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Pairwise $\omega\beta$-Continuous Functions

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Abstract. A subset A of a bitopological space (X,τ_1,τ_2) is said to be $ij-\omega\beta$-open if for every $x\in A$ there exists an $ij-\beta$-open set U containing x such that $U-A$ is countable. In this paper, we introduce and study a new class of functions called pairwise $\omega\beta$-continuous functions by using the notion of $ij-\omega\beta$-open sets, and we give some characterizations of pairwise $\omega\beta$-continuous functions. Also pairwise $\omega\beta$-connectedness and pairwise $\omega\beta$-set connected functions are introduced in bitopological spaces and some of their properties are established.

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1. Introduction and preliminaries

The concept of a bitopological space was first introduced by Kelly [5]. Aljarrah and Noorani defined the notions of $\omega\beta$-open sets and $\omega\beta$-continuity in topological spaces (resp. [1,2]). Also Aljarrah and Noorani [3] extended the notion of $\omega\beta$-open sets to the bitopological spaces.

In the present paper, motivated by [6], we use the notion of $ij-\omega\beta$-open sets to define pairwise $\omega\beta$-continuity in bitopological spaces and we obtain many properties of these functions. Also we define pairwise $\omega\beta$-connected functions and pairwise $\omega\beta$-set connected functions and investigate their properties. Throughout the present paper, the spaces (X,τ_1,τ_2) , (Y,σ_1,σ_2) and (Z,ρ_1,ρ_2) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a bitopological space (X,τ_1,τ_2), $\tau_i-\mathrm{Cl}(A)$(resp. $\tau_i-int(A)$) denotes the closure (resp. interior) of A with respect to τ_i for $i=1,2$. For a nonempty set X, τ_u, τ_{dis} and τ_{coc} will denote, respectively, the usual, discrete and cocountable topologies on X. Also $\mathbb{R}$ and $\mathbb{Q}$ denote the sets of all real and rational numbers. A subset A of a space X is said to be $ij-\beta$-open [4] (briefly $ij-\beta O$) if $A\subseteq\tau_j-\mathrm{Cl}(\tau_i-\mathrm{int}(\tau_j-\mathrm{Cl}(A)))$, and it is said to be $ij-\omega\beta$-open [3] (briefly $ij-\omega\beta O$) if for every $x\in A$ there exists an $ij-\beta$-open set U containing x such that $U-A$ is countable.

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The complement of an $ij-\omega\beta O$ subset is said to be $ij-\omega\beta$-closed (briefly $ij-\omega\beta C$). The intersection of all $ij-\omega\beta C$ sets in X containing A is called the $ij-\omega\beta$-closure of A and is denoted by $ij-\omega\beta\mathrm{Cl}(A)$. The union of all $ij-\omega\beta O$ sets in X contained A is called $ij-\omega\beta$-interior of A and is denoted by $ij-int(A)$.

Here we recall the following known definitions and properties.

Definition 1.1. [7] *A bitopological space (X,τ_1,τ_2) is called pairwise connected if X can not be expressed as the union of two non empty disjoint sets A and B such that $(A\cap\tau_1-\mathrm{Cl}(B))\cup(\tau_2-\mathrm{Cl}(A)\cap B)=\phi$.*

Definition 1.2. [7] *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be pairwise continuous (resp. pairwise open) if the induced functions $f:(X,\tau_1)\to(Y,\sigma_1)$ and $f:(X,\tau_2)\to(Y,\sigma_2)$ are both continuous (resp. open).*

Theorem 1.1. [4] *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a pairwise continuous pairwise open function. If A is an $ij-\beta O$ set in X, then $f(A)$ is an $ij-\beta O$ set in Y.*

Remark 1.1. [3]

(i) If a subset A of (X,τ_1,τ_2) is $ij-\omega\beta O$ and $U\in\omega O(X,\tau_1)\cap\omega O(X,\tau_2)$, then $A\cap U$ is $ij-\omega\beta O$.

(ii) The union of arbitrarily many of $ij-\omega\beta O$ sets is $ij-\omega\beta O$.

Lemma 1.1. [3] *Let A and Y be subsets of (X,τ_1,τ_2) such that $A\subseteq Y$. If A is $ij-\omega\beta O$ in (X,τ_1,τ_2), then A is $ij-\omega\beta O$ in $(Y,\tau_1|_Y,\tau_2|_Y)$. If, in addition, $Y\in\tau_i$, then the converse holds.*

2. Pairwise $\omega\beta$-continuous functions

Definition 2.1. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be pairwise $\omega\beta$-continuous if the inverse image of each σ_i-open subset of Y is an $ij-\omega\beta O$ set in X, where $i\neq j$ and $i,j=1,2$.*

Every pairwise continuous function is pairwise $\omega\beta$-continuous but the converse is not true, as the following example shows.

Example 2.1. Let $X=\mathbb{R}$ with the topologies $\tau_1=\{\phi,\mathbb{R},\mathbb{R}-\mathbb{Q}\}$ and $\tau_2=\tau_u$, let $Y=\{0,1\}$ with the topologies $\sigma_1=\{\phi,Y,\{0\}\}$ and $\sigma_2=\{\phi,Y,\{1\}\}$. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be the function defined as follows

$$f(x)=\begin{cases}1, & x\in\mathbb{R}-\mathbb{Q}\\ 0, & x\in\mathbb{Q}\end{cases}$$

Then f is pairwise $\omega\beta$-continuous but not pairwise continuous.

Remark 2.1. The pairwise $\omega\beta$-continuity of a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is independent of the $\omega\beta$-continuity of the induced functions $f:(X,\tau_1)\to(Y,\sigma_1)$ and $f:(X,\tau_2)\to(Y,\sigma_2)$ as can be seen in the following examples.

Example 2.2. Let $X=\mathbb{R}$ with the topologies $\tau_1=\{\phi,\mathbb{R},\mathbb{Q}\}$ and $\tau_2=\tau_{coc}$, let $Y=\{0,1\}$ with the topologies $\sigma_1=\{\phi,Y,\{0\}\}$ and $\sigma_2=\{\phi,Y,\{1\}\}$. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be the function defined as follows

$$f(x)=\begin{cases}0, & x\in\mathbb{R}-\mathbb{Q}\\ 1, & x\in\mathbb{Q}\end{cases}$$

Then f is pairwise $\omega\beta$-continuous. However, the induced function $f:(X,\tau_2)\to(Y,\sigma_2)$ is not $\omega\beta$-continuous, since $f^{-1}(\{1\})=\mathbb{Q}\notin\omega\beta O(X,\tau_2)$ where $\{1\}\in\sigma_2$.

Example 2.3. Let $X=\mathbb{R}$ with the topologies $\tau_1=\tau_u$ and $\tau_2=\tau_{coc}$, let $Y=\{0,1\}$ with the topologies $\sigma_1=\{\phi,Y,\{1\}\}$ and $\sigma_2=\{\phi,Y,\{0\}\}$. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be the function defined as follows

$$f(x)=\begin{cases}0, & x\in\mathbb{R}-\mathbb{Q}\\ 1, & x\in\mathbb{Q}\end{cases}$$

Then the induced functions $f:(X,\tau_1)\to(Y,\sigma_1)$ and $f:(X,\tau_2)\to(Y,\sigma_2)$ are $\omega\beta$-continuous but f is not pairwise $\omega\beta$-continuous, since $f^{-1}(\{1\})=\mathbb{Q}\notin 12-\omega\beta O$ where $\{1\}\in\sigma_1$.

Theorem 2.1. *For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ the following properties are equivalent:*

(i) *f is pairwise $\omega\beta$-continuous.*
(ii) *The inverse image of each σ_i-closed set of Y is $ij-\omega\beta C$ in X.*
(iii) *For each $x\in X$ and each $V\in\sigma_i$ containing $f(x)$, there exists an $ij-\omega\beta O$ set U of X containing x such that $f(U)\subseteq V$.*
(iv) *$ij-\omega\beta\mathrm{Cl}(f^{-1}(B))\subseteq f^{-1}(i-\mathrm{Cl}(B))$ for every subset B of Y.*
(v) *$f(ij-\omega\beta\mathrm{Cl}(A))\subset i-\mathrm{Cl}(f(A))$ for every subset A of X.*

On each statement above $i\neq j$ and $i,j=1,2$.

Proof. (i) $\leftrightarrow$ (ii) Let V be σ_i-closed in Y, then $Y-V$ is σ_i-open in Y. Therefore, by assumption $f^{-1}(Y-V)$ is $ij-\omega\beta O$ in X, $i\neq j$ and $i,j=1,2$. Hence $f^{-1}(V)$ is $ij-\omega\beta C$ in X. Conversely, let V be σ_i-open in Y, then $Y-V$ is σ_i-closed in Y, by (ii) $f^{-1}(Y-V)$ is $ij-\omega\beta C$ in X, hence $f^{-1}(V)$ is $ij-\omega\beta O$ in X.

(i) $\to$ (iii) Let $x\in X$ and V be a σ_i-open set in Y containing $f(x)$. By (i) $f^{-1}(V)$ is $ij-\omega\beta O$ in X. Now take $U=f^{-1}(V)$, then $x\in U$ and $f(U)\subseteq V$. Therefore, we obtain the result.

(iii) $\to$ (iv) Let B be any subset of Y. Assume that $x\in X-f^{-1}(i-\mathrm{Cl}(B))$. Then $f(x)\in Y-(i-\mathrm{Cl}(B))$ and so there exists a σ_i-open set V of Y containing $f(x)$ such that $V\cap B=\phi$. Therefore, by (iii) there exists U an $ij-\omega\beta O$ set such that $x\in U$ and $f(U)\subseteq V$. Hence we have $U\cap f^{-1}(B)=\phi$ and $x\in X-(ij-\omega\beta\mathrm{Cl}(f^{-1}(B)))$. Thus we obtain the result.

(iv) $\leftrightarrow$ (v) Trivial.

(iv) $\to$ (ii) Let V be σ_i-closed in Y, then by (iv), $ij-\omega\beta\mathrm{Cl}(f^{-1}(V))\subset f^{-1}(V)$. Thus, $f^{-1}(V)$ is $ij-\omega\beta C$ in X. ∎

Definition 2.2. *Let (X,τ_1,τ_2) be a bitopological space and A be a subset of X. The $ij-\omega\beta$ frontier of A is defined as follows:*

$$ij-\omega\beta\mathrm{F_r}(A)=ij-\omega\beta\mathrm{Cl}(A)\cap ij-\omega\beta\mathrm{Cl}(X-A)=(ij-\omega\beta\mathrm{Cl}(A))-(ij-\omega\beta\mathrm{Int}(A)).$$

Theorem 2.2. *Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be bitopological spaces and $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a function. Then $X-(ij-\omega\beta c(f))=\cup\{ij-\omega\beta\mathrm{F_r}(f^{-1}(V)):V\in\sigma_i,f(x)\in V,x\in X\}$, where $ij-\omega\beta c(f)$ denotes the set of points at which f is pairwise $\omega\beta$-continuous.*

Proof. Let $x\in X-(ij-\omega\beta c(f))$. Then there exists a σ_i-open set V of Y containing $f(x)$ such that $U\cap(X-f^{-1}(V))\neq\phi$ for every $ij-\omega\beta O$ set U of (X,τ_1,τ_2) containing x. Thus, $x\in ij-\omega\beta\mathrm{Cl}(X-f^{-1}(V))$. Then $x\in f^{-1}(V)\cap(ij-\omega\beta\mathrm{Cl}(X-f^{-1}(V)))\subseteq ij-\omega\beta\mathrm{F_r}(f^{-1}(V))$. Hence, $X-(ij-\omega\beta c(f))\subseteq\cup\{ij-\omega\beta\mathrm{F_r}(f^{-1}(V)):V\in\sigma_i,f(x)\in$

$V, x \in X\}$. Conversely, let $x \notin X - (ij - \omega\beta c(f))$ and $x \in ij - \omega\beta c(f)$, then for each σ_i-open set V in Y containing $f(x)$, there exists an $ij - \omega\beta O$ set U containing x such that $f(U) \subseteq V$ and hence $x \in U \subseteq f^{-1}(V)$. Therefore, we obtain that $x \in ij - \omega\beta\mathrm{Int}(f^{-1}(V))$ and hence $x \notin ij - \omega\beta\mathrm{F_r}(f^{-1}(V))$ for each σ_i-open set V in Y containing $f(x)$. ∎

Proposition 2.1. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise $\omega\beta$-continuous and $A \in \omega O(X, \tau_1) \cap \omega O(X, \tau_2)$, then the restriction $f|_A : (A, \tau_1|_A, \tau_2|_A) \to (Y, \sigma_1, \sigma_2)$ is pairwise $\omega\beta$- continuous.*

Proof. Since f is pairwise $\omega\beta$-continuous, for any $V \in \sigma_i$ in Y, $f^{-1}(V)$ is $ij - \omega\beta O$ in X. By Remark 1.1(i) $f^{-1}(V) \cap A$ is $ij - \omega\beta O$ in X. Therefore, by Lemma 1.1 $(f|A)^{-1}(V)$ is $ij - \omega\beta O$ in the space $(A, \tau_1|_A, \tau_2|_A)$. ∎

Proposition 2.2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function and $X = \cup\{U_\alpha \in \tau_i | \alpha \in \Delta\}$. If the restriction $f|_{U_\alpha} : (U_\alpha, \tau_1|_{U_\alpha} \tau_2|_{U_\alpha}) \to (Y, \sigma_1, \sigma_2)$ is pairwise $\omega\beta$-continuous for each $\alpha \in \Delta$, then f is pairwise $\omega\beta$-continuous.*

Proof. Let V be any σ_i-open set of Y. Since $f|_{U_\alpha}$ is pairwise $\omega\beta$-continuous for each $\alpha \in \Delta$, $(f|_{U_\alpha})^{-1}(V) = f^{-1}(V) \cap U_\alpha$ is $ij - \omega\beta O$ in U_α. Hence by Lemma 1.1, $f^{-1}(V) \cap U_\alpha$ is $ij - \omega\beta O$ in X for each $\alpha \in \Delta$. Now take $f^{-1}(V) = \bigcup_{\alpha\in\Delta} (f^{-1}(V) \cap U_\alpha)$. By Remark 1.1(ii)$f^{-1}(V) \in ij - \omega\beta O$ in X. Hence f is pairwise $\omega\beta$-continuous. ∎

The composition of two pairwise $\omega\beta$-continuous functions is not a pairwise $\omega\beta$-continuous function in general as shown in the following example.

Example 2.4. Let $X = \mathbb{R}$ with the topologies $\tau_1 = \tau_u$ and $\tau_2 = \tau_{coc}$, let $Y = \{0, 1\}$ with the topologies $\sigma_1 = \{\phi, Y, \{1\}\}$ and $\sigma_2 = \{\phi, Y, \{0\}\}$ and let $Z = \{3, 4\}$ with the topologies $\rho_1 = \{\phi, Z, \{4\}\}$ and $\rho_2 = \{\phi, Z, \{3\}\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the function defined as follows

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases}$$

Let $g : (Y, \sigma_1, \sigma_2) \to (Z, \rho_1, \rho_2)$ be the function defined as follows

$$g(x) = \begin{cases} 3, & x = 1 \\ 4, & x = 0 \end{cases}$$

Then f and g are pairwise $\omega\beta$-continuous. However $g \circ f$ is not pairwise $\omega\beta$-continuous. Note that $(g \circ f)^{-1}(\{4\}) = \mathbb{Q}$ is not $12 - \omega\beta O$ in X and all $12 - \beta O$ sets in X containing $x \in \mathbb{Q}$ are uncountable.

Definition 2.3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:*

(i) *Pairwise $\omega\beta$-irresolute if $f^{-1}(U)$ is an $ij - \omega\beta O$ set in X for each $ij - \omega\beta O$ set U in Y.*

(ii) *Pairwise $\omega\beta$-open if $f(U)$ is an $ij - \omega\beta O$ set in Y for each $ij - \omega\beta O$ set U in X.*

On each statement above, $i \neq j$ and $i, j = 1, 2$.

It is clear that a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise $\omega\beta$-irresolute if and only if $f^{-1}(U)$ is $ij - \omega\beta C$ in X for each $ij - \omega\beta C$ set U in Y, where $i \neq j$ and $i, j = 1, 2$.

Theorem 2.3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \rho_1, \rho_2)$ be two functions. Then, the following properties hold:*

(i) *If f is pairwise $\omega\beta$-continuous and g is pairwise continuous, then $g \circ f$ is pairwise $\omega\beta$-continuous.*
(ii) *If f and g are pairwise $\omega\beta$-irresolute, then $g \circ f$ is pairwise $\omega\beta$-irresolute.*
(iii) *If f is pairwise $\omega\beta$-irresolute and g is pairwise $\omega\beta$-continuous, then $g \circ f$ is pairwise $\omega\beta$-continuous.*
(iv) *If f is pairwise $\omega\beta$-irresolute and g is pairwise continuous, then $g \circ f$ is pairwise $\omega\beta$-continuous.*

Proof. (i) Let $x \in X$ and $V \in \rho_i$ with $(g \circ f)(x) \in V$. Since g is pairwise continuous, there exists $W \in \sigma_i$ with $f(x) \in W$ and $g(W) \subseteq V$. Moreover, since f is pairwise $\omega\beta$-continuous, there exists an $ij-\omega\beta O$ set U in X containing x such that $f(U) \subseteq W$. Therefore, we obtain $(g \circ f)(U) \subseteq g(W) \subseteq V$.

(ii) Let f and g be pairwise $\omega\beta$-irresolute. Let V be $ij-\omega\beta O$ in Z. Since g is pairwise $\omega\beta$-irresolute, $g^{-1}(V)$ is $ij-\omega\beta O$ in Y. Since f is pairwise $\omega\beta$-irresolute, $f^{-1}(g^{-1}(V))$ is $ij-\omega\beta O$ in X. Therefore, $g \circ f$ is pairwise $\omega\beta$-irresolute.

The proofs of (iii) and (iv) are similar to (ii). ∎

Proposition 2.3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise continuous pairwise open function. Then f is pairwise $\omega\beta$-open.*

Proof. Let U be $ij-\omega\beta O$ in X and $y \in f(U)$. Then, there exists $x \in U$ such that $f(x) = y$. Since U is $ij-\omega\beta$-open, there exists $ij-\beta O$ set U_1 in X containing x such that $U_1 - U \subseteq C$, where C is a countable set, hence $f(U_1) - f(U) \subseteq f(C)$. Since f is pairwise continuous pairwise open, by Theorem 1.1 $f(U_1)$ is an $ij-\beta O$ set in Y containing $y = f(x)$ and hence $f(U)$ is $ij-\omega\beta O$ in Y. ∎

Recall that a bitopological space (X, τ_1, τ_2) is said to be pairwise T_2 [5](resp. pairwise $\omega\beta - \mathrm{T}_2$ [3]) if for each pair of distinct points x and y of X, there exist a τ_i-open (resp. $ij-\omega\beta O$) set U containing x and a τ_j-open ($ji-\omega\beta O$) set V containing y such that $U \cap V = \phi$ for $i \neq j$, $i, j = 1, 2$.

Proposition 2.4. *If (Y, σ_1, σ_2) is pairwise T_2 and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise $\omega\beta$-continuous injection, then (X, τ_1, τ_2) is pairwise $\omega\beta - T_2$.*

Proof. Let x and y be two distinct points of X. Then $f(x) \neq f(y)$. Since Y is pairwise T_2, there exist a τ_i-open set U and a τ_j-open set V such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \phi$. Hence $f^{-1}(U) \cap f^{-1}(V) = \phi$. Since f is pairwise $\omega\beta$-continuous, $f^{-1}(U)$ is $ij-\omega\beta O$, $f^{-1}(V)$ is $ji-\omega\beta O$, $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. This implies that (X, τ_1, τ_2) is pairwise $\omega\beta - T_2$. ∎

Lemma 2.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be injective and pairwise $\omega\beta$-irresolute. If (Y, σ_1, σ_2) is a pairwise $\omega\beta - \mathrm{T}_2$ space, then (X, τ_1, τ_2) is pairwise $\omega\beta - T_2$.*

The proof is quite similar to that of Proposition 2.4.

3. Pairwise $\omega\beta$-connected

Definition 3.1. *A bitopological space (X, τ_1, τ_2) is pairwise $\omega\beta$-disconnected if X can be expressed as the union of two nonempty disjoint sets A and B such that $(A \cap ij-\omega\beta \mathrm{Cl}(B)) \cup (ji-\omega\beta \mathrm{Cl}(A) \cap B) = \phi$ and we write $X = A \backslash B$ and call this pairwise $\omega\beta$-separation of X.*

Obviously, if A, B are pairwise $\omega\beta$-separated sets and A_1, B_1 are non empty subsets of A, B respectively, then A_1, B_1 are pairwise $\omega\beta$-separated.

The space (X, τ_1, τ_2) is pairwise $\omega\beta$-connected if and only if it is not pairwise $\omega\beta$-connected. A subset $A \subset X$ is said to be pairwise $\omega\beta$-connected if the space $(A, \tau_1|_A, \tau_2|_A)$ is pairwise $\omega\beta$-connected.

Some characterizations of pairwise $\omega\beta$-connectedness of bitopological spaces will be given next.

Theorem 3.1. *For any bitopological space (X, τ_1, τ_2), the following conditions are equivalent:*

(i) *X is pairwise $\omega\beta$-connected.*
(ii) *X can not be expressed as the union of two nonempty disjoint sets A and B such that A is $ij - \omega\beta O$ and B is $ji - \omega\beta O$.*
(iii) *X contains no nonempty proper subset which is both $ij - \omega\beta O$ and $ji - \omega\beta C$.*

Proof. (i) $\rightarrow$ (ii) Suppose X can be expressed as the union of two nonempty disjoint sets A and B such that A is $ij - \omega\beta O$ and B is $ji - \omega\beta O$. Since $A \cap B = \phi$, Consequently $A \subseteq B^c$. Then $ji - \omega\beta\mathrm{Cl}(A) \subseteq ji - \omega\beta\mathrm{Cl}(B^c) = B^c$. Therefore, $ji - \omega\beta\mathrm{Cl}(A) \cap B = \phi$. Similarly we can prove $A \cap ij - \omega\beta\mathrm{Cl}(B) = \phi$. Hence $(A \cap ij - \omega\beta\mathrm{Cl}(B)) \cup (ji - \omega\beta\mathrm{Cl}(A) \cap B) = \phi$. This is contrary to the fact that X is pairwise $\omega\beta$-connected. Therefore, X can not be expressed as the union of two nonempty disjoint sets A and B such that A is $ij - \omega\beta O$ and B is $ji - \omega\beta O$.

(ii) $\rightarrow$ (iii) Suppose that X contains a nonempty proper subset which is both $ij - \omega\beta O$ and $ji - \omega\beta C$. Then $X = A \cup A^c$, where A is $ij - \omega\beta O$, A^c is $ji - \omega\beta O$ and $A \cap A^c = \phi$. This is contrary to our assumption. Therefore, X contains no nonempty proper subset which is both $ij - \omega\beta O$ and $ji - \omega\beta C$.

(iii) $\rightarrow$ (i) Suppose X is pairwise $\omega\beta$-disconnected. Then X can be expressed as the union of two nonempty disjoint sets A and B such that $(A \cap ij - \omega\beta\mathrm{Cl}(B)) \cup (ji - \omega\beta\mathrm{Cl}(A) \cap B) = \phi$. Since $A \cap B = \phi$, we have $A = B^c$ and $B = A^c$. Since $(ji - \omega\beta\mathrm{Cl}(A) \cap B) = \phi$, we have $ji - \omega\beta\mathrm{Cl}(A) \subseteq B^c$. Hence $ji - \omega\beta\mathrm{Cl}(A) \subseteq A$. Therefore, A is $ji - \omega\beta C$. Similarly, B is $ij - \omega\beta C$. Since $A = B^c$, A is $ij - \omega\beta O$. Therefore, there exists a nonempty proper set A which is both $ij - \omega\beta O$ and $ji - \omega\beta C$. This is contrary to our assumption. Therefore, X is pairwise $\omega\beta$-connected. ∎

Proposition 3.1. *If A is a pairwise $\omega\beta$-connected subset of a bitopological space (X, τ_1, τ_2) which has the pairwise $\omega\beta$-separation $X = C \backslash D$, then either $A \subseteq C$ or $A \subseteq D$.*

Proof. Suppose that (X, τ_1, τ_2) has the pairwise $\omega\beta$-separation $X = C \backslash D$. Then $X = C \cup D$, where C and D are nonempty disjoint sets such that $(C \cap ij - \omega\beta\mathrm{Cl}(D)) \cup (ji - \omega\beta\mathrm{Cl}(C) \cap D) = \phi$. Since $C \cap D = \phi$, we have $C = D^c$ and $D = C^c$. Now $((C \cap A) \cap ij - \omega\beta\mathrm{Cl}(D \cap A)) \cup (ji - \omega\beta\mathrm{Cl}(C \cap A) \cap (D \cap A)) \subseteq (C \cap ij - \omega\beta\mathrm{Cl}(D)) \cup (ji - \omega\beta\mathrm{Cl}(C) \cap D) = \phi$. Hence $A = (C \cap A) \backslash (D \cap A)$ is a pairwise $\omega\beta$-separation of A. Since A is pairwise $\omega\beta$-connected, we have either $C \cap A = \phi$ or $D \cap A = \phi$. Consequently $A \subseteq C^c$ or $A \subseteq D^c$. Therefore, either $A \subseteq C$ or $A \subseteq D$. ∎

Theorem 3.2. *If A is pairwise $\omega\beta$-connected and $A \subseteq B \subseteq (ij - \omega\beta\mathrm{Cl}(A)) \cap (ji - \omega\beta\mathrm{Cl}(A))$, then B is pairwise $\omega\beta$-connected.*

Proof. Suppose B is not pairwise $\omega\beta$-connected. Then $B = C \cup D$, where C and D are two nonempty disjoint sets such that $(C \cap ij - \omega\beta\mathrm{Cl}(D)) \cup (ji - \omega\beta\mathrm{Cl}(C) \cap D) = \phi$. Since A

is pairwise $\omega\beta$-connected, we have from Proposition 3.1 $A \subseteq C$ or $A \subseteq D$. Suppose $A \subseteq C$. Then $D \subseteq D \cap B \subseteq D \cap ji-\omega\beta\mathrm{Cl}(A) \subseteq D \cap ji-\omega\beta\mathrm{Cl}(C) = \phi$. Therefore, $\phi \subseteq D \subseteq \phi$. Consequently, $D = \phi$. Similarly, we can prove $C = \phi$ if $A \subseteq D$. This is contrary to the fact that C and D are nonempty. Therefore, B is pairwise $\omega\beta$-connected. ∎

Proposition 3.2. *Let $\{A_i : i \in \Delta\}$ be a family of pairwise $\omega\beta$-connected subsets of a bitopological space (X, τ_1, τ_2). If $\bigcap\limits_{i\in\Delta} A_i \neq \phi$, then $A = \bigcup\limits_{i\in\Delta} A_i$ is pairwise $\omega\beta$-connected.*

Proof. Suppose that A is not pairwise $\omega\beta$-connected. Then $A = C \cup D$, where C and D are two nonempty disjoint sets such that $(C \cap ij-\omega\beta\mathrm{Cl}(D)) \cup (ji-\omega\beta\mathrm{Cl}(C) \cap D) = \phi$. Since A_i is pairwise $\omega\beta$-connected and $A_i \subseteq A$, we have $A_i \subseteq C$ or $A_i \subseteq D$. Therefore, if $\cup A_i \subseteq C$ or $\cup A_i \subseteq D$, then $A \subseteq C$ or $A \subseteq D$ and hence $A = C$ or $A = D$. Then $D = \phi$ or $C = \phi$. This is a contradiction. Otherwise, there exist $j, k \in \Delta$ such that $A_j \subset C$ and $A_k \subset D$, where $j \neq k$. Since $\cap A_i \neq \phi$, we have $x \in \cap A_i$ and hence $x \in A_j \subset C$ and $x \in A_k \subset D$. Therefore, $C \cap D \neq \phi$ which is a contradiction. Hence A is pairwise $\omega\beta$-connected. ∎

Note that a space (X, τ_1, τ_2) is said to be pairwise disconnected [7] if there exists $U \in \tau_i$ and $V \in \tau_j$ such that $U, V \neq \phi$, $U \cap V = \phi$ and $U \cup V = X$.

Proposition 3.3. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise $\omega\beta$-continuous surjection and (X, τ_1, τ_2) is pairwise $\omega\beta$-connected, then (Y, σ_1, σ_2) is pairwise connected.*

Proof. Suppose that (Y, σ_1, σ_2) is not pairwise connected. Then, there exist $U \in \sigma_i$ and $V \in \sigma_j$ such that $U, V \neq \phi$, $U \cap V = \phi$ and $U \cup V = Y$. Since f is surjection, $f^{-1}(U) \neq \phi$ and $f^{-1}(V) \neq \phi$. Since f is pairwise $\omega\beta$-continuous, $f^{-1}(U)$ is $ij-\omega\beta O$ and $f^{-1}(V)$ is $ji-\omega\beta O$ such that $f^{-1}(U) \cap f^{-1}(V) = \phi$ and $f^{-1}(U) \cup f^{-1}(V) = X$. This shows that (X, τ_1, τ_2) is not pairwise $\omega\beta$-connected, which is a contradiction. Hence (Y, σ_1, σ_2) is pairwise connected. ∎

Corollary 3.1. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise $\omega\beta$-irresolute surjection and (X, τ_1, τ_2) is pairwise $\omega\beta$-connected, then (Y, σ_1, σ_2) is pairwise $\omega\beta$-connected.*

The proof is similar to that of Proposition 3.3.

4. Pairwise $\omega\beta$-set connected functions in bitopological spaces

In this section, we introduce the notion of pairwise $\omega\beta$-set connected functions and study the relationship between these functions and pairwise $\omega\beta$-irresolute functions. If A is both $ij-\omega\beta C$ and $ji-\omega\beta O$ set in X, then it is called an $ij-\omega\beta$-coset, for all $i \neq j$ and $i, j = 1, 2$.

Definition 4.1. *A bitopological space (X, τ_1, τ_2) is said to be $ij-\omega\beta$-connected between A and B, where A and B are nonempty subsets of X, if there exists no $ij-\omega\beta$-coset F such that $A \subset F \subset X - B$. X is said to be pairwise $\omega\beta$-connected between A and B if X is $12-\omega\beta$-connected between A and B and $21-\omega\beta$-connected between A and B.*

Remark 4.1.

(i) X is $12-\omega\beta$-connected between A and B if and only if it is $21-\omega\beta$-connected between B and A.

(ii) If X is $ij-\omega\beta$-connected between A and B and $A \subset C$ and $B \subset D$, then X is $ij-\omega\beta$-connected between C and D.

Definition 4.2. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be pairwise $\omega\beta$-set connected if $f(X)$ is $ij-\omega\beta$-connected between $f(A)$ and $f(B)$ in the bitopological subspace whenever X is $ij-\omega\beta$-connected between A and B, where $i\neq j$, $i,j=1,2$.*

Proposition 4.1. *If a subspace M of a bitopological space X is $ij-\omega\beta$-connected between A and B, then so in the whole space, where $i\neq j$ and $i,j=1,2$.*

Proof. Suppose not, so there exists an $ij-\omega\beta$-coset F in X such that $A\subset F\subset X-B$. Then $F\cap M$ is an $ij-\omega\beta$-coset in M with $A\cap M\subset F\cap M\subset M-B$. Thus M is not $ij-\omega\beta$-connected between A and B. This is a contradiction. Therefore, X is $ij-\omega\beta$-connected between A and B. ∎

Recall that a subset A of a bitopological space (X,τ_1,τ_2) is called an ij-coset in X [6] if A is τ_i-closed and τ_j-open, where $i\neq j$ and $i,j=1,2$.

Lemma 4.1. *If M is a subspace which is a $12-(21-)$coset in X and X is $ij-\omega\beta$-connected between two subsets A and B of M, then M is $ij-\omega\beta$-connected between A and B.*

Proof. Suppose M is not $12-\omega\beta$-connected between A and B. So there exists a $12-\omega\beta$-coset F in M such that $A\subset F\subset M-B$. Since M is a 12-coset in X, by Lemma 1.1, F is a $12-\omega\beta$-coset in X and hence X is not $12-\omega\beta$-connected between A and B. This is a contradiction. Thus M is $12-\omega\beta$-connected between A and B. Now if X is $21-\omega\beta$-connected between A and B, then it is $12-\omega\beta$-connected between B and A and hence M is $12-\omega\beta$-connected between B and A which implies that M is $21-\omega\beta$-connected between A and B. Therefore, we obtain the result. ∎

Theorem 4.1. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is pairwise $\omega\beta$-set connected if and only if $f^{-1}(F)$ is an $ij-\omega\beta$-coset in X for any $ij-\omega\beta$-coset F in $f(X)$.*

Proof. NECESSITY. Let f be pairwise $\omega\beta$-set connected and F be any $ij-\omega\beta$-coset in $f(X)$. Suppose that $f^{-1}(F)$ is not $ij-\omega\beta$-coset in X, then X is $ij-\omega\beta$-connected between $f^{-1}(F)$ and $X-f^{-1}(F)$. Since f is pairwise $\omega\beta$-set connected, $f(X)$ is $ij-\omega\beta$-connected between $f(f^{-1}(F))$ and $f(X-f^{-1}(F))$. But $f(f^{-1}(F))=F\cap f(X)=F$ and $f(X-f^{-1}(F))=f(X)-F$ and by Theorem 3.1 F is not $ij-\omega\beta$-coset in $f(X)$. This is a contradiction. Hence $f^{-1}(F)$ is an $ij-\omega\beta$-coset in X.

SUFFICIENCY. Let $f^{-1}(F)$ be an $ij-\omega\beta$-coset in X for any $ij-\omega\beta$-coset F in $f(X)$ and let X be $ij-\omega\beta$-connected between A and B. Suppose $f(X)$ is not $ij-\omega\beta$-connected between $f(A)$ and $f(B)$, then there exists an $ij-\omega\beta$-coset F in $f(X)$ such that $f(A)\subset F\subset f(X)-f(B)$. But $A\subset f^{-1}(F)\subset X-B$ and $f^{-1}(F)$ is $ij-\omega\beta$-coset in X. This contradicts that X is $ij-\omega\beta$-connected between A and B. Therefore, f is pairwise $\omega\beta$-connected. ∎

Lemma 4.2. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a pairwise $\omega\beta$-set connected function. If X is pairwise $\omega\beta$-connected, then $f(X)$ is pairwise $\omega\beta$-connected.*

Proof. Suppose $f(X)$ is not pairwise $\omega\beta$-connected. Then, by Theorem 3.1 there exists an $ij-\omega\beta$-coset F such that $\phi\neq F\neq f(X)$. Since f is pairwise $\omega\beta$-set connected, by Theorem 4.1 $f^{-1}(F)$ is an $ij-\omega\beta$-coset in X. This contradicts that X is pairwise $\omega\beta$-connected. Therefore, $f(X)$ is pairwise $\omega\beta$-connected. ∎

It is clear that every function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ such that $f(X)$ is $\omega\beta$-connected, is pairwise $\omega\beta$-set connected.

Lemma 4.3. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be pairwise $\omega\beta$-set connected and $A\subset X$ be such that $f(A)$ is an ij-coset in $f(X)$. Then the restriction $f|_A : A\to Y$ is pairwise $\omega\beta$-set connected.*

Proof. Let A be $ij-\omega\beta$-connected between C and D. Then by Proposition 4.1, X is $ij-\omega\beta$-connected between C and D and $f(X)$ is $ij-\omega\beta$-connected between $f(C)$ and $f(D)$ (since f is pairwise $\omega\beta$-set connected). Since $f(A)$ is an ij-coset in $f(X)$, by Lemma 4.1, $f(A)$ is $ij-\omega\beta$-connected between $f(C)$ and $f(D)$. Therefore, we obtain the result. ∎

Theorem 4.2. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a pairwise $\omega\beta$-set connected, pairwise $\omega\beta$-open surjection and $f^{-1}(y)$ be pairwise $\omega\beta$-connected for each $y\in Y$. Then for any ij-coset F in Y, F is pairwise $\omega\beta$-connected if and only if $f^{-1}(F)$ is pairwise $\omega\beta$-connected.*

Proof. NECESSITY. Let $f^{-1}(F)$ be not pairwise $\omega\beta$-connected for some $ij-\omega\beta$-coset F in Y. Then by Theorem 3.1 there exists an $ij-\omega\beta$-coset K with $\phi\neq K\neq f^{-1}(F)$, in the bitopological subspace $(f^{-1}(F),\tau_1|_{f^{-1}(F)},\tau_2|_{f^{-1}(F)})$. We show that $f(K)$ is an $ij-\omega\beta$-coset in F with $\phi\neq f(K)\neq F$. Since $f^{-1}(y)$ is pairwise $\omega\beta$-connected, by Proposition 3.1 either $f^{-1}(y)\subset K$ or $f^{-1}(y)\subset f^{-1}(F)-K$ for all $y\in F$ and so $\phi\neq f(K)\neq F$ and $f(K)\cap f(f^{-1}(F)-K)=\phi$. Since f is surjective, $f(K)\cup f(f^{-1}(F)-K)=F$ and hence $f(f^{-1}(F)-K)=F-f(K)$. Since f is $ij-\omega\beta$-open, $f|_{f^{-1}(F)}$ is $ij-\omega\beta$-open onto F and hence $f(K)$ is an $ij-\omega\beta$-coset in F. This implies that F is not pairwise $\omega\beta$-connected. Therefore, if F is pairwise $\omega\beta$-connected then $f^{-1}(F)$ is pairwise $\omega\beta$-connected.

SUFFICIENCY. Since $f(f^{-1}(F))=F$ and F is an ij-coset in Y, by Lemma 4.3, the restriction $f|_{f^{-1}(F)}:f^{-1}(F)\to Y$ is pairwise $\omega\beta$-set connected. Now by Lemma 4.2, it follows that $f|_{f^{-1}(F)}(f^{-1}(F))=F$ is pairwise $\omega\beta$-connected if $f^{-1}(F)$ is pairwise $\omega\beta$-connected. Therefore, we obtain the result. ∎

Definition 4.3. *A bitopological space (X,τ_1,τ_2) is said to be $ij-\omega\beta$-extremally disconnected if the $ji-\omega\beta$-closure of any $ij-\omega\beta O$ set is $ij-\omega\beta O$, where $i\neq j$ and $i,j=1,2$. The space is said to be pairwise $\omega\beta$-extremally disconnected if it is $12-\omega\beta$-extremally disconnected and $21-\omega\beta$-extremally disconnected.*

Theorem 4.3. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be pairwise $\omega\beta$-set connected. If Y is pairwise $\omega\beta-\mathrm{T}_2$ and $ij-\omega\beta$-extremally disconnected, then $f|_C:C\to Y$ is constant for every pairwise $\omega\beta$-connected subset C of X.*

Proof. Let $x,y\in C$ and $x\neq y$. Suppose $f(x)\neq f(y)$ in Y. Since Y is pairwise $\omega\beta-\mathrm{T}_2$ and $ij-\omega\beta$-extremally disconnected, there exists $ji-\omega\beta$-coset V in Y such that $f(x)\in V$ and $f(y)\notin V$. Now $f^{-1}(V)$ is $ji-\omega\beta$-coset in X as f is pairwise $\omega\beta$-set connected. Therefore, by Lemma 1.1 $f^{-1}(V)\cap C$ is a nonempty proper $ji-\omega\beta$-coset in the subspace C and by Theorem 3.1 C is not pairwise $\omega\beta$-connected. This is a contradiction. Hence $f(x)=f(y)$, for all $x,y\in C$ and hence $f|_C:C\to Y$ is constant. ∎

Definition 4.4. *A bitopological space (X,τ_1,τ_2) is said to be pairwise $\omega\beta-C$-compact if given an $ij-\omega\beta C$ set A of X and a cover $\{V_\alpha:\alpha\in\Delta\}$ of A by $ji-\omega\beta O$ sets of X, then there exists a finite subset $\Delta_\circ$ of Δ such that $A\subset\cup\{ij-\omega\beta\mathrm{Cl}(V_\alpha:\alpha\in\Delta_\circ)\}$, $i,j=1,2$, $i\neq j$.*

Theorem 4.4. *Let Y be pairwise $\omega\beta$-extermally disconnected, pairwise $\omega\beta-C$-compact and pairwise $\omega\beta-\mathrm{T}_2$. Then $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is pairwise $\omega\beta$-irresolute if and only if it is pairwise $\omega\beta$-set connected.*

Proof. NECESSITY. It is obvious.

SUFFICIENCY. Let f be not pairwise $\omega\beta$-irresolute. Then, there exists an $ij-\omega\beta C$ set F in Y such that $f^{-1}(F)$ is not an $ij-\omega\beta C$ in X. Let $x \in ij-\omega\beta\mathrm{Cl}(f^{-1}(F))-f^{-1}(F)$. Then X is $ij-\omega\beta$-connected between $f^{-1}(F)$ and x. Hence $f(X)$ is $ij-\omega\beta$-connected between $f(f^{-1}(F))$ and $f(x)$. By Proposition 4.1 and Remark 4.1(ii), Y is $ij-\omega\beta$-connected between F and $f(x)$. Now since Y is pairwise $\omega\beta-\mathrm{T}_2$, for each $y \in F$ there exists a $ji-\omega\beta O$ set V_y containing y in Y such that $f(x) \notin ij-\omega\beta\mathrm{Cl}(V_y)$. Then the family $\{V_y : y \in F\}$ is a cover of F by $ji-\omega\beta O$ sets in Y. Since Y pairwise $\omega\beta-C$-compact, there exist a finite number of points $y_1, y_2, ..., y_n$ in F such that $F \subset \bigcup\limits_{k=1}^{n} ij-\omega\beta\mathrm{Cl}(V_{y_k}) = V$ (say). Then V is an $ij-\omega\beta$-coset in Y since Y is pairwise $\omega\beta$-extermally disconnected. Also, $f(x) \notin V$ since $f(x) \notin ij-\omega\beta\mathrm{Cl}(V_y)$ for any $y \in F$. This contradicts that Y is $ij-\omega\beta$-connected between F and $f(x)$. Hence f is pairwise $\omega\beta$-irresolute. ∎

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