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On Generalized Douglas-Weyl Spaces

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Abstract. In this paper, we show that the class of R-quadratic Finsler spaces is a proper subset of the class of generalized Douglas-Weyl spaces. Then we prove that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the non-Riemannian quantity **H**, generalizing result previously only known in the case of R-quadratic metric. Also, this yields an extension of well-known Numata's Theorem.

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1. Introduction

In Finsler geometry, there are several well-known projective invariants of Finsler metrics namely, Douglas curvature, Weyl curvature, and another projective invariant which is due to Akbar-Zadeh [2, 14, 15]. Douglas curvature is a non-Riemannian projective invariant constructed from the Berwald curvature [8]. The notion of Douglas curvature was proposed by Bácsó and Matsumoto as a generalization of Berwald curvature [4]. The Douglas curvature vanishes for Riemannian spaces, therefore it is plays a role only outside the Riemannian world [9]. Finsler metrics with ${D^i}_{jkl} = 0$ are called *Douglas metrics* and Finsler metrics with $W^i_k = 0$ are called *Weyl metrics*. There is another projective invariant in Finsler geometry, namely ${D^i}_{jkl|m}y^m = T_{jkl}y^i$ that is hold for some tensor T_{jkl}, where ${D^i}_{jkl|m}$ denotes the horizontal covariant derivatives of ${D^i}_{jkl}$ with respect to the Berwald connection of F. This equation is equivalent to that for any linearly parallel vector fields $u = u(t)$, $v = v(t)$ and $w = w(t)$ along a geodesic $c(t)$, there is a function $T = T(t)$ such that $d/dt\big[D_{\dot c}(u,v,w)\big] = T\dot c$. The geometric meaning of this identity is that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic [12].

For a manifold M, let $\mathscr{G}DW(M)$ denote the class of all Finsler metrics satisfying in above relation for some tensor T_{jkl} (T_{jkl} not fixed). In [6], Bácsó-Papp show that $\mathscr{G}DW(M)$ is closed under projective changes.

A natural question is: how large is $\mathscr{G}DW(M)$ and what kind of interesting metrics does it have? It is obvious that all Douglas metrics belong to this class. On the other hand, all

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Weyl metrics (metrics of scalar flag curvature) also belong to this class. The later is really a surprising result, due to Sakaguchi [17].

In this paper, we show that the class of generalized Douglas-Weyl metrics contains the class of R-quadratic metrics as a special case.

Theorem 1.1. *Every R-quadratic Finsler metric is a generalized Douglas-Weyl metric.*

A Finsler metric is said to be R-quadratic if its Riemann curvature $\mathbf{R}_y$ is quadratic in $y \in T_xM$ [5]. The notion of R-quadratic metric was introduced by Shen [18]. There are many non-Riemann R-quadratic Finsler metrics. For example, all Berwald metrics are R-quadratic. Some non-Berwaldian R-quadratic Finsler metrics have been constructed in [10]. Thus R-quadratic Finsler metrics form a rich class of Finsler spaces.

In [2], Akbar-Zadeh considered a non-Riemannian quantity $\mathbf{H}$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. In the class of Weyl metrics, vanishing this quantity reasults that the Finsler metric is of constant flag curvature and this fact clarifies its geometric meaning [2, 13]. Recently Li-Shen prove that every R-quadratic Randers metric has constant non-Rieman-nian invariant $\mathbf{S}$-curvature, hence it has vanishing non-Riemannian invariant $\mathbf{H}$ [10]. Then Mo extend their result and show that every R-quadratic Finsler metric has vanishing $\mathbf{H}$-curvature [11]. In this paper, we get an extension of these results and prove that every generalized Douglas-Weyl space with vanishing Landsberg curvature satisfies $\mathbf{H} = 0$.

Theorem 1.2. *Let (M,F) be a generalized Douglas-Weyl space. Suppose that F is a Landsberg metric. Then $\mathbf{H} = 0$.*

According to Theorem 1.2, every Landsberg metric F of scalar flag curvature $\mathbf{K}$ satisfies $\mathbf{H} = 0$ and then F is of constant flag curvature. By Akbar-Zadeh Theorem, the Cartan tensor of F satisfies $\ddot{A}_{ijk} + \mathbf{K}A_{ijk} = 0$ [1]. Since F is a Landsberg metric, then $\mathbf{K}A_{ijk} = 0$. If we suppose that F is of non-zero scalar flag curvature, then F is Riemannian. Therefore, we get the following.

Corollary 1.1. *Every Landsberg metric of non-zero scalar flag curvature is Riemannian.*

Corollary 1.1 was proved by Numata [16]. Theorem 1.2 can be regarded as a generalization of the Numata Theorem. The converse of Theorem 1.2 is not true. For example, consider following Finsler metric on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$,

$$F(y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2}, \quad y \in T_x\mathbb{B}^n = \mathbb{R}^n$$

where $|.|$ and $\langle,\rangle$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. F is called the *Funk metric* which is a Randers metric on $\mathbb{B}^n$ [19]. Funk metric is a generalized Douglas-Weyl metric satisfies $\mathbf{H} = 0$ while $\mathbf{L} \neq 0$.

There are many connections in Finsler geometry [20, 21]. In this paper, we set the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle of

M. A Finsler metric on M is a function $F: TM \to [0,\infty)$ which has the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y+su+tv)\right]|_{s,t=0}, \quad u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u,v)\right]|_{t=0}, \quad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = \mathbf{0}$ if and only if F is Riemannian.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}.$$

where $G^i(x,y) := 1/4\, g^{il}(x,y)\{[F^2]_{x^ky^l}y^k - [F]^2_{x^l}\}$. $\mathbf{G}$ is called the associated spray to (M,F). The projection of an integral curve of $\mathbf{G}$ is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}\Big|_x, \quad \mathbf{E}_y(u,v) := E_{ij}(y)u^iv^j,$$

where $B^i{}_{jkl}(y) := (\partial^3G^i)/(\partial y^j\partial y^k\partial y^l)(y)$, $E_{ij}(y) := 1/2\ B^m{}_{ijm}(y)$, $u = u^i\partial/(\partial x^i)|_x$, $v = v^i\partial/(\partial x^i)|_x$ and $w = w^i\partial/(\partial x^i)|_x$. $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature respectively. A Finsler metric is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = 0$ and $\mathbf{E} = 0$, respectively [19, 22].

Define $\tilde{\mathbf{B}}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{H}_y : T_xM \otimes T_xM \to \mathbb{R}$ by

$$\tilde{\mathbf{B}}_y(u,v,w) := \tilde{B}^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}\Big|_x, \quad \mathbf{H}_y(u,v) := H_{ij}(y)u^iv^j,$$

where $\tilde{B}^i{}_{jkl} := B^i{}_{jkl|s}y^s$ and $H_{ij} := E_{ij|s}y^s$. Then $\tilde{\mathbf{B}}_y$ and $\mathbf{H}_y$ are defined as the covariant derivative of $\mathbf{B}$ and $\mathbf{E}$ along geodesics, respectively [13].

Define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{D}_y(u,v,w) := D^i{}_{jkl}(y)u^iv^jw^k\partial/(\partial x^i)|_x$ where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\left\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\right\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y\in TM_0}$ the Douglas curvature. A Finsler metric with $\mathbf{D} = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [4].

Define $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ and $\tilde{\mathbf{L}}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^iv^jw^k$ and $\tilde{\mathbf{L}}_y(u,v,w) := \tilde{L}_{ijk}(y)u^iv^jw^k$ where

$$L_{ijk} := C_{ijk|s}y^s \quad \text{and} \quad \tilde{L}_{ijk} := L_{ijk|s}y^s.$$

The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. F is called a Landsberg metric if $\mathbf{L} = 0$ [23].

Theorem 2.1. [3] *For a Douglas metric F on a manifold M, if* $\mathbf{L}=0$, *then* $\mathbf{B}=0$.

By using the notion of Landsberg curvature, we define $\Sigma_y : T_xM\otimes T_xM\otimes T_xM\otimes T_xM \to \mathbb{R}$ by $\Sigma_y(u,v,w,z) := \Sigma_{ijkl}(y)u^iv^jw^kz^l$ where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}).$$

A Finsler metric is said to be stretch metric if $\Sigma = 0$. In [7], Berwald showed that stretch curvature vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram.

3. Proof of Theorem 1.1

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics. For a vector $y \in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) := R^i{}_k(y)u^k\partial/(\partial x^i)$, where

$$R^i{}_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $R := \{R_y\}_{y\in TM_0}$ is called the Riemann curvature [19]. A Finsler metric F is said to be R-quadratic if R_y is quadratic in $y \in T_xM$ at each point $x \in M$. Let

$$R^i{}_{jkl}(x,y) := \frac{1}{3}\frac{\partial}{\partial y^j}\left\{\frac{\partial R^i{}_k}{\partial y^l} - \frac{\partial R^i{}_l}{\partial y^k}\right\},$$

where $R^i{}_{jkl}$ is the Riemann curvature of Berwald connection. We have

$$R^i{}_k = R^i{}_{jkl}(x,y)y^jy^l.$$

Then $R^i{}_k$ is quadratic in $y \in T_xM$ if and only if $R^i{}_{jkl}$ are functions of position alone.

In this section, we prove that every R-quadratic Finsler metric is a generalized Douglas-Weyl metric. To prove this, we need the following.

Lemma 3.1.

$$R^i{}_{jkl|m} + R^i{}_{jlm|k} + R^i{}_{jmk|l} = B^i{}_{jku}R^u{}_{lm} + B^i{}_{jlu}R^u{}_{km} + B^i{}_{klu}R^u{}_{jm}, \tag{3.1}$$

$$B^i{}_{jkl|m} - B^i{}_{jmk|l} = R^i{}_{jml,k}, \tag{3.2}$$

$$B^i{}_{jkl,m} = B^i{}_{jkm,l}. \tag{3.3}$$

Proof. The curvature form of Berwald connection is

$$\Omega^i{}_j = d\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k = \frac{1}{2}R^i{}_{jkl}\omega^k\wedge\omega^l - B^i{}_{jkl}\omega^k\wedge\omega^{n+l}. \tag{3.4}$$

For the Berwald connection, we have the following structure equation

$$dg_{ij} - g_{jk}\Omega^k_i - g_{ik}\Omega^k_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}. \tag{3.5}$$

Differentiating (3.5) yields the following Ricci identity

$$\begin{aligned} g_{pj}\Omega^p_i - g_{pi}\Omega^p_j = {}& -2L_{ijk|l}\omega^k\wedge\omega^l - 2L_{ijk,l}\omega^k\wedge\omega^{n+l} - 2C_{ijl|k}\omega^k\wedge\omega^{n+l} \\ & - 2C_{ijl,k}\omega^{n+k}\wedge\omega^{n+l} - 2C_{ijp}\Omega^p_l y^l. \end{aligned} \tag{3.6}$$

Differentiating of (3.4) yields

$$d\Omega_i^{\ j} - \omega_i^{\ k}\wedge\Omega_k^{\ j} + \omega_k^{\ j}\wedge\Omega_i^{\ k} = 0. \tag{3.7}$$

Define ${B^i}_{jkl|m}$ and ${B^i}_{jkl,m}$ by

$$dB^i_{jkl} - B^i_{mkl}\omega^m_i - B^i_{jml}\omega^m_k - B^i_{jkm}\omega^m_l + B^i_{jkl}\omega^i_m = B^i_{jkl|m}\omega^m + B^i_{jkl,m}\omega^{n+m}. \tag{3.8}$$

Similarly, we define ${R^i}_{jkl|m}$ and ${R^i}_{jkl,m}$ by

$$dR^i_{jkl} - R^i_{mkl}\omega^m_i - B^i_{jml}\omega^m_k - R^i_{jkm}\omega^m_l + R^i_{jkl}\omega^i_m = R^i_{jkl|m}\omega^m + R^i_{jkl,m}\omega^{n+m}. \tag{3.9}$$

From (3.6), (3.7), (3.8) and (3.9), one obtains the above Bianchi identity. ∎

Proposition 3.1. *Every R-quadratic Finsler metric is a generalized Douglas-Weyl metric.*

Proof.

$${D^i}_{jkl} = {B^i}_{jkl} - \frac{2}{n+1}\{E_{jk}{\delta^i}_l + E_{kl}{\delta^i}_j + E_{lj}{\delta^i}_k + E_{jk,l}y^i\}. \tag{3.10}$$

Then

$$D^i_{jkl|m}y^m = B^i_{jkl|m}y^m - \frac{2}{n+1}\{H_{jk}\delta^i_l + H_{kl}\delta^i_j + H_{lj}\delta^i_k + E_{jk,l|m}y^m y^i\}. \tag{3.11}$$

By (3.2), it follows that

$${B^i}_{jkl|m}y^m = {R^i}_{jml,k}y^m, \tag{3.12}$$

which yields

$$H_{jk} = {R^p}_{jmp,k}y^m. \tag{3.13}$$

We obtain

$$h^i_\alpha D^\alpha_{jkl|m}y^m = h^i_\alpha R^\alpha_{jml,k}y^m - \frac{2}{n+1}\{R^p_{jmp,k}h^i_l + R^p_{lmp,j}h^i_k + R^p_{kmp,l}h^i_j\}y^m. \tag{3.14}$$

F is R-quadratic, then we have

$$h^i_\alpha D^\alpha_{jkl|m}y^m = 0.$$

It means that F is a generalized Douglas-Weyl metric. ∎

The following examples shows that there is a generalized Douglas-Weyl metric which is not R-quadratic.

Example 3.1. Let $X = (x,y,z) \in \mathbb{B}^3(1) \subset \mathbb{R}^3$ and $Y = (u,v,w) \in T_x\mathbb{B}^3(1)$. Put

$$A := (x^2+y^2+z^2)u - 2x(xu+yv+zw),$$
$$B := 1-(x^2+y^2+z^2)^2,$$
$$C := u^2+v^2+w^2.$$

Define $F = F(x,y)$ by

$$F := \alpha+\beta = \frac{\sqrt{A^2+BC}}{B} + \frac{A}{B}.$$

The flag curvature of F is given by

$$K = \frac{-3u}{F} + x^2 - 2y^2 - 2z^2.$$

It means that F is of scalar flag curvature and then F is a generalized Douglas-Weyl metric on $\mathbb{B}^3(1)$. It is easy to show that F is not R-quadratic metric.

4. Proof of Theorem 1.2

In these section, we will prove a generalized version of Theorem 1.2. Indeed, we study compact generalized Douglas-Weyl spaces with vanishing stretch curvature and prove the following.

Theorem 4.1. *Every compact generalized Douglas-Weyl space with vanishing stretch curvature satisfies* $\mathbf{H} = 0$.

The most elegant importance of studying Finsler metrics, is to obtain non-Riemannian PDEs in the sence that they hold trivially for Riemannian metrics. To prove Theorem 4.1, we find a PDE on mean Berwald curvature of generalized Douglas-Weyl metrics with vanishing stretch tensor. For this reason, we need the following:

Lemma 4.1. *Let* (M,F) *be a generalized Douglas-Weyl space. Then*

$$\tilde{B}^l{}_{ijk|h} = \frac{-2y^l}{F^2}\tilde{L}_{ijk|h} + \frac{2}{n+1}\{H_{ij|h}h^l_k + H_{jk|h}h^l_i + H_{ik|h}h^l_j\}. \tag{4.1}$$

Proof.

$$D^i{}_{jkl} = B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\}. \tag{4.2}$$

Then

$$h^m_i D^i{}_{jkl|s}y^s = h^m_i B^i{}_{jkl|s}y^s - \frac{2}{n+1}\{H_{jk}h^m_l + H_{kl}h^m_j + H_{lj}h^m_k\}. \tag{4.3}$$

By assumption we get

$$h^m_i \tilde{B}^i{}_{jkl} = \frac{2}{n+1}\{H_{jk}h^m_l + H_{kl}h^m_j + H_{lj}h^m_k\}. \tag{4.4}$$

Taking a horizontal derivative of (4.4) yields

$$h^m_i \tilde{B}^i{}_{jkl|h} = \frac{2}{n+1}\{H_{jk|h}h^m_l + H_{kl|h}h^m_j + H_{lj|h}h^m_k\}. \tag{4.5}$$

Using

$$g_{ip}y^p B^i{}_{jkl} = -2L_{jkl}, \tag{4.6}$$

one can yields

$$h^m_i \tilde{B}^i{}_{jkl|h} = (h^m_i B^i{}_{jkl})_{|s|h}y^s = (B^m{}_{jkl} + \frac{2y^m}{F^2}L_{jkl})_{|s|h}y^s = \tilde{B}^m{}_{jkl|h} + \frac{2y^m}{F^2}\tilde{L}_{jkl|h}. \tag{4.7}$$

By (4.5) and (4.7) we obtain (4.1). ∎

Lemma 4.2. *Let* (M,F) *be a generalized Douglas-Weyl space. Suppose that* F *is a stretch metric. Then for any geodesic* $c(t)$ *and any parallel vector field* $U(t)$ *along* c*, the following function*

$$\mathbf{E}(t) = \mathbf{E}_{\dot{c}}(U(t),U(t)), \tag{4.8}$$

satisfying in the following equation

$$\mathbf{E}(t) = \mathbf{H}(0)t + \mathbf{E}(0). \tag{4.9}$$

Proof. Since F is a stretch metric, then we have $L_{ijk|l} = L_{ijl|k}$. Contracting it with y^l yields $\tilde{L}_{ijk} = 0$. By considering Lemma 4.1, we have

$$\tilde{B}^l{}_{ijk|h} - \tilde{B}^l{}_{ijh|k} = \frac{2}{n+1}\{(H_{jk|h} - H_{jh|k})h^l_i + (H_{ik|h} - H_{ih|k})h^l_j\} + \frac{2}{n+1}\{H_{ij|h}h^l_k - H_{ij|k}h^l_h\}. \tag{4.10}$$

Putting $j = l$ in (4.10), we get

$$H_{ik|h} - H_{ih|k} = \frac{2}{n+1}\{H_{ik|h} - H_{ih|k}\}, \tag{4.11}$$

which yields

$$H_{ik|h} = H_{ih|k}. \tag{4.12}$$

Contacting (4.12) with y^h

$$H_{ik|h}y^h = 0. \tag{4.13}$$

Let

$$\mathbf{H}(t) = \mathbf{H}_{\dot{c}}(U(t), U(t)). \tag{4.14}$$

From the definition of $\mathbf{H}_y$, we have

$$\mathbf{H}(t) = \mathbf{E}^{'}(t). \tag{4.15}$$

By (4.13) we have $\mathbf{H}^{'}(t) = 0$ which implies that

$$\mathbf{H}(t) = \mathbf{H}(0).$$

Then by (4.15), we get the equation (4.9). ∎

Remark 4.1. Let (M,F) be a Finsler space and $c : [a,b] \to M$ be a geodesic. For a parallel vector field $V(t)$ along c, we have $g_{\dot{c}}(V(t),V(t)) = constant$.

Proof of Theorem 4.1. Take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Let $c(t)$ be the geodesic with $\dot{c}(0) = y$ and $V(t)$ the parallel vector field along c with $V(0) = v$. Define $\mathbf{E}(t)$ and $\mathbf{H}(t)$ as in (4.8) and (4.14), respectively. Then by Lemma 4.2, we have $\mathbf{E}(t) = t\,\mathbf{H}(0) + \mathbf{E}(t)$. Suppose that $\mathbf{E}_y$ is bounded, i.e., there is a constant $N < \infty$ such that

$$||\mathbf{E}||_x := \sup_{y\in T_xM_0} \sup_{v\in T_xM} \frac{\mathbf{E}_y(v,v)}{[g_y(v,v)]^{\frac{3}{2}}} \leq N. \tag{4.16}$$

By Remark 4.1, we know that $T := g_{\dot{c}}(V(t),V(t)) = constant$ is positive constant. Thus

$$|\mathbf{E}(t)| \leq NT^{\frac{3}{2}} < \infty,$$

and $\mathbf{E}(t)$ is a bounded function on $[0,\infty)$. This implies

$$\mathbf{H}_y(v,v) = \mathbf{H}(0) = 0.$$

Therefore $\mathbf{H} = 0$. ∎

By Theorem 4.1, every compact generalized Douglas-Weyl space with vanishing Landsberg curvature satisfies $\mathbf{H} = 0$. By a similar way, it follows that every compact Douglas space with vanishing stretch curvature satisfy in $\mathbf{H} = 0$.

Proof of Theorem 1.2. By (4.4) we have

$$h^m_i \tilde{B}^i{}_{jkl} = \frac{2}{n+1}\{H_{jk}h^m_l + H_{kl}h^m_j + H_{lj}h^m_k\}. \tag{4.17}$$

Using (4.6) we get

$$h^m_i \tilde{B}^i{}_{jkl} = \tilde{B}^m{}_{jkl} + \frac{2}{F^2}\tilde{L}_{jkl}y^m. \tag{4.18}$$

From assumption and the relations (4.17) and (4.18), we obtain

$$\tilde{B}^m{}_{jkl} = \frac{2}{n+1}\{H_{jk}h^m_l + H_{kl}h^m_j + H_{lj}h^m_k\}. \tag{4.19}$$

By putting $m = k$ in (4.19), we conclude that $\mathbf{H} = 0$. ∎

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