}\) is a Lockett class, \(\mathfrak{F} \subseteq \mathfrak{H}\) and \((\mathfrak{F}, \mathfrak{H})\) is a Lockett pair, then \(\mathfrak{F}\) satisfies Lockett conjecture in \(\mathfrak{H}\). In particular, if \(\mathfrak{F}\) is a Lockett class, \(\mathfrak{F} \subseteq \mathfrak{G}\), \(\mathfrak{H} = \mathfrak{G}\) and \((\mathfrak{F}, \mathfrak{H})\) is a Lockett pair, then \(\mathfrak{F}\) satisfies Lockett conjecture in \(\mathfrak{G}\). We say that \(\mathfrak{F}\) is an \(\mathcal{L}_{\mathfrak{G}}\)-class if \(\mathfrak{F}\) satisfies Lockett conjecture in \(\mathfrak{G}\). If \(\mathfrak{F}\) satisfies Lockett conjecture in \(\mathfrak{G}\), then \(\mathfrak{F}\) is called an \(\mathcal{L}\)-class. Recall that a Fitting class \(\mathfrak{F}\) is said to be \(S\)-closed if every subgroup of \(G \in \mathfrak{F}\) is in \(\mathfrak{F}\).

Bryce and Cossey \([6]\) in the universe \(\mathfrak{G}\) proved the existence of Lockett pairs. They showed that if \(\mathfrak{F}\) and \(\mathfrak{H}\) are \(S\)-closed Fitting classes, then \((\mathfrak{F}, \mathfrak{H})\) is a Lockett pair. In connection with this, the following problem arises.

**Problem 1.1.** Which Fitting classes \(\mathfrak{F}\) and \(\mathfrak{H}\) satisfy that \((\mathfrak{F}, \mathfrak{H})\) is a Lockett pair? In particular, which Fitting classes \(\mathfrak{F}\) satisfies Lockett conjecture in \(\mathfrak{H}\)?

Note that, up to now, the problem was resolved only in the following special cases:

1. \(\mathfrak{F} \in \{\mathfrak{M}_\pi, \mathfrak{X}\mathfrak{G}_\pi\}\), where \(\mathfrak{X}\) is some nonempty soluble Fitting class, \(\mathfrak{H}\) is \(\mathfrak{F}\mathfrak{G}_\pi\)-injector and \(\mathcal{L}_p(\mathfrak{F})\)-injector closed for all primes \(p\) (see Beidleman and Hauck \([1]\));
2. \(\mathfrak{F} = \mathfrak{G}_\pi\) and \(\mathfrak{H} \supseteq \mathfrak{G}_\pi\) (see Brison \([4]\));
3. \(\mathfrak{F}\) is an arbitrary soluble local Fitting class, and \(\mathfrak{H}\) is \(\psi\)-injector closed, where \(\psi\) is a local function of \(\mathfrak{F}\) (see Vorob’ev \([15]\));
4. \(\mathfrak{F} = \mathfrak{G}\), and \(\mathfrak{H} = \mathfrak{E}\) is the class of all finite groups (see Berger \([3]\));
5. \(\mathfrak{F}\) is a Fischer class (i.e., \(\mathfrak{F}\) is a Fitting class which is closed under taking subgroups of the form \(PN\) where \(P\) is a Sylow subgroup and \(N\) is a normal subgroup) or is closed under taking \(\mathfrak{F}\)-subgroups, whose intersection with \(\mathfrak{F}\)-radical of \(G\) is a normal subgroup of \(G\), \(\mathfrak{F}\) satisfies the property that \(\mathfrak{X}E_p \subseteq \mathfrak{F} \subseteq \mathfrak{X}E_p E_p'\) for some Fitting class \(\mathfrak{X}\) and all \(p \in \text{Char}(\mathfrak{X})\) (see Gallego \([9]\));
6. \(\mathfrak{F}\) is \(\omega\)-local with \(\text{char}(\mathfrak{F}) \subseteq \omega\) and \(\mathfrak{H} = \mathfrak{E}\) is the class of all finite groups (see \([12]\)).

In this paper, we will construct a new family of Lockett pairs in the class \(\mathfrak{G}^\pi\) of all finite \(\pi\)-soluble groups. In order to achieve the purpose, in Section 3, we will give the concept of \(\pi\)-HR-closed Fitting class for some Fitting class \(\mathfrak{X}\), which, in fact, is a generalized \(S\)-closed Fitting class. Base on this, in Section 4, we obtain a family of Lockett pairs and also give a new characteristic of the validity of Lockett conjecture. As application, some known results in \([1,3,4,9,12,15]\) are obtained as corollaries of our results. Throughout this paper, all groups are finite. All unexplained notation and terminology are standard. The reader is referred to \([8,10]\) if necessary.

### 2. Preliminaries

Recall that a class \(\mathfrak{F}\) of groups is called a Fitting class provided the following two conditions are satisfied:

1. if \(G \in \mathfrak{F}\) and \(N \unlhd G\), then \(N \in \mathfrak{F}\).
2. if \(N_1, N_2 \subseteq G\) and \(N_1, N_2 \in \mathfrak{F}\), then \(N_1N_2 \in \mathfrak{F}\).

From the condition (ii) in the definition, we see that, for every non-empty Fitting class \(\mathfrak{F}\), every group \(G\) has a largest normal \(\mathfrak{F}\)-subgroup which is called the \(\mathfrak{F}\)-radical of \(G\) and denote by \(G_{\mathfrak{F}}\). The product \(\mathfrak{F}S\) of two Fitting classes \(\mathfrak{F}\) and \(\mathfrak{H}\) is the class \((G : G/G_{\mathfrak{F}} \in \mathfrak{H})\).
It is well known that the product of any two Fitting classes is also a Fitting class and the multiplication of Fitting classes satisfies associative law (see [8, IX.1.12]).

Recall that a class $\mathcal{F}$ of groups is said to be a saturated homomorph if the following conditions hold:

1. $\mathcal{F}$ is closed about homomorphic images, that is, if $G \in \mathcal{F}$ and $N \trianglelefteq G$, then $G/N \in \mathcal{F}$;
2. If $G/\Phi(G) \in \mathcal{F}$, then $G \in \mathcal{F}$.

A nonempty Fitting class $\mathcal{F}$ is said to be a Fischer class if $H \in \mathcal{F}$ whenever $K \trianglelefteq G$ and $H/K$ is a nilpotent subgroup of $G/K$ (see [8, IX.3.3]). Obviously, any 𝑆-closed Fitting class is a Fischer class.

We here cite some properties of the operators ’∗’ and ’∗∗’, which are used in later proof.

**Lemma 2.1.** ([13] and [8, XI]). Let $\mathcal{F}$ and $\mathcal{G}$ be two non-empty Fitting classes. Then:

1. If $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{F}^* \subseteq \mathcal{G}^*$ and $\mathcal{F}_{**} \subseteq \mathcal{G}_{**}$;
2. $(\mathcal{F}_*)_* = (\mathcal{F}^*)^* \subseteq \mathcal{G}^*$;
3. $\mathcal{F}^* \subseteq \mathcal{G}_a$, where $\mathcal{A}$ is the class of all abelian groups;
4. If $\{\mathcal{F}_i \mid i \in I\}$ is a set of Fitting classes, then $(\bigcap_{i \in I} \mathcal{F}_i)^* = \bigcap_{i \in I} \mathcal{F}_i^*$.
5. If $\mathcal{F}$ is a saturated homomorph, then $(\mathcal{F}^*^*^*) = \mathcal{F}^*$.
6. If $\mathcal{F}$ is a homomorph (in particular, a formation) or a Fischer class, then $\mathcal{F}$ is a Lockett class.

Suppose that $G$ is a group, $\mathcal{X}$ is a class of groups and $\mathbb{P}$ is the set of all primes. Then we let $\sigma(G) = \{p \in \mathbb{P} : p \mid |G|\}$, $\sigma(\mathcal{X}) = \cup\{\sigma(G) : G \in \mathcal{X}\}$ and $\text{Char}(\mathcal{X}) = \{p \in \mathbb{P} : Z_p \in \mathcal{X}\}$.

**Lemma 2.2.** [8, X.1.20]. $\text{Char}(\mathcal{F}^*) = \text{Char}(\mathcal{F})$ and $\sigma(\mathcal{F}^*) = \sigma(\mathcal{F})$ for every Fitting class $\mathcal{F}$.

Let $\emptyset \neq \omega \subseteq \mathbb{P}$. We denote by $\mathcal{E}_\omega$ the class of all finite $\omega$-groups, $\mathcal{N}$ denotes the class of all finite nilpotent groups, $\mathcal{S}$ denotes the class of all finite soluble groups. For a class $\mathcal{F}$ of groups, put $\mathcal{F}_\omega = \mathcal{F} \cap \mathcal{E}_\omega$. Following [14], a map

$$f : \omega \cup \{\omega'\} \rightarrow \{\text{Fitting class}\}$$

is said to be a $\omega$-local Hatley function (or in brevity, a $\omega$-local $H$-function). Then $LR_{\omega}(f)$ denotes the Fitting class $(\cap_{p \in \omega} E_{p'}) \cap (\cap_{p \in \omega} f(p) \mathcal{N}_p E_{p'}) \cap (\cap_{p \in \omega} f(\omega'))$, where $p_1 = \text{Supp}(f) \cap \omega$, $\pi_2 = \omega \setminus \pi_1$. Here, $\text{Supp}(f) := \{a \in \omega \cup \{\omega\} : f(a) \neq \emptyset\}$ is called the support of the $\omega$-local $H$-function $f$. A Fitting class $\mathcal{F}$ is said to be a $\omega$-local if there exists an $\omega$-local $H$-function $f$ such that $\mathcal{F} = LR_{\omega}(f)$. If $\omega = \mathbb{P}$, then the $\omega$-local Fitting class $\mathcal{F}$ is said to be local.

**Lemma 2.3.** [17, Theorem]. If $\mathcal{F}$ is an $\omega$-local Fitting class and a Lockett class as well, then $\mathcal{F} = LR_{\omega}(F)$ and $F(a)$ is a Lockett class for all $a \in \omega \cup \{\omega'\}$. Moreover, $F(p)\mathcal{N}_p = F(p) \subseteq \mathcal{F} \subseteq F(p)\mathcal{E}_{p'}$, for all $p \in \omega$.

For constructing Lockett pairs, we need the concept of the normal subgroup $N(G)$ of $G$ and its properties which given by Gallego [9]. We use $\text{Snemb}(S \rightarrow G)$ denote the set of all subnormal embedding of $S$ in $G$. Then let $N(G) = \langle x^{-1} x^\alpha : x \in S \text{ sn } G$ and $\alpha \in \text{Snemb}(S \rightarrow G)\rangle$.

**Lemma 2.4.** [9, Proposition (3.1)(3.2)]. Let $G$ be a group and $\mathcal{F}$ a Fitting class. Then
Lemma 2.6. (Čunićin [7]) Every \( \pi \)-soluble group \( G \) has a Hall \( \pi \)-subgroup and any two Hall \( \pi \)-subgroups of \( G \) are conjugate in \( G \).

Lemma 2.7. Let \( G \) be a group, \( H \in \mathcal{H}(G) \), \( H \in \mathcal{R} \) and \( X \) a Fitting class. Then \( H \cap N(G) \leq N(HG_X) \).

Proof. \( H_0 = \langle x^{-1}x^\alpha : x \in S \text{sn} G, \alpha \in \text{Snemb}(S \to G) \rangle \).

We first prove that \( H \cap N(G) = H_0 \). By the definition of \( N(G) \), every generator of \( N(G) \) has the form \( g^{-1}g^\alpha \), where \( g \in S \text{sn} G \) and \( \alpha \in \text{Snemb}(S \to G) \). Let \( g = xy \), where \( x, y \in \langle g \rangle \leq S \) such that \( x \) is a \( \pi \)-element and \( y \) is a \( \pi' \)-element. Since \( H \in \mathcal{H}(G) \), there exist elements \( a \) and \( b \) of \( G \) such that \( x^a \in H \) and \( (x^a)^b \in H \). It is clear that

\[
g^{-1}g^\alpha O^\pi(G)G' = x^{-1}x^\alpha O^\pi(G)G' = (x^a)^{-1}(x^a)^b O^\pi(G)G'.
\]

Note that \( S^a \text{sn} G \), \( (S^a)^b \text{sn} G \) and there exists an isomorphism from \( S^a \) onto \( (S^a)^b \) such that the image of \( x^a \) is \( (x^a)^b \). Therefore \( (x^a)^{-1}(x^a)^b \in H_0 \) and \( g^{-1}g^\alpha \in H_0 O^\pi(G)G' \). Since \( N(G) \) is generated by such elements \( g^{-1}g^\alpha \), we have \( N(G) \leq H_0 O^\pi(G)G' \) and hence \( H \cap N(G) \leq H_0 (H \cap O^\pi(G)G') \). Since \( O^\pi(G)G'/G' \) is a \( \pi' \)-group, \( H \cap O^\pi(G)G' = H \cap G' \).

By [2, 21.3(2)], \( H \cap G' \subseteq H_0 \). Hence \( H \cap N(G) \leq H_0 \). On the other hand, obviously, \( H_0 \leq H \cap N(G) \).

Therefore \( H \cap N(G) = H_0 \). This shows that \( H \cap N(G) \) is generated by the elements \( x^{-1}x^a \), where \( x \in S \text{sn} G, \alpha \in \text{Snemb}(S \to G) \) and \( x, x^a \in H \). Note that the subgroup \( \langle x \rangle S_{\tilde{G}} = \langle x \rangle (S \cap G_{\tilde{G}}) = S \cap (x)G_{\tilde{G}} \) is subnormal in \( HG_{\tilde{G}} \). Analogously, \( (\langle x \rangle S_{\tilde{G}})^\alpha = (x^a)S_{\tilde{G}}^\alpha \) is subnormal in \( HG_{\tilde{G}} \). Therefore \( x^{-1}x^a \in N(HG_{\tilde{G}}) \) and \( H \cap N(G) \leq N(HG_{\tilde{G}}) \).

3. HR-classes

In order to construct a new family of Lockett pairs, in this section, we will define the following generalized S-closed Fitting classes.

Definition 3.1. Suppose that \( \pi \subseteq \mathcal{P} \) and \( X \) is a Fitting class.

(a) A subgroup \( T \) of \( G \) is called a \( \pi \)-HR-subgroup if \( T = HG_X \) for some Hall \( \pi \)-subgroup \( H \) of \( G \).

(b) A Fitting class \( \mathcal{F} \) is called \( \pi \)-HR-closed if every \( \pi \)-HR-subgroup of \( G \) belongs to \( \mathcal{F} \) whenever \( G \in \mathcal{F} \). If \( \mathcal{F} \) is \( \sigma \)-HR-closed for any \( \sigma \subseteq \mathcal{P} \), then \( \mathcal{F} \) is called an HR-class.

(c) If \( X = (1) \), we write "\( \pi \)-class" instead "\( \pi \)-HR-class", and write "\( H \)-class" instead "\( H \)-HR-class".
The following examples show that the family of the Fitting classes defined in Definition 3.1 is wide.

**Example 3.1.**

(1) Suppose that $\mathcal{F}$ is a $S$-closed Fitting class. Then, obviously, $\mathcal{F}$ is an $\pi$-$HGX$-class, for any nonempty Fitting class $\mathcal{X}$.

(2) Recall that for a Fitting class $\mathcal{F}$ and a group $G \in \mathcal{F}$, if $H \in \mathcal{F}$ for every $H \in \text{Hall}_\pi(G)$, then $\mathcal{F}$ is said to be $\pi$-Hall closed [4]. Obviously, a Fitting class $\mathcal{F}$ is a $\pi$-$H$-class if and only if it is $\pi$-Hall closed.

(3) Let $\pi = \mathbb{P}$ and $\mathcal{S}_*$ be the smallest normal Fitting class. By the result in [6], $\mathcal{S}_*$ is an $H$-class.

(4) For any set $\pi \subseteq \mathbb{P}$ and any Fitting class $\mathcal{S}_\pi$, the Fitting class $\mathcal{S}_\pi(\mathcal{S}_\pi) = (G \in \mathcal{S}_\pi : \text{Hall}_\pi(G) \subseteq \mathcal{S}_\pi)$ was defined in [11] (see Lemma 2.6(2)). Obviously, $\mathcal{S}_\pi(\mathcal{S}_\pi)$ is $\pi$-Hall closed if and only if $\mathcal{S}_\pi(\mathcal{S}_\pi)$. Moreover, by the proof of [5, Proposition 4.4], we can see that for any $\tau \subseteq \pi$, the Fitting class $\mathcal{S}_\tau(\mathcal{S}_\tau)$ is $\pi$-Hall closed for any $\pi$-$H$-closed Fitting class $\mathcal{F}$.

4. On problem of the construction of $\mathcal{L}$-pairs and $\mathcal{L}$-classes

In this section, we construct a family of Lockett pairs and give a new characteristic of the validity of Lockett conjecture.

**Definition 4.1.** Let $\mathcal{F}$ and $\mathcal{S}_\pi$ be two Fitting classes.

(i) We say that $\mathcal{F}$ and $\mathcal{S}_\pi$ satisfy Property $(\alpha_\sigma)$ if $\sigma \subseteq \pi$ and there exists a Fitting class $\mathcal{X}$ such that $\mathcal{X} \mathcal{S}_\sigma \subseteq \mathcal{F} \subseteq \mathcal{X} \mathcal{S}_\sigma \mathcal{S}_\sigma^\pi$, $\mathcal{S}_\pi \subseteq \mathcal{R}_\sigma(M)$ and $\mathcal{S}_\pi$ is a $\sigma$-$HR$-class.

(ii) Let $\text{Char}(\mathcal{F})$ be the characteristic of $\mathcal{F}$ and $\text{Char}(\mathcal{S}_\pi) = \bigcup_{i \in I} \sigma_i$, where $\sigma_i \neq \emptyset$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i, j \in I(i \neq j)$. We say that $\mathcal{F}$ and $\mathcal{S}_\pi$ satisfy Property $(\alpha)$ if $\mathcal{F}$ and $\mathcal{S}_\pi$ satisfy Property $(\alpha_\sigma)$ for all $i \in I$.

**Lemma 4.1.** Let $\mathcal{F}$ and $\mathcal{S}_\pi$ be Fitting classes. If $\mathcal{F}$ and $\mathcal{S}_\pi$ satisfies Property $(\alpha_\sigma)$, then $\mathcal{F} \cap \mathcal{S}_\pi \subseteq (\mathcal{F} \cap \mathcal{S}_\pi), \mathcal{S}_\pi^\sigma$.

**Proof.** Let $(\mathcal{F} \cap \mathcal{S}_\pi), \mathcal{S}_\pi^\sigma = \mathcal{M}$. In order to prove $\mathcal{F} \cap \mathcal{S}_\pi \subseteq \mathcal{M}$, by Lemma 2.5, we only need to prove that $N(G) \cap \mathcal{S}_\pi \subseteq \mathcal{M}$ for every $G \in \mathcal{S}_\pi$. Suppose that $G \in \mathcal{S}_\pi$ and $H \in \text{Hall}_\pi(G)$. Then $HG_X/G_X \in \text{Hall}_\pi(G/G_X)$ and so $HG_X/G_X \in \mathcal{S}_\sigma$. Hence, from [8, IX, 1.11], we have $H \in \mathcal{X} \mathcal{S}_\sigma \subseteq \mathcal{S}_\sigma$. Since $\mathcal{S}_\sigma$ is a $\sigma$-$HR$-class by hypothesis, $HG_X \in \mathcal{S}_\sigma$. Hence, $HG_X \in \mathcal{F} \cap \mathcal{S}_\sigma$. Besides, since $H$ is a Hall $\sigma$-subgroup, by Lemma 2.7, we have $H \cap N(G) \subseteq N(HG_X)$. Hence, $H \cap N(G) \cap \mathcal{S}_\sigma \subseteq N(HG_X) \cap \mathcal{S}_\sigma$.

We claim that $N(HG_X) \cap \mathcal{S}_\sigma \subseteq (HG_X \cap \mathcal{S}_\sigma)_{(\mathcal{S}_\Pi \cap \Pi_X)^*}$. In fact, since $HG_X \in \mathcal{F} \cap \mathcal{S}_\pi$, by Lemma 2.4 $N(HG_X) \subseteq (HG_X)_{(\mathcal{S}_\Pi \cap \Pi_X)^*}$, and so $N(HG_X) \cap \mathcal{S}_\sigma \subseteq (HG_X)_{(\mathcal{S}_\Pi \cap \Pi_X)^*} \cap \mathcal{S}_\sigma$. But, because $HG_X \cap \mathcal{S}_\sigma \subseteq HG_X$, by [8, IX, 1.11(a)], we have

$$(HG_X \cap \mathcal{S}_\sigma)_{(\mathcal{S}_\Pi \cap \Pi_X)^*} = (HG_X)_{(\mathcal{S}_\Pi \cap \Pi_X)^*} \cap (HG_X \cap \mathcal{S}_\sigma) = (HG_X)_{(\mathcal{S}_\Pi \cap \Pi_X)^*} \cap \mathcal{S}_\sigma.$$ 

Hence $N(HG_X) \cap \mathcal{S}_\sigma \subseteq (HG_X)_{(\mathcal{S}_\Pi \cap \Pi_X)^*} \cap G_{\mathcal{S}_\sigma} = (HG_X \cap G_{\mathcal{S}_\sigma})_{(\mathcal{S}_\Pi \cap \Pi_X)^*}$.

Now we prove $HG_X \cap G_{\mathcal{S}_\sigma} \subseteq G$. In fact, since $G_{\mathcal{S}_\sigma} \in \mathcal{X} \mathcal{S}_\sigma \mathcal{S}_\sigma^\pi$, $G_{\mathcal{S}_\sigma}/G_X \mathcal{S}_\sigma \in \mathcal{S}_\sigma^\pi$. Then, by theorem [8, IX.1.12],

$$\frac{G_{\mathcal{S}_\sigma}/G_X}{(G_X \mathcal{S}_\sigma)/G_X} = \frac{(G_{\mathcal{S}_\pi}/G_X)}{(G/G_X) \mathcal{S}_\sigma} \in \mathcal{S}_\sigma^\pi.$$
and so $G_{\mathfrak{X} \circ \sigma} / G_{\mathfrak{X}} \in \text{Hall}_G(G_{\mathfrak{X}} / G_{\mathfrak{X}})$. On the other hand, since $H \in \text{Hall}_G(G_{\mathfrak{X}})$, $(H \cap G_{\mathfrak{X}}) / G_{\mathfrak{X}} \in \text{Hall}_G(G_{\mathfrak{X}} / G_{\mathfrak{X}})$. Hence $G_{\mathfrak{X} \circ \sigma} = (H \cap G_{\mathfrak{X}})G_{\mathfrak{X}} = HG_{\mathfrak{X}} \cap G_{\mathfrak{X}}$ and $HG_{\mathfrak{X}} \cap G_{\mathfrak{X}} \subseteq G$. This implies that $(HG_{\mathfrak{X}} \cap G_{\mathfrak{X}}) \cap G_{\mathfrak{X} \circ \sigma} \subseteq G_{\mathfrak{X} \circ \sigma}$. Therefore, $H \cap (N(G) \cap G_{\mathfrak{X}}) \subseteq G_{\mathfrak{X} \circ \sigma}$ and $N(G) \cap G_{\mathfrak{X}} \subseteq G_{\mathfrak{X} \circ \sigma}$. By the Lemma 2.4(i), $N(G) \subseteq G$. Hence $G_{\mathfrak{X} \circ \sigma} \cap (N(G) \cap G_{\mathfrak{X}}) = (N(G) \cap G_{\mathfrak{X}}) \cap G_{\mathfrak{X} \circ \sigma}$, by [8, IX, 1.1 (a)]. It follows that $H \cap (N(G) \cap G_{\mathfrak{X}}) \subseteq (N(G) \cap G_{\mathfrak{X}}) \cap G_{\mathfrak{X} \circ \sigma}$. Since $H_1 := H \cap (N(G) \cap G_{\mathfrak{X}})$ is a $\mathfrak{X}$-subgroup of $N(G) \cap G_{\mathfrak{X}}$, $|N(G) \cap G_{\mathfrak{X}} : H_1|$ is an $\sigma'$-number. But since $|H_1| \mid |(N(G) \cap G_{\mathfrak{X}}) \cap G_{\mathfrak{X} \circ \sigma}|$, $|(N(G) \cap G_{\mathfrak{X}}) \cap G_{\mathfrak{X} \circ \sigma} \cap G_{\mathfrak{X} \circ \sigma} | \in \mathfrak{S}_{\sigma'}$. Hence $N(G) \cap G_{\mathfrak{X}} \in (\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \mathfrak{S}_{\sigma'} = \mathfrak{M}$. Consequently $N(G) \cap G_{\mathfrak{X}} \subseteq G_{\mathfrak{M}}$. This completes that proof.

The following theorem describes a new and wide family of Lockett pairs. In particular, the theorem give some new Fitting classes which satisfy Lockett conjecture.

**Theorem 4.1.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two Fitting classes. If $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy Property $(\alpha)$, then $(\mathfrak{X}, \mathfrak{Y})$ is an $\mathfrak{L}$-pair. In particular, if $\mathfrak{X} \subseteq \mathfrak{Y}$, then $\mathfrak{X}$ is an $\mathfrak{L}_{\mathfrak{Y}}$-class, that is, $\mathfrak{X}$ satisfies Lockett conjecture in $\mathfrak{Y}$.

**Proof.** By Lemma 4.1, we only need to prove that if $\mathfrak{X} \cap \mathfrak{Y} \subseteq (\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \mathfrak{S}_{\sigma'}$, for every $i \in I$, then $\mathfrak{X} \cap \mathfrak{Y} \subseteq (\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \mathfrak{S}_{\sigma'}$. Firstly, by Lemma 2.1, we have $(\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \subseteq \mathfrak{X} \cap \mathfrak{Y}$.

Conversely, assume that it is not true and let $G$ be a group in $\mathfrak{X} \cap \mathfrak{Y} \setminus (\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \mathfrak{S}_{\sigma'}$, of minimal order. Then $G$ has a unique maximal normal subgroup $M = G_{\mathfrak{X} \cap \mathfrak{Y}}$. Since $G \in \mathfrak{X} \cap \mathfrak{Y}$, $G \in \mathfrak{X} \cap \mathfrak{Y}$ by Lemma 2.1(a)(b). Then by using Lemma 2.1(b)(c), we obtain that $G \in (\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \mathfrak{A}$, where $\mathfrak{A}$ is the class of all abelian groups. Hence, $G/M$ has a unique maximal normal subgroup of order $p$ and so $G/M \simeq Z_p$. Since $G \in \mathfrak{X} \cap \mathfrak{Y}$, $p \in \text{Char}(\mathfrak{X} \cap \mathfrak{Y})$ by [8, Lemma IX.1.7]. It follows from Lemma 2.2 that there exists $\sigma_0 \subseteq \text{Char}(\mathfrak{X} \cap \mathfrak{Y})$ for $i_0 \in I$ such that $p \in \sigma_0 \subseteq \text{Char}(\mathfrak{X} \cap \mathfrak{Y})_{\sigma_0}$. Therefore $G/M \in \mathfrak{S}_{\sigma_0}$.

On the other hand, since $\sigma$, $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy conditions of Lemma 4.1, $\mathfrak{X} \cap \mathfrak{Y} \subseteq (\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \mathfrak{S}_{\sigma'}$. Since $G \in \mathfrak{X} \cap \mathfrak{Y}$ and $M = G_{\mathfrak{X} \cap \mathfrak{Y}}$, we have $G/M \in \mathfrak{S}_{\sigma'}$. This implies that $G = M \in (\mathfrak{X} \cap \mathfrak{Y})_{\sigma_0}$. This contradiction shows that $(\mathfrak{X}, \mathfrak{Y})$ is an $\mathfrak{L}$-pair.

Now assume that $\mathfrak{X} \subseteq \mathfrak{Y}$. In order to prove that $\mathfrak{X}$ is an $\mathfrak{L}$-class (that is, $\mathfrak{X}$ satisfies Lockett conjecture), clearly, we only need to prove that $\mathfrak{X}$ is a Lockett class (that is, $\mathfrak{X}^* = \mathfrak{X}$). By the hypothesis, $\mathfrak{X}$ satisfies Property $(\alpha)$. Hence $\mathfrak{X} \subseteq \mathfrak{X}$ for all $i \in I$. Then by Lemma 2.1(a), we see that $\mathfrak{X}^* \subseteq (\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \mathfrak{S}_{\sigma'}$. By [15, Corollary], $\mathfrak{X} \subseteq \mathfrak{X} \cap \mathfrak{Y}$ is local and so it is a Lockett class by [15, Lemma 5]. Therefore, $(\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \mathfrak{S}_{\sigma'} = \mathfrak{X} \cap \mathfrak{Y}$ and thereby $\mathfrak{X}^* \subseteq \mathfrak{X} \cap \mathfrak{Y}$ for all $i \in I$. By Lemma 2.2, $\text{Char}(\mathfrak{X}) = \text{Char}(\mathfrak{X}^*)$. Now analogous proof from $\mathfrak{X} \cap \mathfrak{Y} \subseteq (\mathfrak{X} \cap \mathfrak{Y})_{\sigma} \mathfrak{S}_{\sigma'}$, to $\mathfrak{X} \cap \mathfrak{Y} \subseteq (\mathfrak{X} \cap \mathfrak{Y})_{\sigma_0}$, we obtain that $\mathfrak{X}^* = \mathfrak{X}$. This completes the proof.

**5. Applications**

By using Theorem 4.1, we immediately obtain the following known results about description of $\mathfrak{L}$-pair and $\mathfrak{L}$-class.

**Corollary 5.1.** (Bryce, Cossey [6]). If $\mathfrak{X}$ and $\mathfrak{Y}$ are soluble $S$-closed Fitting classes, then $(\mathfrak{X}, \mathfrak{Y})$ is an $\mathfrak{L}$-pair. In particular, $(\mathfrak{X}, \mathfrak{S})$ is an $\mathfrak{L}$-pair, that is, every $S$-closed Fitting class is an $\mathfrak{L}$-class.
Proof. By [17, Theorem], $\mathcal{F}$ and $\mathcal{H}$ are local Fitting classes. Put $\sigma_i = \{p\}$ in Theorem 4.1, for all $p \in \text{Char}(\mathcal{F})$ and $i \in I$. Since $\mathcal{H}$ is $S$-closed, $\mathcal{H}$ is an $HR$-class for any Fitting class $\mathcal{X}$. Besides, by Lemma 2.3, for $\omega = \mathbb{P}$ and every $p \in \text{Char}(\mathcal{F})$, we have $F(p)\mathcal{M}_p \subseteq \mathcal{F} \subseteq F(p)\mathcal{M}_p\mathcal{S}_{\mathcal{F}}$. Thus, by Theorem 4.1, $(\mathcal{F}, \mathcal{H})$ is an $L$-pair. Besides, by Lemma 2.1(f), $\mathcal{F} = \mathcal{F}^*$. If $\mathcal{H} = \mathcal{G}$, then $\mathcal{F}$ is an $L$-class, that is, $\mathcal{F}$ satisfies Lockett conjecture.

Corollary 5.2. [see 12, Theorem B]. If $\mathcal{F} = LR_\omega(F)$ is $\omega$-local Fitting class with $\text{Char}(\mathcal{F}) \subseteq \omega$, then $(\mathcal{F}, \mathcal{S}_{\mathcal{F}})$ is an $L$-pair and $\mathcal{F}$ is an $L$-class.

Proof. By Lemma 2.3, $F(p)\mathcal{M}_p \subseteq \mathcal{F} \subseteq F(p)\mathcal{M}_p\mathcal{S}_{\mathcal{F}}$ for all $p \in \text{Char}(\mathcal{F})$. Hence, if put $\sigma_i = \{p\}$ for all $p \in \text{Char}(\mathcal{F})$ and $i \in I$ and let $\mathcal{H} = \mathcal{S}_{\mathcal{F}}$, the $\mathcal{F}$ and $\mathcal{H}$ satisfy the hypothesis of Theorem 4.1. Thus, by Theorem 4.1, $(\mathcal{F}, \mathcal{S}_{\mathcal{F}})$ is an $L$-pair and so, clearly, $\mathcal{F}$ is an $L$-class since $\mathcal{F} \subseteq \mathcal{H}$.

Put $\omega = \pi$, then by Corollary 5.2, we have

Corollary 5.3. If $\mathcal{F}$ is a local Fitting class, then $(\mathcal{F}, \mathcal{S}_{\mathcal{F}})$ is an $L$-pair.

Put $\omega = \pi = \mathbb{P}$, then by Corollary 5.2, we obtain

Corollary 5.4. [15]. Lockett conjecture holds for every soluble local Fitting class $\mathcal{G}$, that is, if $\mathcal{G} \subseteq \mathcal{S}$, then pair $(\mathcal{G}, \mathcal{S})$ is an $L$-pair.

Corollary 5.5. [1]. Let $\mathcal{F} \in \{ \mathcal{X}\mathcal{M}, \mathcal{X}\mathcal{G}_{\pi}\mathcal{S}_{\pi'} \}$, where $\mathcal{X}$ is some nonempty soluble Fitting class. Then $\mathcal{F}$ satisfies Lockett conjecture.

Proof. By [15, Corollary], $\mathcal{X}\mathcal{M}$ and $\mathcal{X}\mathcal{G}_{\pi}\mathcal{S}_{\pi'}$ are all local Fitting classes. Hence by Corollary 5.3, the statement holds.

Corollary 5.6. Let $\sigma \subseteq \pi$ and $\mathcal{F}$, $\mathcal{H}$ be Fitting classes such that $\mathcal{F} = \mathcal{G}_\sigma \subseteq \mathcal{H}$. Then $(\mathcal{F}, \mathcal{H})$ is an $L$-pair and $\mathcal{G}_\sigma$ is an $L_{\mathcal{F}_\sigma}$-class.

Proof. Obviously, $\mathcal{F}$ is local. By Lemma 2.3, for $\omega = \mathbb{P}$, $\mathcal{F}$ satisfies the related conditions of Theorem 4.1 for $\mathcal{F}$ if put $\sigma_i = \{p\}$ for all $p \in \text{Char}(\mathcal{F})$ and $i \in I$. Besides, by [15, Lemma 5], $\mathcal{F}$ is a Lockett class, that is, $\mathcal{F}^* = \mathcal{F}$.

Now we prove that $(\mathcal{F}, \mathcal{H})$ is an $L$-pair, that is, $\mathcal{F} \cap \mathcal{H} = (\mathcal{F} \cap \mathcal{H})_*$. If $\sigma = \emptyset$, then $\mathcal{F} = (1)$ and so it is trivial. If $\sigma = \{p\}$, then by [8, X.1.23], $(\mathcal{M}_p)_* = \mathcal{M}_p$ and so $\mathcal{M}_p \cap \mathcal{H} = (\mathcal{M}_p \cap \mathcal{H})_* = (\mathcal{M}_p)_*$. Put $|\sigma| \geq 2$. Since $\mathcal{G}_\sigma \subseteq \mathcal{H}$, then $\mathcal{H}$ is a $p$-$H$-class and $\mathcal{H} \subseteq \mathcal{K}_p(\mathcal{H})$ for every $p \in \text{Char}(\mathcal{F}) = \sigma$. Thus by Theorem 4.1, $(\mathcal{G}_\sigma, \mathcal{H})$ is an $L$-pair.

For $\pi = \mathbb{P}$ we have

Corollary 5.7. [4]. Let $\mathcal{F}$, $\mathcal{H}$ be soluble Fitting classes and $\sigma \subseteq \mathbb{P}$. If $\mathcal{F} = \mathcal{G}_\sigma \subseteq \mathcal{H}$, then $(\mathcal{F}, \mathcal{H})$ is an $L$-pair and $\mathcal{F}$ is an $L_{\mathcal{G}_\sigma}$-class.

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References


