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New Characterizations of p-Nilpotency and Sylow Tower Groups

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Abstract. We introduce a new subgroup embedding property of finite groups called s^*-permutably embedding. By using this embedding property and formation theory, we obtain some new characterizations of p-nilpotency and Sylow tower groups of supersolvable type. Some recent results are unified and generalized.

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1. Introduction

Throughout this article, all groups are finite. Our notation is standard and the reader is referred to [12, 14] if necessary. Recall that a class of groups $\mathcal{F}$ is a formation if $\mathcal{F}$ is closed under homomorphic images and subdirect product. A formation $\mathcal{F}$ is said to be saturated if it contains each group G with $G/\Phi(G) \in \mathcal{F}$. A formation $\mathcal{F}$ is said to be s-closed if every subgroup of G belongs to $\mathcal{F}$ whenever $G \in \mathcal{F}$. In this paper, $\mathcal{U}$, $\mathcal{N}_p$ will denote the class of all supersolvable groups and the class of all p-nilpotent groups, respectively. As well-known results, $\mathcal{U}$, $\mathcal{N}_p$ are saturated formations. Let $\mathcal{F}$ be a formation. We say a subgroup H of a group G is $\mathcal{F}$-supplemented in G if G has a subgroup $T \in \mathcal{F}$ such that $G = HT$. In this case, we say T is an $\mathcal{F}$-supplement of H in G.

The relationship between the properties of subgroups of the Sylow subgroups of G and the structure of G has been investigated by many authors in the literature (see [7, 9, 13, 17, 19, 22, 23, 27, 29]). In particular, some results about p-nilpotency of finite groups were obtained. For example, a well-known theorem due to Itô (see [14, IV, 5.5]) asserts that a group G is p-nilpotent if all cyclic subgroups of G of order p or 4 (when $p = 2$) lie in the center. Recently, we can find the following results: Let P a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. If the maximal subgroups of P are either all c-normal [8, Theorem 3.4], or all c-supplemented [9, Theorem 3.2], or all s-quasinormally embedded [1, Theorem 3.1], or all weakly s-permutable [20, Theorem 3.1], or all weakly s-permutably embedded [19, Theorem 3.1] in G, then G is p-nilpotent. In fact, it is easy

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to see that s-permutable subgroups [15], c-normal subgroups [27], c-supplemented [3], s-quasinormally embedded subgroups [4], weakly s-permutable [26], weakly s-permutable embedded subgroups [19] are a series of generalizations of normal subgroups. We think it is very necessary and interesting to unify above subgroups, so the following conception is introduced naturally:

Definition 1.1. *A subgroup H of a group G is said to be s^*-permutably embedded in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{se}$, where H_{se} is an s-permutably embedded subgroup of H contained in H.*

We now give some examples to show that the new subgroup embedding property is different from the previous ones which are generalized.

Example 1.1. Suppose that $G = S_4$, the symmetric group of degree 4. Take $\alpha = (34)$ and $\beta = (123)$. Then $G = \langle\alpha\rangle A_4$ and $\langle\alpha\rangle \cap A_4 = 1$, and hence $\langle\alpha\rangle$ is s^*-permutably embedded in G. However $\langle\alpha\rangle$ is not s-quasinormally embedded in G. In fact, if $\langle\alpha\rangle$ is a Sylow 2-subgroup of some s-permutable subgroup K of G, then $K\langle\beta\rangle$ is a group. Since $|K\langle\beta\rangle : \langle\beta\rangle| = 2$, we have $\langle\beta\rangle \lhd K\langle\beta\rangle$ and so $\langle\alpha\rangle\langle\beta\rangle = \langle\beta\rangle\langle\alpha\rangle$, which is a contradiction.

Example 1.2. Suppose that $G = A_5$, the alternative group of degree 5. Then the Sylow 2-subgroups of G are s^*-permutably embedded in G, but they are neither weakly s-permutable in G nor c-supplemented in G.

Example 1.3. Suppose that $G = S_5$, the symmetric group of degree 5. Let $H = \langle(123),(124)\rangle$. Then H is s^*-permutably embedded in G, but H is not weakly s-permutably embedded in G.

In this article, we give some new characterizations about p-nilpotent groups and Sylow tower groups of supersoluble type by assumption that some second maximal subgroups or second minimal subgroups of the Sylow are s^*-permutably embedded. As an application of our results, some recent results are generalized, such as in [2,7,11,18,21,28].

2. Preliminaries

For convenience, we list here some known results which are crucial in proving our main result.

Lemma 2.1. [4, Lemma 1] *Suppose that H is s-permutably embedded in a group G.*

(1) *If $H \leq L \leq G$, then H is s-permutably embedded in L.*
(2) *If $N \lhd G$, then HN is s-permutably embedded in G and HN/N is s-permutably embedded in G/N.*

Lemma 2.2. *Let H be an s^*-permutably embedded subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is s^*-permutably embedded in L.*
(2) *If $N \lhd G$ and $N \leq H \leq G$, then H/N is s^*-permutably embedded in G/N.*
(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is s^*-permutably embedded in G/N.*

Proof. By the hypothesis, there are a subgroup K of G and an s-permutably embedded subgroup H_{se} of G such that $G = HK$ and $H \cap K \leq H_{se}$.

(1) $L = L \cap HK = H(L \cap K)$ and $H \cap (L \cap K) = H \cap K \leq H_{se}$. By Lemma 2.1(1), H_{se} is s-permutably embedded in L. Hence H is s^*-permutably embedded in L.

(2) $G/N = H/N \cdot NK/N$ and $(H/N)\cap(KN/N) = (H\cap KN)/N = (H\cap K)N/N \le H_{se}N/N$. By Lemma 2.1(2), $H_{se}N/N$ is s-permutably embedded in G/N. Hence H/N is s^*-permutably embedded in G/N.

(3) Since $(|G:K|,|N|) = 1$, $N \le K$. It is easy to see that $G/N = HN/N \cdot KN/N = HN/N \cdot K/N$ and $(HN/N)\cap(K/N) = (HN\cap K)/N = (H\cap K)N/N \le H_{se}N/N$. By Lemma 2.1(2), $H_{se}N/N$ is s-permutably embedded in G/N. Hence HN/N is s^*-permutably embedded in G/N. ∎

Lemma 2.3. *Let $\mathfrak{F}$ be a formation and H is an $\mathfrak{F}$-supplemented subgroup of G.*

(1) *If $H \le L \le G$, then H is $\mathfrak{F}$-supplemented in L.*
(2) *If $N \lhd G$, then HN/N is $\mathfrak{F}$-supplemented in G/N.*

Lemma 2.4. [30, Lemma 2.4] *Let p be the smallest prime dividing the order of a group G and H a normal subgroup of G such that G/H is p-nilpotent. If $|H_p| \leqslant p^2$ and G is A_4-free, then G is p-nilpotent.*

Lemma 2.5. [24, Lemma 2.12] *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if P has a non-trivial proper subgroup D such that every subgroup E of P with $|E| = |D|$ is $\mathcal{N}_p$-supplemented in G.*

Lemma 2.6. [5, A, 1.2] *Let U,V, and W be subgroups of a group G. Then the following statements are equivalent:*

(1) $U\cap VW = (U\cap V)(U\cap W)$.
(2) $UV\cap UW = U(V\cap W)$.

Lemma 2.7. [18, Lemma 2.3] *Suppose that H is s-permutable in G, P a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-permutable in G.*

Lemma 2.8. [18, Lemma 2.4] *Suppose P is a p-subgroup of G contained in $O_p(G)$. If P is s-permutably embedded in G, then P is s-permutable in G.*

Lemma 2.9. [25, Lemma A] *If P is an s-permutable p-subgroup of G for some prime p, then $N_G(P) \ge O^p(G)$.*

Lemma 2.10. [16, Lemma 2.6] *Let H be a solvable normal subgroup of a group G $(H \neq 1)$. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.11. [10, Lemma 3.16] *Let $\mathfrak{F}$ be the class of groups with Sylow tower of supersolvable type. Also let P be a normal p-subgroup of a group G such that $G/P \in \mathfrak{F}$. If G is A_4-free and $|P| \leqslant p^2$, then $G \in \mathfrak{F}$.*

Lemma 2.12. [29, Lemma 2.8] *Let M be a maximal subgroup of G, P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P\cap M$ is a normal subgroup of G.*

3. New characterizations of p-nilpotent groups

Theorem 3.1. *Suppose that p is the smallest prime dividing the order of a group G and H is a normal subgroup of G such that G/H is p-nilpotent. If G is A_4-free and there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is either s^*-permutably embedded or $\mathcal{N}_p$-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $O_{p'}(G) = 1$.

If $T = O_{p'}(G) \neq 1$, we consider $\overline{G} = G/T$. Clearly, $\overline{G}/\overline{H} \cong G/HT$ is p-nilpotent because G/H is, where $\overline{H} = HT/T$. Let $\overline{P_2} = P_2T/T$ be a 2-maximal subgroup of PT/T. We may assume that P_2 is a 2-maximal subgroup of P. Since P_2 is either s^*-permutably embedded or $\mathcal{N}_p$-supplemented in G, the subgroup P_2T/T is either s^*-permutably embedded or $\mathcal{N}_p$-supplemented in G/T by Lemmas 2.2(3) and 2.3(2). The minimality of G implies that $\overline{G}$ is p-nilpotent, and so G is also p-nilpotent, a contradiction.

(2) $H = G$.

Suppose that $H < G$. By Lemmas 2.2(1) and 2.3(1), every 2-maximal subgroup of P is either s^*-permutably embedded or $\mathcal{N}_p$-supplemented in H. Hence H satisfies the hypothesis of the theorem. The choice of G yields that H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \lhd G$. By Step (1), $H_{p'} \leq O_{p'}(G) = 1$. This shows that $H = P$. Let N be a minimal normal subgroup of G contained in P. Then N is an elementary p-group. It is easy to see that G/N satisfies the hypotheses of the theorem, hence G/N is p-nilpotent by the minimality of G. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $P \cap \Phi(G) = 1$. Thus, there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Now $P \cap M \lhd G$ by Lemma 2.12, $P \cap M = 1$ and $N = P$. Since $P \lhd G$, we may pick a 2-maximal N_2 of N such that $N_2 \lhd G_p$, where G_p is a Sylow p-subgroup of G. Then N_2 is either s^*-permutably embedded or $\mathcal{N}_p$-supplemented in G. Let T be any supplement of N_2 in G, i.e., $N_2T = G$. Thus $G = NT$ and $N = N \cap N_2T = N_2(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in G and N is a minimal normal subgroup of G, $N \cap T = N$ and $T = G$. This shows that N_2 can not be $\mathcal{N}_p$-supplemented in G, and so is s^*-permutably embedded in G. Furthermore, N_2 must be s-permutably embedded in G. By Lemma 2.8, N_2 is s-permutable in G. By Lemma 2.9, $O^p(G) \leq N_G(N_2)$. Thus $N_2 \lhd G_pO^p(G) = G$. It follows that $N_2 = 1$, and so $|N| = p^2$. By Lemma 2.4, G is p-nilpotent, a contradiction.

(3) G is not a non-abelian simple group.

By Lemma 2.4, $p^3 \| P|$ and so there exists a non-identity 2-maximal subgroup of P. By Lemma 2.5, P has a 2-maximal subgroup P_2 which is not $\mathcal{N}_p$-supplemented in G. By the hypothesis, P_2 is s^*-permutably embedded in G. Then there is a non-p-nilpotent subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. Thence there is an s-permutable subgroup K of G such that $(P_2)_{se}$ is a Sylow p-subgroup of K. Since K is s-permutable in G, we have K is subnormal in G. If G is simple, then $K = 1$, and so $(P_2)_{se} = 1$. It follows that $P_2 \cap T = 1$. By Lemma 2.4, T is p-nilpotent, a contradiction.

(4) G has a unique minimal normal subgroup N such that G/N is p-nilpotent. Moreover, $\Phi(G) = 1$.

Let N be a minimal subgroup of G and we verify that the hypothesis holds for G/N. Since P is a Sylow p-subgroup of G, PN/N is a Sylow p-subgroup of G/N. If $|PN/N| \leqslant p^2$, then G/N is p-nilpotent by Lemma 2.4. So we suppose $|PN/N| \geqslant p^3$. Let M_2/N be a 2-maximal subgroup of PN/N. Then $M_2 = P_2N$ for some 2-maximal subgroup P_2 of P and $P_2 \cap N = P \cap N$ is a Sylow p-subgroup of N. If P_2 is $\mathcal{N}_p$-supplemented in G, then M_2/N is $\mathcal{N}_p$-supplemented in G/N by Lemma 2.3(2). If P_2 is s^*-permutably embedded in G, then there is a subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. Thus $G/N = P_2N/N \cdot TN/N$. Since $(|N : P_2 \cap N|, |N : T \cap N|) = 1$, $(P_2 \cap N)(T \cap N) = N = N \cap G = N \cap (P_2T)$. By Lemma 2.6, $(P_2N) \cap (TN) = (P_2 \cap T)N$. It follows that $(P_2N/N) \cap (TN/N) = (P_2N \cap TN)/N = (P_2 \cap T)N/N \leq (P_2)_{se}N/N$. Since $(P_2)_{se}N/N$ is s-permutably embedded in G/N by Lemma 2.1(2), M_2/N is s^*-permutably embedded in G/N. Therefore, G/N satisfies the hypothesis of the theorem. The minimal choice of G yields that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

(5) $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. Then, by Step (4), $N \leq O_p(G)$ and G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Let P_1 be an arbitrary maximal subgroup of P. We will show P_1 is $\mathcal{N}_p$-supplemented in G. Pick some 2-maximal subgroup P_2 of P such that $P_2 < P_1$ and $P_2 \lhd P$. If P_2 is $\mathcal{N}_p$-supplemented in G, then P_1 is also $\mathcal{N}_p$-supplemented in G obviously. By the hypothesis of the theorem, we only need prove that if P_2 is s^*-permutably embedded in G, then P_2 is also $\mathcal{N}_p$-supplemented in G. We assume that P_2 is s^*-permutably embedded in G. Then there is a subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. Thus there is an s-permutable subgroup K of G such that $(P_2)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \leq (P_2)_{se} \leq P_2$, and so $G = NM = P_2M$. Since M is p-nilpotent, P_2 is $\mathcal{N}_p$-supplemented in G. If $K_G = 1$, by Lemma 2.7, $(P_2)_{se}$ is s-permutable in G. From Lemma 2.9 we have $O^p(G) \leq N_G((P_2)_{se})$. Thus $(P_2)_{se} \leq ((P_2)_{se})^G = ((P_2)_{se})^{O^p(G)P} = ((P_2)_{se})^P \leq P_2$. It follows that $((P_2)_{se})^G = 1$ or $N \leq ((P_2)_{se})^G \leq P_2$. If $((P_2)_{se})^G = 1$, then $P_2 \cap T = 1$ and so $|T|_p = p^2$. Hence T is p-nilpotent by Lemma 2.4, and so P_2 is $\mathcal{N}_p$-supplemented in G. If $N \leq P_2$, then P_2 is also $\mathcal{N}_p$-supplemented in G as above. From above argument, we know every maximal subgroup of P should be $\mathcal{N}_p$-supplemented in G, hence G is p-nilpotent by Lemma 2.5, a contradiction.

(6) N is not p-nilpotent.

Assume N is p-nilpotent and let $N_{p'}$ be the normal p-complement of N. Since $N_{p'}$ *char* $N \lhd G$, we have $N_{p'} \lhd G$ and so $N_{p'} \leq O_{p'}(G) = 1$ by Step (1). It follows that N is a p-group. Then $N \leq O_p(G) = 1$ by step (5), contrary to Step (3).

(7) $G = NP$.

By Lemmas 2.2(1) and 2.3(1), every 2-maximal subgroup of P is either s^*-permutably embedded or $\mathcal{N}_p$-supplemented in NP. Since NP is also A_4-free and P is a Sylow p-subgroup of NP too, NP satisfies the hypothesis of the theorem. If $NP < G$, then the choice of G yields that NP is p-nilpotent. It follows that N is p-nilpotent, contrary to Step (6).

(8) If G has Hall p'-subgroups, then any two Hall p'-subgroups of G are conjugate in G.

If p is odd, then G is solvable by Feit-Thompson's Theorem, contrary to Steps (1) and (5). Thus $p = 2$. By applying a deep result of Gross [6, Main Theorem], any two Hall p'-subgroups of G are conjugate in G.

(9) Final contradiction.

If $N \cap P \leq \Phi(P)$, then N is p-nilpotent by J. Tate's theorem [14, IV, 4.7], a contradiction. Consequently, there is a maximal subgroup P_1 of P such that $P = (N \cap P)P_1$. Take a 2-maximal subgroup P_2 of P such that $P_2 < P_1$. By the hypothesis of the theorem, P_2 is either s^*-permutably embedded or $\mathcal{N}_p$-supplemented in G.

First, we assume that there is a p-nilpotent subgroup T of G such that $G = P_2T$. By Step (7), the normal Hall p'-subgroup $T_{p'}$ of T is also contained in N. By Frattini's argument, $G = NN_G(T_{p'}) = (P\cap N)N_G(T_{p'})$ and so $P = (P\cap N)(P\cap N_G(T_{p'}))$. If $P\cap N_G(T_{p'}) = P$, then $N_G(T_{p'}) = G$ and so $T_{p'} \lhd G$, a contradiction. Hence there is a maximal subgroup G_1 of P such that $P\cap N_G(T_{p'}) \leq G_1$. Then $P = (P\cap N)G_1$. Pick a 2-maximal subgroup P_0 of P such that $P_0 < G_1$. By the hypothesis of the theorem, P_0 is either s^*-permutably embedded or $\mathcal{N}_p$-supplemented in G. We now prove that if P_0 is s^*-permutably embedded in G, then P_0 is also $\mathcal{N}_p$-supplemented in G. Let L is a subgroup of G such that $G = P_0L$ and $P_0\cap L \leq (P_0)_{se}$. So there is an s-permutable subgroup K of G such that $(P_0)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $(P_0)_{se}\cap N$ is a Sylow p-subgroup of N. We know $(P_0)_{se}\cap N \leq P_0\cap N \leq P\cap N$ and $P\cap N$ is a Sylow p-subgroup of N, so $(P_0)_{se}\cap N = P_0\cap N = P\cap N$. Consequently, $P = (N\cap P)G_1 = (P_0\cap N)G_1 = G_1$, a contradiction. Therefore $K_G = 1$. By Lemma 2.7, $(P_0)_{se}$ is s-permutable in G and so $(P_0)_{se} \lhd\lhd G$. Hence $P_0\cap L \leq (P_0)_{se} \leq O_p(G) = 1$. Since $|L|_p = p^2$, L is p-nilpotent by Lemma 2.4. Hence P_0 is $\mathcal{N}_p$-supplemented in G. Let $L_{p'}$ be the normal p-complement of L, then $L_{p'}$ is a Hall p'-subgroups of G. By Step (8), $L_{p'}$ and $T_{p'}$ are conjugate in G. Since $L_{p'}$ is normalized by L, there exists $g \in P_0$ such that $L_{p'}^g = T_{p'}$. Hence $G = (P_0L)^g = P_0L^g = P_0N_G(L_{p'}^g) = P_0N_G(T_{p'})$ and $P = P\cap P_0N_G(T_{p'}) = P_0(P\cap N_G(T_{p'})) \leq G_1$, a contradiction.

Hence G has a non-p-nilpotent subgroup B of G such that $G = P_2B$ and $P_2\cap B \leq (P_2)_{se}$. Then there is an s-permutable subgroup K of G such that $(P_2)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $(P_2)_{se}\cap N = P\cap N$ is a Sylow p-subgroup of N. Consequently, $P = ((P_2)_{se}\cap N)P_1 = P_1$, a contradiction. Hence $K_G = 1$. By Lemma 2.7, $(P_2)_{se}$ is s-permutable in G, and so $(P_2)_{se} \lhd\lhd G$. Hence $P_2\cap B \leq (P_2)_{se} \leq O_p(G) = 1$. It follows that $|B|_p = p^2$. By Lemma 2.4, B is p-nilpotent, a contradiction. ∎

Theorem 3.2. *Suppose that p is the smallest prime dividing the order of a group G and H is a normal subgroup of G such that G/H is p-nilpotent. If G is A_4-free and every subgroup of H with order p^2 is either $\mathcal{N}_p$-supplemented or s^*-permutably embedded in G, then G is p-nilpotent.*

Proof. Assume that the Theorem is false and let G be a counterexample of minimal order. Then:

(1) Every proper subgroup of G is p-nilpotent.

By Lemma 2.4, we see that $|H|_p > p^2$. Let L be a arbitrary proper subgroup of G. Since $L/(L\cap H) \cong LH/H \leq G/H$, $L/(L\cap H)$ is p-nilpotent. If $|L\cap H|_p \leqslant p^2$, then L is p-nilpotent by Lemma 2.4 If $|L\cap H|_p > p^2$, then every subgroup of $L\cap H$ of order p^2 is either $\mathcal{N}_p$-supplemented or s^*-permutably embedded in L by Lemmas 2.2(1) and 2.3(1). Hence L is p-nilpotent by the choice of G. This shows that G is a minimal non-p-nilpotent group.

(2) By Step (1) and [14, Theorem IV. 5.4], G is a minimal non-nilpotent group. Hence G has the following properties:
   - (i) $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G;
   - (ii) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(3) For every subgroup L of P with order p^2, if there is a subgroup T of G such that $G = LT$, then $T = G$.

Obviously, $P = P \cap G = P \cap LT = L(P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \lhd G/\Phi(P)$. By Step (2)(ii), $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$. If $P \cap T \leq \Phi(P)$, then $L = P \lhd G$. Since G/P is p-nilpotent, G is p-nilpotent by Lemma 2.4, a contradiction. Hence $P = P \cap T$ and $T = G$.

(4) For every subgroup L of P with order p^2, then L is s-permutable in G.

By the hypothesis of the theorem, L is either $\mathcal{N}_p$-supplemented or s^*-permutably embedded in G. By Step (3), L must be s^*-permutably embedded in G. Furthermore, L is s-permutably embedded in G. Since $L \leq P \leq O_p(G)$, L is s-permutable in G by Lemma 2.8.

(5) Final contradiction.

By [18, Theorem 4.4], G is p-nilpotent, a contradiction. ∎

4. New characterizations of Sylow tower groups

Theorem 4.1. *Suppose that $\mathcal{F}$ is the class of groups with Sylow tower of supersolvable type and G is A_4-free. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every 2-maximal subgroup of any Sylow subgroup of H is either $\mathcal{U}$-supplemented or s^*-permutably embedded in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order.

Let p be smallest prime dividing $|H|$. By Lemmas 2.2(1) and 2.3(1), every 2-maximal subgroup of any Sylow p-subgroup of H is either $\mathcal{U}$-supplemented or s^*-permutably embedded in H. By Theorem 3.1, H is p-nilpotent. Let $H_{p'}$ be the normal p'-complement of H. By repeating the above argument on $H_{p'}$, one can find finally that H is Sylow tower group of supersolvable type. Again let q be the largest prime dividing $|H|$ and Q a Sylow q-subgroup of H. Then Q must be a normal subgroup of G and every 2-maximal subgroup of Q is either $\mathcal{U}$-supplemented or s^*-permutably embedded in G. It is easy to see that all 2-maximal subgroups of every Sylow subgroup of H/Q are either $\mathcal{U}$-supplemented or s^*-permutably embedded in G/P by Lemmas 2.2(3) and 2.3(1). By the minimality of G, we have $G/Q \in \mathcal{F}$. Let N be a minimal normal subgroup of G contained in Q.

(1) N is not a Sylow q-subgroup of H.

Suppose that $N = Q$. Since $N \lhd G$, we may take some 2-maximal N_2 of N such that $N_2 \lhd G_q$, where G_q is a Sylow q-subgroup of G. By the hypothesis of the theorem, N_2 is either s^*-permutably embedded or $\mathcal{U}$-supplemented in G. Let T be any supplement of N_2 in G, i.e., $N_2T = G$. Thus $G = NT$ and $N = N \cap N_2T = N_2(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in G and N is a minimal normal subgroup of G, we have $N \cap T = N$ and $T = G$. This shows that N_2 can not be $\mathcal{U}$-supplemented in G, and so is s^*-permutably embedded in G. Furthermore, N_2 must be s-permutably embedded in G. By Lemma 2.8, N_2 is s-permutable in G since $N_2 \leq Q \leq O_q(G)$. By Lemma 2.9, $O^q(G) \leq N_G(N_2)$. Thus $N_2 \lhd G_q O^q(G) = G$. It follows that $N_2 = 1$, and so $|N| = q^2$. By Lemma 2.11, $G \in \mathcal{F}$, a contradiction.

(2) Final contradiction.

By Step (1), $N < Q$. Then $(G/N)/(Q/N) \cong G/Q \in \mathcal{F}$. We will show that $G/N \in \mathcal{F}$. If $|Q/N| \leqslant q^2$, then $G/N \in \mathcal{F}$ by Lemma 2.11. If $|Q/N| > q^2$, then every 2-maximal subgroup of Q/N is either $\mathcal{U}$-supplemented or s^*-permutably embedded in G/N by Lemmas 2.2(2) and 2.3(2). By the minimality of G, we have $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation,

N is the unique minimal normal subgroup of G contained in Q and $N \nleq \Phi(G)$. By Lemma 2.10, it follows that $Q = F(Q) = N$, contrary to Step (1). ∎

Theorem 4.2. *Suppose that $\mathfrak{F}$ is the class of groups with Sylow tower of supersolvable type and G is A_4-free. Then $G \in \mathfrak{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and every subgroup of H of prime square order is either $\mathfrak{U}$-supplemented or s^*-permutably embedded in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. Let p be smallest prime dividing $|H|$. By Lemmas 2.2(1) and 2.3(1), every subgroup of any Sylow p-subgroup of H with order p^2 is either $\mathfrak{U}$-supplemented or s^*-permutably embedded in H. By Theorem 3.2, H is p-nilpotent, and so H is solvable.

(1) $G^{\mathfrak{F}}$ is a p-group and $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G, where $G^{\mathfrak{F}}$ is the $\mathfrak{F}$-residual of G.

Since $G/H \in \mathfrak{F}$, $G^{\mathfrak{F}} \leq H$. Let M be a maximal subgroup of G such that $G^{\mathfrak{F}} \nsubseteq M$ (that is, M is an $\mathfrak{F}$-abnormal maximal subgroup of G). Then $G = MH$. We claim that the hypothesis holds for $(\mathfrak{F}, M)$. In fact, $M/M \cap H \cong MH/H = G/H \in \mathfrak{F}$ and every subgroup of $M \cap H$ of prime square order is s^*-permutably embedded in M. Thus the hypothesis holds for $(\mathfrak{F}, M)$. By the choice of G, $M \in \mathfrak{F}$. Thus (1) holds by [12, Theorem 3.4.2].

(2) For every subgroup L of $G^{\mathfrak{F}}$ with order p^2, if there is a subgroup T of G such that $G = LT$, then $T = G$.

Clearly, $G^{\mathfrak{F}} = G^{\mathfrak{F}} \cap LT = L(G^{\mathfrak{F}} \cap T)$. Since $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is abelian, $(G^{\mathfrak{F}} \cap T)\Phi(G^{\mathfrak{F}})/\Phi(G^{\mathfrak{F}}) \lhd G/\Phi(G^{\mathfrak{F}})$. Since $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a chief factor of G, we have $G^{\mathfrak{F}} \cap T \leq \Phi(G^{\mathfrak{F}})$ or $G^{\mathfrak{F}} = (G^{\mathfrak{F}} \cap T)\Phi(G^{\mathfrak{F}}) = G^{\mathfrak{F}} \cap T$. If the former holds, then $L = G^{\mathfrak{F}} \lhd G$. Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$ and $|G^{\mathfrak{F}}| = p^2$, $G \in \mathfrak{F}$ by Lemma 2.11, a contradiction. Therefore $G^{\mathfrak{F}} = G^{\mathfrak{F}} \cap T$, and so $T = G$.

(3) For every subgroup L of $G^{\mathfrak{F}}$ with order p^2, then L is s-permutable in G.

By the hypothesis of the theorem, L is either $\mathfrak{N}_p$-supplemented or s^*-permutably embedded in G. By Step (2), L must be s^*-permutably embedded in G. Furthermore, L is s-permutably embedded in G. Since $L \leq G^{\mathfrak{F}} \leq O_p(G)$, L is s-permutable in G by Lemma 2.8.

(4) Final contradiction.

Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$ and every subgroup of $G^{\mathfrak{F}}$ of prime square order is s-permutable in G by Step (3), $G \in \mathfrak{F}$ by [18, Theorem 4.8], a contradiction. ∎

5. Some Applications

Corollary 5.1. [2, Theorem 3] *Let p be the smallest prime dividing the order of a group G. If G is A_4-free and every 2-maximal subgroup of any Sylow p-subgroup of G is complemented in G, then G is p-nilpotent.*

Corollary 5.2. [11, Theorem 3.2] *Let p be the smallest prime dividing the order of a group G. If G is A_4-free and every 2-maximal subgroup of any Sylow p-subgroup of G is c-normal in G, then G is p-nilpotent.*

Corollary 5.3. [18, Theorem 3.3] *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every 2-maximal subgroup of P is π-quasinormally embedded in G and G is G is A_4-free, then G is p-nilpotent.*

Corollary 5.4. [28, Theorem 4.2] *Let G be a group and p the smallest prime dividing $|G|$. If G is A_4-free and every 2-maximal subgroup of any sylow p-subgroup of G is c-supplemented in G, then $G/O_p(G)$ is p-nilpotent.*

Corollary 5.5. [7, Theorem 3.4] *Let p be the smallest prime dividing the order of a group G. If G is A_4-free and every 2-maximal subgroup of a Sylow p-subgroup of G is c-supplemented in G, then G is p-nilpotent.*

Corollary 5.6. [21, Theorem 3.4] *Suppose that p is the smallest prime dividing the order of a group G and H is a normal subgroup of G such that G/H is p-nilpotent. If G is A_4-free and every subgroup of H with order p^2 is c-supplemented in G, then G is p-nilpotent.*

Corollary 5.7. [7, Corollary 3.6] *Let G be a group of odd order, and N a normal subgroup of G such that G/N is a Sylow tower group of supersolvable type. If, for every prime p dividing the order of N and $P \in Syl_p(N)$, every 2-maximal subgroup of P is c-supplemented in G, then G is a Sylow tower group of supersolvable type.*

Corollary 5.8. [18, Corollary 3.5] *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and N a normal subgroup of a group G. Suppose that G is A_4-free. If, for every prime p dividing the order of N and $P \in Syl_p(N)$, every 2-maximal subgroup of P is s-permutably embedded in G, then G belongs to $\mathcal{F}$.*

Corollary 5.9. [21, Theorem 3.1] *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and N a normal subgroup of a group G. Suppose that G is A_4-free. If, for every prime p dividing the order of N and $P \in Syl_p(N)$, every 2-maximal subgroup of P is c-supplemented in G, then G belongs to $\mathcal{F}$.*

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