BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(3) (2013), 657–670*

Asymptotic Distributions of the Generalized and the Dual Generalized Extremal Quotient

[1]H. M. BARAKAT, [2]E. M. NIGM AND [3]A. M. ELSAWAH

[1,2,3] Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt
[1]hbarakat2@hotmail.com, [2]s_nigm@yahoo.com, [3]A_Elsawah85@yahoo.com

Abstract. Necessary and sufficient conditions for the weak convergence of the generalized and the dual generalized extremal quotient are obtained. The class of possible non-degenerate limit distribution functions of quotient of generalized and its dual extreme order statistics is characterized. Some illustrative examples are obtained.

2010 Mathematics Subject Classification: 62G30

Keywords and phrases: Weak convergence, generalized order statistics, dual generalized order statistics, extremal quotient.

1. Introduction

Consider a sequence of independent and identically distributed random variables (rv's) $\{X_n : n \geq 1\}$ with distribution function (df) F. Let $M_n = \max\{X_1,...,X_n\}$ and $L_n = \min\{X_1,...,X_n\}$. The extremal quotient is defined by $q_n = M_n/L_n$ (see, Galambos and Simonelli [10]). This statistic is obviously not affected by a change of scale. Therefore, its use may be of interest in cases where the scale plays no role, e.g., in climatic study (see Canard [7]). The extremal quotient has used in several fields, most notably in industrial quality control, life testing, small-area variation analysis and the classical heterogeneity of variance situation. For example, a quality engineer might use this statistic as a basic measurement in controlling the roundness of a circular component in a production process. Also, Wong and Wong [17] used the extremal quotient to test the hypothesis that the population of a sample has an exponential df. The same authors [16] used this statistic for testing the shape parameter of the Weibull df. The limit laws for the extremal quotient were fully characterized by Barakat [5]. Cramer and Kamps [8] have used the extremal quotient in the framework of sequential order statistics. Moreover, some additional relevant references of this statistic are given these.

Generalized order statistics (gos) have been introduced by Kamps [12] as a unification of several models of ascendingly ordered rv's. The gos $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k)$,

Communicated by M. Ataharul Islam.
Received: December 3, 2010; *Revised:* June 9, 2011.

$...,X(n,n,\tilde{m},k)$ based on a df F are defined by their probability density function (pdf)

$$f^{(\tilde{m},k)}_{1,2,...,n:n}(x_1,...,x_n) = k\left(\prod_{i=1}^{n-1}\gamma_i\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^{m_i}\right)(1-F(x_n))^{k-1}\left(\prod_{i=1}^{n}f(x_i)\right),$$

on the cone $\{(x_1,...,x_n): x_0 = F^{-1}(0) < x_1 \le ... \le x_n < F^{-1}(1) = x^0\}$, where $x_0 = \inf\{x : F(x) > 0\} \ge -\infty$ and $x^0 = \sup\{x : F(x) < 1\} \le \infty$. The parameters $\gamma_1,...,\gamma_n$ are defined by $\gamma_n = k > 0$ and $\gamma_r = k+n-r+\sum_{j=r}^{n-1} m_j > 0$, $r = 1,2,...,n-1$, where $\tilde{m} = (m_1,...,m_{n-1}) \in \Re^{n-1}$. Particular choice of the parameters $\gamma_1,...,\gamma_n$ leads to different models, e.g., ordinary order statistics (oos) ($m_1 = m_2 = ... = m_{n-1} = 0, k = 1$); order statistics with non-integral sample size ($m_1 = m_2 = ... = m_{n-1} = 0, k = \alpha - n+1$, and $\alpha > n-1$); kth record values ($m_1 = m_2 = ... = m_{n-1} = -1$ and k is any positive integer) and sequential order statistics (sos) ($m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1, 1 \le i \le n-1, k = \alpha_n$ and $\alpha_1,\alpha_2,...,\alpha_n > 0$).

The concept of dual generalized order statistics (dgos) is introduced in Burkschat *et al.* [6] to enable a common approach to descendingly ordered rv's like reversed order statistics and lower records models. The dgos $X_d(1,n,\tilde{m},k), X_d(2,n,\tilde{m},k),...,X_d(n,n,\tilde{m},k)$ based on a df F are defined by their pdf

$$f^{d(\tilde{m},k)}_{1,2,...,n.n}(x_1,...,x_n) = k\left(\prod_{i=1}^{n-1}\gamma_i\right)\left(\prod_{i=1}^{n-1}F^{m_i}(x_i)\right)(F^{k-1}(x_n))\left(\prod_{i=1}^{n}f(x_i)\right),$$

where $x^0 = F^{-1}(1) > x_1 \ge ... \ge x_n > F^{-1}(0) = x_0$.

In this work, we consider a wide subclass of gos (dgos), by assuming $\gamma_j - \gamma_{j+1} = m+1 \ge 0$. This subclass is known as m-gos (m-dgos). Clearly, many important practical models of m-gos are included such as oos, order statistics with non-integer sample size, upper record values and sos. The marginal df's of the rth and $(n-r+1)$th m-gos (c.f. Nasri-Roudsari [13], see also Barakat [3]) are represented by $\Phi^{(m,k)}_{r:n}(x) = I_{G(x)}(r,N-r+1)$ and $\Phi^{(m,k)}_{n-r+1:n}(x) = I_{G(x)}(N-R+1,R)$, respectively, where $I_x(n,m) = 1/B(n,m)\int_0^x t^{n-1}(1-t)^{m-1}dt$ is the incomplete beta ratio function, $G(x) = 1-(1-F(x))^{m+1}, N = \ell+n-1$, $R = \ell+r-1$ and $\ell = k/(m+1)$. Similarly, by using the results of Kamps [12] and Burkschat *et al.* [6], the marginal df's of the rth and $(n-r+1)$th m-dgos are given by $\Phi^{(m,k)}_{d,r:n}(x) = I_{F^{m+1}(x)}(N-r+1,r)$ and $\Phi^{(m,k)}_{d,n-r+1:n}(x) = I_{F^{m+1}(x)}(R,N-R+1)$, respectively.

The central result of the classical extreme value theory is that the class of possible limit df's of each of the lower and upper extreme order statistics is restricted to essentially three different types. Namely, for some suitable normalizing constants $\alpha_n, a_n > 0$ and $\beta_n, b_n \in \Re$, we have

$$\Phi^{(0,1)}_{r:n}(\alpha_n x+\beta_n) = I_{F(\alpha_n x+\beta_n)}(r,n-r+1) \xrightarrow[n]{w} 1-\Gamma_r(U_{i,\alpha}(x)),\ i \in \{1,2,3\}, \tag{1.1}$$

if, and only if, $nF(\alpha_n x+\beta_n) \to U_{i,\alpha}(x)$, as $n \to \infty$, and

$$\Phi^{(0,1)}_{n-r+1:n}(a_n x+b_n) = I_{F(a_n x+b_n)}(n-r+1,r) \xrightarrow[n]{w} \Gamma_r(V_{j,\beta}(x)),\ j \in \{1,2,3\}, \tag{1.2}$$

if, and only if, $n(1-F(a_n x+b_n) \to V_{j,\beta}(x))$, as $n \to \infty$, where $\xrightarrow[n]{w}$ denotes the weak convergence, as $n \to \infty$, $\Gamma_r(x) = 1/(\Gamma(r))\int_x^\infty t^{r-1}e^{-t}dt$,

$$\text{Type I}: U_{1,\alpha}(x) = \begin{cases} (-x)^{-\alpha}, & x < 0, \alpha > 0, \\ \infty, & x \ge 0, \end{cases} \quad \text{Type II}: U_{2,\alpha}(x) = \begin{cases} x^{\alpha}, & x \ge 0, \alpha > 0, \\ 0, & x < 0, \end{cases}$$

(1.3) $$\text{Type III}: \ U_{3,0}(x) = e^{x}, \ \forall x,$$

and

$$\text{Type I}: V_{1,\beta}(x) = \begin{cases} x^{-\beta}, & x > 0, \beta > 0, \\ \infty, & x \le 0, \end{cases} \quad \text{Type II}: V_{2,\beta}(x) = \begin{cases} (-x)^{\beta}, & x \le 0, \beta > 0, \\ 0, & x > 0, \end{cases} .$$

(1.4) $$\text{Type III}: \ V_{3,0}(x) = e^{-x}, \ \forall x.$$

The following two theorems, due to Nasri-Roudsari [13] and Nasri-Roudsari and Cramer [14] (see also Barakat [3]), extend the above result to the $X(r,n,m,k)$, $X_d(n-r+1,n,m,k)$, $X(n-r+1,n,m,k)$ and $X_d(r,n,m,k)$.

Theorem 1.1. *Let $m_1 = m_2 = ... = m_{n-1} = m > -1$ and r be fixed integer with respect to n. Then, there exist normalizing constants $\alpha_{n,m}, \tilde{\alpha}_{n,m} > 0$ and $\beta_{n,m}, \tilde{\beta}_{n,m}$, for which*

(1.5) $$\Phi_{r:n}^{(m,k)}(\alpha_{n,m}x + \beta_{n,m}) \ \xrightarrow[n]{w} \ \underline{H}_{i,\alpha}^{(m,k)}(x)$$

and

(1.6) $$\Phi_{d,n-r+1:n}^{(m,k)}(\tilde{\alpha}_{n,m}x + \tilde{\beta}_{n,m}) \ \xrightarrow[n]{w} \ \overline{H}_{i,\alpha}^{d(m,k)}(x),$$

where $\underline{H}_{i,\alpha}^{(m,k)}(x)$ and $\overline{H}_{i,\alpha}^{d(m,k)}(x)$ are non-degenerate df's if, and only if, (1.1) is satisfied. In this case, $\underline{H}_{i,\alpha}^{(m,k)}(x) = \underline{H}_{i,\alpha}^{(0,1)}(x) = 1 - \Gamma_r(U_{i,\alpha}(x))$ and $\overline{H}_{i,\alpha}^{d(m,k)}(x) = 1 - \Gamma_R(U_{i,\alpha}^{m+1}(x))$ (and we say that the df F belongs to the domain of attraction of each of the limits $\underline{H}_{i,\alpha}^{(m,k)}$ and $\overline{H}_{i,\alpha}^{d(m,k)}$, written $F \in D(\underline{H}_{i,\alpha}^{(m,k)})$ and $F \in D(\overline{H}_{i,\alpha}^{d(m,k)})$, respectively). Moreover, the normalizing constants can be chosen such that $\alpha_{n,m} = \alpha_{\phi(n)}, \beta_{n,m} = \beta_{\phi(n)}, \tilde{\alpha}_{n,m} = \alpha_{\psi(n)}$ and $\tilde{\beta}_{n,m} = \beta_{\psi(n)}$, where $\phi(n) = n(m+1)$ and $\psi(n) = n^{1/(m+1)}$. Finally, equivalent necessary and sufficient conditions for (1.5) and (1.6) to be satisfied are $NG(\alpha_{n,m}x + \beta_{n,m}) \to U_{i,\alpha}(x)$ and $NF^{m+1}(\tilde{\alpha}_{n,m}x + \tilde{\beta}_{n,m}) \to U_{i,\alpha}^{m+1}(x)$, as $n \to \infty$, respectively.

Theorem 1.2. *Let $m_1 = m_2 = ... = m_{n-1} = m > -1$ and r be fixed integer with respect to n. Then, there exist normalizing constants $a_{n,m}, \tilde{a}_{n,m} > 0$ and $b_{n,m}, \tilde{b}_{n,m}$, for which*

(1.7) $$\Phi_{n-r+1:n}^{(m,k)}(a_{n,m}x + b_{n,m}) \ \xrightarrow[n]{w} \ \overline{H}_{j,\beta}^{(m,k)}(x)$$

and

(1.8) $$\Phi_{d,r:n}^{(m,k)}(\tilde{a}_{n,m}x + \tilde{b}_{n,m}) \ \xrightarrow[n]{w} \ \underline{H}_{j,\beta}^{d(m,k)}(x),$$

where $\overline{H}_{j,\beta}^{(m,k)}(x)$ and $\underline{H}_{j,\beta}^{d(m,k)}(x)$ are non-degenerate df's if, and only if, (1.2) is satisfied. In this case, $\overline{H}_{j,\beta}^{(m,k)}(x) = \Gamma_R(V_{j,\beta}^{m+1}(x))$ and $\underline{H}_{j,\beta}^{d(m,k)}(x) = \underline{H}_{j,\beta}^{d(0,1)}(x) = \Gamma_r(V_{j,\beta}(x))$ (and we say that the df F belongs to the domain of attraction of each of the limits $\overline{H}_{j,\beta}^{(m,k)}$ and $\underline{H}_{j,\beta}^{d(m,k)}$, written $F \in D(\overline{H}_{j,\beta}^{(m,k)})$ and $F \in D(\underline{H}_{j,\beta}^{d(m,k)})$, respectively). Moreover, the normalizing constants can be chosen such that $a_{n,m} = a_{\psi(n)}, b_{n,m} = b_{\psi(n)}, \tilde{a}_{n,m} = a_{\phi(n)}$ and $\tilde{b}_{n,m} = b_{\phi(n)}$. Finally, equivalent necessary and sufficient conditions for (1.7) and (1.8) to be satisfied are $N(1 - G(a_{n,m}x + b_{n,m})) \to V_{j,\beta}^{m+1}(x)$ and $N(1 - F^{m+1}(\tilde{a}_{n,m}x + \tilde{b}_{n,m})) \to V_{j,\beta}(x)$, as $n \to \infty$, respectively.

The normalizing constants in Theorems 1.1 and 1.2 can be determined by the following lemma, see Barakat [5].

Lemma 1.1. *The normalizing constants a_n, b_n, α_n and β_n can be chosen such as:*
Type I: $a_n = |\overline{\gamma}(n)|, b_n = 0$ *and* $\alpha_n = |\underline{\gamma}(n)|, \beta_n = 0$, *where* $x^0 = -x_0 = \infty$, $\overline{\gamma}(t) = \inf\{x : F(x) \geq 1 - 1/t\} \uparrow x^0$, *and* $\underline{\gamma}(t) = \sup\{x : F(x) \leq 1/t\} \downarrow x_0$, *as* $n \to \infty$.
Type II: $a_n = |x^0 - \overline{\gamma}(n)|, b_n = x^0$ *and* $\alpha_n = |x_0 - \underline{\gamma}(n)|, \beta_n = x_0$, *where* $-\infty < x_0 < x^0 < \infty$.
Type III: $a_n = \overline{g}(b_n), b_n = |\overline{\gamma}(n)|$ *and* $\alpha_n = \underline{g}(b_n), \beta_n = |\underline{\gamma}(n)|$, *where* $\overline{g}(t) = 1/(1-F(t)) \int_t^{x^0} (1-F(y))dy, t < x^0 \leq \infty$, *and* $\underline{g}(t) = 1/(F(t)) \int_{x_0}^t F(y)dy, -\infty \leq x_0 < t$.

Our aim in this paper is to derive the class of possible non-trivial and trivial limit df's of the suitably normalized generalized and dual generalized extremal quotient $q_n^*(m,k) = A_{n,m}^{-1}(q_n(m,k) - B_{n,m})$ and $q_{d,n}^*(m,k) = \tilde{A}_{n,m}^{-1}(q_{d,n}(m,k) - \tilde{B}_{n,m})$, respectively, where $A_{n,m}$, $\tilde{A}_{n,m} > 0, B_{n,m}, \tilde{B}_{n,m} \in \Re$, $q_n(m,k) = M_n(m,k)/L_n(m,k) = X(n,n,m,k)/X(1,n,m,k)$, $q_{d,n}(m,k) = M_{d,n}(m,k)/L_{d,n}(m,k) = X_d(n,n,m,k)/X_d(1,n,m,k)$ and the trivial convergence takes place, when one of the extremes outweighs the other (see de Haan [9]). Since, any result of dgos can be easily deduced from the corresponding result of gos (as Burkschat *et al.* [6], and Theorems 1.1, 1.2 have shown), the emphasis of our study will be mainly on gos.

2. The generalized extremal quotient (the case $m > -1$)

In this section and in the sequel the limit df's $\underline{H}_{i,\alpha}^{(m,k)}$ and $\overline{H}_{i,\beta}^{(m,k)}$($\underline{H}_{i,\beta}^{d(m,k)}$ and $\overline{H}_{i,\alpha}^{d(m,k)}$), $i = 1,2,3$, are considered as the limit df's of the minimum and maximum gos (dgos), respectively, e.g., in the sequel the limit df's $\underline{H}_{i,\alpha}^{(m,k)}$ and $\overline{H}_{i,\beta}^{(m,k)}, i = 1,2,3$, are defined in Theorems 1.1 and 1.2, with $r = 1$. The following two theorems fully characterize the possible non-trivial and trivial limit df's of $q_n^*(m,k)$.

Theorem 2.1. *(Non-trivial types).*
Part 1. *If* $F \in D(\underline{H}_{1,\alpha}^{(m,k)})$ *and* $F \in D(\overline{H}_{1,\beta}^{(m,k)})$, *then*

$$P(q_n^*(m,k) \leq q) \ \xrightarrow[n]{w} \ Q_{1,1,\beta,\alpha}^{(m,k)}(q) = \begin{cases} 1, & q \geq 0, \\ 1 - \int_0^\infty \Gamma_\ell\left((|q|y^{-1/\alpha})^{-(m+1)\beta}\right)e^{-y}dy, & q < 0. \end{cases}$$

Moreover, If $F \in D(\underline{H}_{1,\alpha}^{(m,k)})$, $F \in D(\overline{H}_{2,\beta}^{(m,k)})$ *and* $x^0 = 0$, *then*

$$P(q_n^*(m,k) \leq q) \ \xrightarrow[n]{w} \ Q_{2,1,\beta,\alpha}^{(m,k)}(q) = \begin{cases} 0, & q < 0, \\ 1 - \int_0^\infty \Gamma_\ell\left((|q|y^{-1/\alpha})^{(m+1)\beta}\right)e^{-y}dy, & q \geq 0. \end{cases}$$

Finally, if $F \in D(\underline{H}_{2,\alpha}^{(m,k)})$, $F \in D(\overline{H}_{1,\beta}^{(m,k)})$ *and* $x_0 = 0$, *then*

$$P(q_n^*(m,k) \leq q) \ \xrightarrow[n]{w} \ Q_{1,2,\beta,\alpha}^{(m,k)}(q) = \begin{cases} 0, & q < 0, \\ \int_0^\infty \Gamma_\ell\left((qy^{1/\alpha})^{-(m+1)\beta}\right)e^{-y}dy, & q \geq 0. \end{cases}$$

In the above three cases, we can take $A_{n,m} = a_{n,m}/\alpha_{n,m}$ *and* $B_{n,m} = 0$.
Part 2. *Let* $F \in D(\underline{H}_{2,\alpha}^{(m,k)})$ *and* $F \in D(\overline{H}_{2,\beta}^{(m,k)})$, *where* $x_0, x^0 \neq 0$ *and* $\alpha = \beta(m+1)$. *Then, the df of* $q_n^*(m,k)$ *converges weakly to a non-degenerate df if, and only if,*

$$\eta = \lim_{n\to\infty} \frac{a_{n,m}}{\alpha_{n,m}} \ \textit{exists, with } 0 \leq \eta \leq \infty. \tag{2.1}$$

The convergence is non-trivial if, and only if, $0<\eta<\infty$. *In this case we can take* $A_{n,m}=a_{n,m}/|\beta_{n,m}|$ *and* $B_{n,m}=b_{n,m}/\beta_{n,m}$. *The non-trivial types are*
Type I: $Q^{(m,k):1}_{2,2,\beta,\alpha}(q)=\int_{-\infty}^{\infty}(\underline{H}^{(m,k)}_{2,\alpha}(\eta|x_0/x^0|(q-z)))\,d(1-\overline{H}^{(m,k)}_{2,\beta}(-z))$, *if* $x^0<0$.
Type II: $1-Q^{(m,k):1}_{2,2,\beta,\alpha}(-q)$, *if* $x_0>0$.
Type III: $Q^{(m,k):3}_{2,2,\beta,\alpha}(q)=1-Q^{(m,k):2}_{2,2,\beta,\alpha}(-q)$, *if* $x_0<0<x^0$,
where $Q^{(m,k):2}_{2,2,\beta,\alpha}(q)=\int_{-\infty}^{\infty}(\underline{H}^{(m,k)}_{2,\alpha}(\eta\mid x_0/x^0\mid(q-z)))d\overline{H}^{(m,k)}_{2,\beta}(z)$.

Part 3. *Let* $F\in D(\underline{H}^{(m,k)}_{3,0})$ *and* $F\in D(\overline{H}^{(m,k)}_{3,0})$. *In order that* $q_n^*(m,k)$ *converges weakly to a non-degenerate df, it is necessary and sufficient that*

$$\zeta=\lim_{n\to\infty}a^{-1}_{n,m}\,\alpha_{n,m}\mid b_{n,m}\beta^{-1}_{n,m}\mid \tag{2.2}$$

exists, with $0\le\zeta\le\infty$. *The convergence is non-trivial if, and only if,* $0<\zeta<\infty$. *The limit type in this case is given by* $Q^{(m,k)}_3(q)=\underline{H}^{(m,k)}_3(\zeta^{-1}q)*\overline{H}^{(m,k)}_3(q)$, *where "$*$"denotes the convolution operator,*

$$\overline{H}^{(m,k)}_3(q)=\begin{cases}\overline{H}^{(m,k)}_{3,0}(q), & x_0\ge 0,\\ 1-\overline{H}^{(m,k)}_{3,0}(-q), & x_0<0,\end{cases}\quad\text{and }\underline{H}^{(m,k)}_3(q)=\begin{cases}\underline{H}^{(m,k)}_{3,0}(q), & x^0\le 0,\\ 1-\underline{H}^{(m,k)}_{3,0}(-q), & x^0>0.\end{cases}$$

Corollary 2.1. *Let* $F\in D(\underline{H}^{(m,k)}_{3,0})$, $F\in D(\overline{H}^{(m,k)}_{3,0})$ *and* $x^0,x_0\in(-\infty,\infty)|\{0\}$. *Then* $q_n^*(m,k)$ *converges weakly to a non-degenerate df if, and only if, the condition (2.1) is satisfied. In this case, we can take* $A_{n,m}=a_{n,m}/|\beta_{n,m}|$ *and* $B_{n,m}=b_{n,m}/\beta_{n,m}$. *Moreover, the non-trivial types are*
Type I: $Q^{(m,k):1}_3(q)=\int_{-\infty}^{\infty}(\underline{H}^{(m,k)}_{3,0}(\eta\mid x_0/x^0\mid(q-z)))d(1-\overline{H}^{(m,k)}_{3,0}(-z))$, *if* $x^0<0$.
Type II: $1-Q^{(m,k):1}_3(-q)$, *if* $x_0>0$.
Type III: $Q^{(m,k):3}_3(q)=1-Q^{(m,k):2}_3(-q)$, *if* $x_0<0<x^0$,
where $Q^{(m,k):2}_3(q)=\int_{-\infty}^{\infty}(\underline{H}^{(m,k)}_{3,0}(\eta\mid x_0/x^0\mid(q-z)))d(\overline{H}^{(m,k)}_{3,0}(z))$.

Theorem 2.2 (Trivial types)**.** ***Part 1.*** *If* $F\in D(\underline{H}^{(m,k)}_{3,0})$ *and* $F\in D(\overline{H}^{(m,k)}_{1,\beta})$, *then*

$$P(q_n^*(m,k)\le q)\ \xrightarrow[n]{w}\ Q^{(m,k)}_{1,3,\beta,0}(q)=\begin{cases}\overline{H}^{(m,k)}_{1,\beta}(q), & x_0>0,\\ 1-\overline{H}^{(m,k)}_{1,\beta}(-q), & x_0<0.\end{cases}$$

Moreover, if $F\in D(\underline{H}^{(m,k)}_{2,\alpha}),F\in D(\overline{H}^{(m,k)}_{1,\beta})$ *and* $x_0\neq 0$, *then the df of* $q_n^*(m,k)$ *converges weakly to the trivial df* $Q^{(m,k)}_{1,2,\beta,\alpha}(q)=Q^{(m,k)}_{1,3,\beta,0}(q)$. *Finally, the normalizing constants in the above two cases can be chosen as* $A_{n,m}=\frac{a_{n,m}}{|\beta_{n,m}|}$ *and* $B_{n,m}=0$.

Part 2. *If* $F\in D(\underline{H}^{(m,k)}_{1,\alpha})$ *and* $F\in D(\overline{H}^{(m,k)}_{3,0})$, *then*

$$P(q_n^*(m,k)\le q)\ \xrightarrow[n]{w}\ Q^{(m,k)}_{3,1,0,\alpha}(q)=\begin{cases}(1-e^{-q^{\alpha}})I_{[0,\infty)}(q), & x^0<0,\\ e^{-|q|^{\alpha}}I_{(-\infty,0)}(q)+I_{[0,\infty)}(q), & x^0>0.\end{cases}$$

Moreover, if $F\in D(\underline{H}^{(m,k)}_{1,\alpha})$, $F\in D(\overline{H}^{(m,k)}_{2,\beta})$ *and* $x^0\neq 0$, *then the df of* $q_n^*(m,k)$ *converges weakly to the trivial df* $Q^{(m,k)}_{2,1,\beta,\alpha}(q)=Q^{(m,k)}_{3,1,0,\alpha}(q)$. *In the above two cases we can take* $A_{n,m}=$

$\frac{|b_{n,m}|}{\alpha_{n,m}}$ *and* $B_{n,m}=0$.

Part 3. *Let* $F\in D(\underline{H}^{(m,k)}_{3,0})$ *and* $F\in D(\overline{H}^{(m,k)}_{2,\beta})$. *Then the df of* $q^*_n(m,k)$ *converges weakly to the trivial df*

$$Q^{(m,k)}_{2,3,\beta,0}(q)=\begin{cases}1-\overline{H}^{(m,k)}_{2,\beta}(-q), & \text{if } x^0=0, \text{ with } A_{n,m}=\frac{a_{n,m}}{|\beta_{n,m}|},\ B_{n,m}=0,\\ \underline{H}^{(m,k)}_{3,0}(q), & \text{if } x^0<0, \text{ with } A_{n,m}=\frac{|b_{n,m}|\alpha_{n,m}}{|\beta_{n,m}|^2},\ B_{n,m}=\frac{b_{n,m}}{\beta_{n,m}},\\ 1-\underline{H}^{(m,k)}_{3,0}(-q), & \text{if } x^0>0, \text{ with } A_{n,m}=\frac{|b_{n,m}|\alpha_{n,m}}{|\beta_{n,m}|^2},\ B_{n,m}=\frac{b_{n,m}}{\beta_{n,m}}.\end{cases}$$

Part 4. *Let* $F\in D(\underline{H}^{(m,k)}_{2,\alpha})$ *and* $F\in D(\overline{H}^{(m,k)}_{3,0})$. *Then the df of* $q^*_n(m,k)$ *converges weakly to the trivial df*

$$Q^{(m,k)}_{3,2,0,\alpha}(q)=\begin{cases}e^{-q^{-\alpha}}I_{[0,\infty)}(q), & \text{if } x_0=0, \text{ with } A_{n,m}=\frac{b_{n,m}}{\alpha_{n,m}},\ B_{n,m}=0,\\ 1-\overline{H}^{(m,k)}_{3,0}(-q), & \text{if } x_0<0, \text{ with } A_{n,m}=\frac{a_{n,m}}{|\beta_{n,m}|},\ B_{n,m}=\frac{b_{n,m}}{\beta_{n,m}},\\ \overline{H}^{(m,k)}_{3,0}(q), & \text{if } x_0>0, \text{ with } A_{n,m}=\frac{a_{n,m}}{|\beta_{n,m}|},\ B_{n,m}=\frac{b_{n,m}}{\beta_{n,m}}.\end{cases}$$

Part 5. *Let* $F\in D(\underline{H}^{(m,k)}_{2,\alpha})$ *and* $F\in D(\overline{H}^{(m,k)}_{2,\beta})$. *Then the df of* $q^*_n(m,k)$ *converges weakly to the trivial df*

$$Q^{(m,k)}_{2,2,\beta,\alpha}(q)=\begin{cases}e^{-q^{-\alpha}}I_{[0,\infty)}(q), & \text{if} x_0=0, \text{with} A_{n,m}=\frac{|b_{n,m}|}{\alpha_{n,m}}, B_{n,m}=0,\\ 1-\overline{H}^{(m,k)}_{2,\beta}(-q), & \text{if} x^0=0, \text{with} A_{n,m}=\frac{a_{n,m}}{|\beta_{n,m}|}, B_{n,m}=0,\\ \overline{H}^{(m,k)}_{2,\beta}(q), & \text{if} x_0>0, \beta(m+1)>\alpha, \text{with} A_{n,m}=\frac{a_{n,m}}{|\beta_{n,m}|}, B_{n,m}=\frac{b_{n,m}}{\beta_{n,m}},\\ 1-\overline{H}^{(m,k)}_{2,\beta}(-q), & \text{if} x_0<0, \beta(m+1)>\alpha, \text{with} A_{n,m}=\frac{a_{n,m}}{|\beta_{n,m}|}, B_{n,m}=\frac{b_{n,m}}{\beta_{n,m}},\\ \underline{H}^{(m,k)}_{2,\alpha}(q), & \text{if} x^0<0, \beta(m+1)<\alpha, \text{with} A_{n,m}=\frac{|b_{n,m}|\alpha_{n,m}}{|\beta_{n,m}|^2}, B_{n,m}=\frac{b_{n,m}}{\beta_{n,m}},\\ 1-\underline{H}^{(m,k)}_{2,\alpha}(-q), & \text{if} x^0>0, \beta(m+1)<\alpha, \text{with} A_{n,m}=\frac{|b_{n,m}|\alpha_{n,m}}{|\beta_{n,m}|^2}, B_{n,m}=\frac{b_{n,m}}{\beta_{n,m}}.\end{cases}$$

2.1. Proofs

The proof of the preceding two theorems depends on the following lemmas.

Lemma 2.1. *Let* $F\in D(\underline{H}^{(m,k)}_{3,0}), F\in D(\overline{H}^{(m,k)}_{3,0})$ *and* $q^*_n(m,k)$ *converges weakly to a non-degenerate df* Q, *then* $a^{-1}_{n,m}\alpha_{n,m}|b_{n,m}\beta^{-1}_{n,m}|\to\zeta$, *as* $n\to\infty, 0\le\zeta\le\infty$.

Proof. Suppose that the df of $q^\star_n(m,k)$ converges weakly to a non-degenerate limit df. Take $A_{n,m}=a_{n,m}/|\beta_{n,m}|$ and $B_{n,m}=b_{n,m}/\beta_{n,m}$. Then

$$q^\star_n(m,k)=\frac{M^\star_n(m,k)-(a^{-1}_{n,m}\alpha_{n,m}b_{n,m}\beta^{-1}_{n,m})\ L^\star_n(m,k)}{|\beta_{n,m}|^{-1}L_n(m,k)},$$

where $M^\star_n(m,k)=a^{-1}_{n,m}(M_n(m,k)-b_{n,m})$ and $L^\star_n(m,k)=\alpha^{-1}_{n,m}(L_n(m,k)-\beta_{n,m})$. On the other hand, we have $b_{n,m}\uparrow x^0$ and $\beta_{n,m}\downarrow x_0$, as $n\to\infty$ (c.f. Gnedenko [11]). Thus, on account of Lemma 3.3 in Barakat [5], $|\beta_{n,m}|^{-1}L_n(m,k)\ \xrightarrow[n]{p}\ 1$, if $x_0\ge0$, $|\beta_{n,m}|^{-1}L_n(m,k)\ \xrightarrow[n]{p}\ -1$, if $x_0<0$, where $\xrightarrow[n]{p}$ denotes the convergence in probability, as $n\to\infty$.

After some algebra, we get, for sufficiently large n, the following representation

$$q^\star_n(m,k)\ \overset{w}{\underset{n}{=}}\ S^0\Big(\frac{M_n(m,k)-b_{n,m}}{a_{n,m}}\Big)+S_0(a^{-1}_{n,m}\ \alpha_{n,m}\mid b_{n,m}\beta^{-1}_{n,m}\mid)\Big(\frac{L_n(m,k)-\beta_{n,m}}{\alpha_{n,m}}\Big),\qquad \text{(I)}$$

where "$X_n \overset{w}{\underset{n}{=}} Y_n$"means that the rv's X_n and Y_n have the same limit df,

$$S_0 = \begin{cases} -1, & x^0 > 0, \\ +1, & x^0 \le 0, \end{cases} \quad \text{and } S^0 = \begin{cases} 1, & x_0 \ge 0, \\ -1, & x_0 < 0. \end{cases}$$

Now, in view of Theorems 1.1 and 1.2, we have $F \in D(\underline{H}_{3,0}^{(0,1)})$ and $F \in D(\overline{H}_{3,0}^{(0,1)})$. Therefore, by replacing n in (α_n, β_n) and (a_n, b_n) respectively by $\phi(n)$ and $\psi(n)$, we can easily see that the remaining part of the proof is the same as the proof of Lemma 3.4 in Barakat [5]. ∎

Remark 2.1. In the proof of Lemma 2.1, if we take $A_{n,m} = |\, b_{n,m} \,| \alpha_{n,m}/\beta_{n,m}^2$ and $B_{n,m} = b_{n,m}/\beta_{n,m}$, we get the asymptotic relation

$$q_n^\star(m,k) \overset{w}{\underset{n}{=}} S^0(|\, b_{n,m}^{-1}\, \beta_{n,m} \,|\, a_{n,m}\alpha_{n,m}^{-1})(\frac{M_n(m,k) - b_{n,m}}{a_{n,m}}) + S_0(\frac{L_n(m,k) - \beta_{n,m}}{\alpha_{n,m}}). \qquad \text{(II)}$$

Remark 2.2. If $F \in D(\underline{H}_{2,\alpha}^{(m,k)})$, $F \in D(\overline{H}_{2,\beta}^{(m,k)})$ and $x_0, x^0 \neq 0$, the asymptotic representations (I) and (II) hold with $\beta_{n,m} = x_0$ and $b_{n,m} = x^0$.

Lemma 2.2. *For any $\varepsilon > 0$, as $n \to \infty$, we have*

(i) *If $F \in D(\overline{H}_{1,\beta}^{(m,k)})$, then $a_{n,m} n^{-(\beta(m+1))^{-1}+\varepsilon} \longrightarrow \infty$ and $a_{n,m} n^{-(\beta(m+1))^{-1}-\varepsilon} \longrightarrow 0$;*

(ii) *If $F \in D(\underline{H}_{1,\alpha}^{(m,k)})$, then $\alpha_{n,m} n^{-(\alpha)^{-1}+\varepsilon} \longrightarrow \infty$ and $\alpha_{n,m} n^{-(\alpha)^{-1}-\varepsilon} \longrightarrow 0$;*

(iii) *If $F \in D(\overline{H}_{2,\beta}^{(m,k)})$, then $a_{n,m} n^{(\beta(m+1))^{-1}+\varepsilon} \longrightarrow \infty$ and $a_{n,m} n^{(\beta(m+1))^{-1}-\varepsilon} \longrightarrow 0$;*

(iv) *If $F \in D(\underline{H}_{2,\alpha}^{(m,k)})$, then $\alpha_{n,m} n^{(\alpha)^{-1}+\varepsilon} \longrightarrow \infty$ and $\alpha_{n,m} n^{(\alpha)^{-1}-\varepsilon} \longrightarrow 0$;*

(v) *If $F \in D(\overline{H}_{3,0}^{(m,k)})$, then $a_{n,m} n^{+\varepsilon} \longrightarrow \infty$ and $a_{n,m} n^{-\varepsilon} \longrightarrow 0$;*

(vi) *If $F \in D(\underline{H}_{3,0}^{(m,k)})$, then $\alpha_{n,m} n^{+\varepsilon} \longrightarrow \infty$ and $\alpha_{n,m} n^{-\varepsilon} \longrightarrow 0$.*

Proof. In view of Theorems 1.1 and 1.2 and by replacing n in α_n and a_n respectively by $\phi(n)$ and $\psi(n)$, we can easily see that the proof of this lemma is exactly the same as the proof of Lemma 3.5 in Barakat [5]. ∎

Lemma 2.3. *For any $\varepsilon > 0$, as $n \to \infty$, we have*

(i) *If $F \in D(\overline{H}_{3,0}^{(m,k)})$, then $\overline{d}_{n,m} n^{+\varepsilon} \to \infty$ and $\overline{d}_{n,m} n^{-\varepsilon} \to 0$, where $\overline{d}_{n,m} = a_{n,m}/|b_{n,m}|$.*

(ii) *If $F \in D(\underline{H}_{3,0}^{(m,k)})$, then $\underline{d}_{n,m} n^{+\varepsilon} \to \infty$ and $\underline{d}_{n,m} n^{-\varepsilon} \to 0$, where $\underline{d}_{n,m} = \alpha_{n,m}/|\beta_{n,m}|$.*

Proof. Again since $F \in D(\overline{H}_{3,0}^{(m,k)})$ and $F \in D(\underline{H}_{3,0}^{(m,k)})$, then in view of Theorems 1.1, 1.2, we get $F \in D(\overline{H}_{3,0}^{(0,1)})$ and $F \in D(\underline{H}_{3,0}^{(0,1)})$. Therefore, by using Lemma 3.6 in Barakat [5], we get $\overline{d}_n n^{+\varepsilon} \to \infty$; $\overline{d}_n n^{-\varepsilon} \to 0$ and $\underline{d}_n n^{+\varepsilon} \to \infty$; $\underline{d}_n n^{-\varepsilon} \to 0$, where, $\overline{d}_n = a_n/|b_n|$ and $\underline{d}_n = \alpha_n/|\beta_n|$. Then by replacing $n^{1/(m+1)}$ and $n(m+1)$ instead of n we get the result. ∎

Proof of Theorem 2.1. The proof of Part 1 follows, after some algebra, by using Lemma 1.1, Lemma 3.1 in Barakat [5] and Khinchine's convergence to types theorem. To prove Part 2, we use Remark 2.2 with Lemma 2.1 and the result of de Haan [9, Theorem 2]). In this case, it is easy to show that $q_n^*(m,k)$ converges to a non-degenerate df if, and only if, (2.1) is satisfied with $0 < \eta < \infty$. Finally, to prove Part 3, we use Lemmas 2.1, 2.3 and the result of de Haan [9, Theorem 2]. It is easy in this case to show that $q_n^*(m,k)$ converges to a non-degenerate df if, and only if, (2.2) is satisfied, with $0 < \zeta < \infty$. ∎

Proof of Corollary 2.1. Under the assumption of the corollary, we have $\lim_{n\to\infty} a_{n,m}/\alpha_{n,m}=\eta$. Therefore, the proof follows by using the representation (I) and by applying the result of Theorem 2.1, Part 3. ∎

Proof of Theorem 2.2. Under the assumptions of Parts 1 and 2 and by using Lemma 1.1, Lemma 3.3 in Barakat [5], we get after some algebra the two representations

$$q_n^\star(m,k) \overset{w}{\underset{n}{=}} \begin{cases} (\frac{M_n(m,k)}{a_{n,m}}), & x_0>0,\\ -(\frac{M_n(m,k)}{a_{n,m}}), & x_0<0. \end{cases} \quad \text{and} \quad q_n^\star(m,k) \overset{w}{\underset{n}{=}} \begin{cases} (\frac{L_n(m,k)}{\alpha_{n,m}})^{-1}, & x^0>0,\\ -(\frac{L_n(m,k)}{\alpha_{n,m}})^{-1}, & x^0<0, \end{cases}$$

respectively. The proof of the first two parts follows by using the above two representations. The proof of Part 3 follows, by virtue of Lemmas 1.1, 2.2, 2.3 and the representation in (II), if $x^0\neq 0$, or by virtue of the relation $q_n^\star(m,k) \overset{w}{\underset{n}{=}} -(M_n(m,k)/a_{n,m})$, if $x^0=0$. The proof of Part 4 follows, by virtue of Lemmas 1.1, 2.2, 2.3 and the representation in (I), if $x_0\neq 0$, or by virtue the relation $q_n^\star(m,k) \overset{w}{\underset{n}{=}} (L_n(m,k)/\alpha_{n,m})^{-1}$, if $x_0=0$. Finally, the proof of Part 5 can be obtained after some algebra from the asymptotic representations (I) (or (II)) by choosing the normalizing constants, as stated in this part. ∎

The class of possible non-trivial and trivial limit df's of $q^*_{d,n}(m,k)$ can be obtained in a simple way as Theorems 2.1 and 2.2, e.g.,

Theorem 2.3 (Non-trivial types for $q^*_{d,n}(m,k)$)**. *Part 1.*** *Let $F\in D(\underline{H}^{d(m,k)}_{1,\beta})$ and $F\in D(\overline{H}^{d(m,k)}_{1,\alpha})$. Then the df of $P(q^*_{d,n}(m,k)\le q) \overset{w}{\underset{n}{\longrightarrow}} Q^{(m,k)}_{1,1,\alpha,\beta}(q)$, $\tilde{A}_{n,m}=\tilde{\alpha}_{n,m}/\tilde{a}_{n,m}$ and $\tilde{B}_{n,m}=0$. Moreover, if $F\in D(\underline{H}^{d(m,k)}_{1,\beta})$, $F\in D(\overline{H}^{d(m,k)}_{2,\alpha})$ and $x_0=0$, then $P(q^*_{d,n}(m,k)\le q) \overset{w}{\underset{n}{\longrightarrow}} Q^{(m,k)}_{2,1,\alpha,\beta}(q)$. Finally, if $F\in D(\underline{H}^{d(m,k)}_{2,\beta})$, $F\in D(\overline{H}^{d(m,k)}_{1,\alpha})$ and $x^0=0$. then $P(q^*_{d,n}(m,k)\le q) \overset{w}{\underset{n}{\longrightarrow}} Q^{(m,k)}_{1,2,\alpha,\beta}(q)$.*

In the above three cases, we can take $\tilde{A}_{n,m}=\tilde{\alpha}_{n,m}/\tilde{a}_{n,m}$ and $\tilde{B}_{n,m}=0$.

Part 2. *Let $F\in D(\underline{H}^{d(m,k)}_{2,\alpha})$, $F\in D(\overline{H}^{d(m,k)}_{2,\beta})$, where $x_0,x^0\neq 0$ and $\beta=\alpha(m+1)$. Then $q^*_{d,n}(m,k)$ converges weakly to a non-degenerate df, it is necessary and sufficient that $\tau=\lim_{n\longrightarrow\infty}\tilde{a}_{n,m}/\tilde{\alpha}_{n,m}$ exists, with $0\le\tau\le\infty$. The convergence is non-trivial if, and only if, $0<\tau<\infty$. Moreover, we can take $\tilde{A}_{n,m}=\tilde{a}_{n,m}|\tilde{\beta}_{n,m}|/|\tilde{b}_{n,m}|^2$ and $\tilde{B}_{n,m}=|\tilde{\beta}_{n,m}/\tilde{b}_{n,m}|$. The non-trivial types are*

Type I: $Q^{d(m,k):1}_{2,2,\alpha,\beta}(q)=\int_{-\infty}^{\infty}\overline{H}^{d(m,k)}_{2,\alpha}(\tau|x_0/x^0|(q-z))d(1-\underline{H}^{d(m,k)}_{2,\beta}(-z))$, if $x_0>0$.

Type II: $1-Q^{d(m,k):1}_{2,2,\alpha,\beta}(-q)$, if $x^0<0$.

Type III: $Q^{d(m,k):3}_{2,2,\alpha,\beta}(q)=1-Q^{d(m,k):2}_{2,2,\alpha,\beta}(-q)$, if $x_0<0<x^0$,

where $Q^{d(m,k):2}_{2,2,\alpha,\beta}(q)=\int_{-\infty}^{\infty}(\overline{H}^{d(m,k)}_{2,\alpha}(\tau|x_0/x^0|(q-z)))d\underline{H}^{d(m,k)}_{2,\beta}(z)$.

Part 3. *Let $F\in D(\underline{H}^{d(m,k)}_{3,0})$ and $F\in D(\overline{H}^{d(m,k)}_{3,0})$. In order that $q^*_{d,n}(m,k)$ converges weakly to a non-degenerate df, it is necessary and sufficient that $\xi=\lim_{n\downarrow\infty}\tilde{a}^{-1}_{n,m}\tilde{\alpha}_{n,m}|\tilde{b}_{n,m}\tilde{\beta}^{-1}_{n,m}|$ exists, with $0\le\xi\le\infty$. The convergence is non-trivial if, and only if, $0<\xi<\infty$. Moreover, the limit type in this case is given by $Q^{d(m,k)}_3(q)=\overline{H}^{d(m,k)}_3(\xi^{-1}q)*\underline{H}^{d(m,k)}_3(q)$, where*

$$\underline{H}^{d(m,k)}_3(q)=\begin{cases}\underline{H}^{d(m,k)}_{3,0}(q), & x^0\le 0,\\ 1-\underline{H}^{d(m,k)}_{3,0}(-q), & x^0>0,\end{cases} \quad \text{and } \overline{H}^{d(m,k)}_3(q)=\begin{cases}\overline{H}^{(m,k)}_{3,0}(q), & x_0\ge 0,\\ 1-\overline{H}^{d(m,k)}_{3,0}(-q), & x_0<0.\end{cases}$$

3. The generalized extremal quotient (the case $m = -1$, i.e., record values)

The upper (lower) record model can be obtained as a special case of gos (dgos) model by putting $m = -1$ and $k = 1$. In this section we consider the limit behavior of the statistic $q_n^*(-1,1) = C_n^{-1}(q_n(-1,1) - D_n)$, where $C_n > 0$ and D_n are suitable sequences of normalizing constants. Beside the definition of the record values based on the concept of the gos, the upper record values (or simply a record) can be defined as an observation X_j, such that $X_j > \max(X_1,...,X_{j-1})$. By convention X_1 is a record value. The indices at which record values occur are given by the rv's $T_n = \min\{j : j > T_{n-1},\ X_j > X_{j-1},\ n > 1\}$ and $T_1 = 1$. Thus, the record value sequence $\{R_n\}$ is then defined by $R_n = X_{T_n},\ n \geq 1$. Consequently, the record extremal quotient is defined by $q_n(-1,1) = R_n/R_1 = R_n/X_1$. Therefore, we expect that the limit df of $q_n^\star$ will be depend on the population df $F(x)$. The explicit form of the df of R_n is given by

$$P(R_n \leq x) = \begin{cases} 1 - \Gamma_n(H(x)), & \text{if } n > 1, \\ F(x), & \text{if } n = 1, \end{cases}$$

where $H(x) = -\log(1 - F(x))$ is the hazard function of the df F (see Arnold *et al.* [2]).

Resnick [15] showed that the possible limiting record value distributions of the suitably normalized record $R_n^\star = c_n^{-1}(R_n - d_n),\ c_n > 0,\ d_n \in \Re$, are

$$\overline{H}_{i,\beta}^{(-1,1)}(x) = \mathcal{N}(-\log(-\log \overline{H}_{i,\beta}^{(0,1)}(x))) = \mathcal{N}(-\log(V_{i,\beta}(x))),\ i = 1,2,3,$$

where $\mathcal{N}(.)$ is the standard normal distribution, $\overline{H}_{i,\beta}^{(0,1)}$ is an maximum value distribution and the functions $V_{i,\beta},\ i = 1,2,3$, are defined in (1.4). In this case we say that F is in the domain of record attract of $\overline{H}_{i,\beta}^{(-1,1)}$ and write $F \in D_R(\overline{H}_{i,\beta}^{(-1,1)})$. The following theorem due to Resnick [15] (see Arnold *et al.*, [2]) is a basic tool of our study in this section.

Theorem 3.1 (Duality Theorem). *If an associated df F_a is defined by $F_a = 1 - \exp(-\sqrt{H(x)})$ and $\Psi_F(n) = \inf\{y : F(y) > 1 - e^{-n}\} = F^{-1}(1 - e^{-n}) \to x^0$, as $n \to \infty$, then the following limit implications hold:*

(i) $F \in D_R(\overline{H}_{1,\alpha}^{(-1,1)})$ *if, and only if,* $F_a \in D(\overline{H}_{1,\alpha/2}^{(0,1)})$ *and in this case we may use as normalizing constants* $c_n = \Psi_F(n)$ *and* $d_n = 0$;

(ii) $F \in D_R(\overline{H}_{2,\alpha}^{(-1,1)})$ *if, and only if,* $F_a \in D(\overline{H}_{2,\alpha/2}^{(0,1)})$. *In this case* $F^{-1}(1) = x^0$ *is necessarily finite (see Lemma 1.1) and we may use as normalizing constants* $c_n = x^0 - \Psi_F(n)$ *and* $d_n = x^0$;

(iii) $F \in D_R(\overline{H}_{3,0}^{(-1,1)})$ *if, and only if,* $F_a \in D(\overline{H}_{3,0}^{(0,1)})$ *and in this case we may use as normalizing constants* $c_n = \Psi_F(n + \sqrt{n}) - \Psi_F(n)$ *and* $d_n = \Psi_F(n)$.

The following theorem fully characterizes the possible limit df's of $q_n^\star(-1,1)$.

Theorem 3.2. *Let $C_n > 0$ and D_n be suitable normalizing constants. Furthermore, let $q_n^\star(-1,1) = C_n^{-1}(q_n(-1,1) - D_n)$. Then, we have the following implications:*
(i) If $F \in D_R(\overline{H}_{1,\alpha}^{(-1,1)})$, then

$$P(q_n^\star(-1,1) \leq q) \ \underset{n}{\overset{w}{\to}} \ \begin{cases} F(0) + \int_0^\infty \mathcal{N}(\alpha \log qx))dF(x), & \text{if } q \geq 0, \\ F(0) - \int_{-\infty}^0 \mathcal{N}(\alpha \log qx))dF(x), & \text{if } q \leq 0, \end{cases}$$

with $C_n = c_n = \Psi_F(n)$ and $D_n = -1$.
(ii) If

(a) $F \in D_R(\overline{H}^{(-1,1)}_{2,\alpha})$, $x^0 > 0$ *or*

(b) $F \in D_R(\overline{H}^{(-1,1)}_{3,0})$, $0 < x^0 < \infty$ *or*

(c) $F \in D_R(\overline{H}^{(-1,1)}_{3,0})$, $x^0 = \infty$, $\frac{\Psi_F(n+\sqrt{n})}{\Psi_F(n)} \to 1$, *as* $n \to \infty$, *then*

$$P(q^\star_n(-1,1) \le q) \ \underset{n}{\overset{w}{\to}} \ P(W \le q+1),$$

where $W = 1/X_1$, *with* $C_n = d_n$ *and* $D_n = d_n$.

Proof. First, we notice that the condition $x^0 > 0$, in Part (ii), guarantees that the scale normalizing constant $C_n = d_n$ will be positive (at least for large n, namely, $d_n = x^0 > 0$, in Part (a) and $C_n = d_n = \Psi_F(n) \to x^0 > 0$, as $n \to \infty$, in Part (b)). Now, it is easy to check the validity of the representation

$$q^\star_n(-1,1) \ \underset{n}{\overset{w}{=}} \ \begin{cases} \frac{R^\star_n}{X_1}, & \text{if } C_n = c_n,\ D_n = d_n = 0, \\ \frac{c_n\ d_n^{-1} R^\star_n - (X_1 - 1)}{X_1}, & \text{if } C_n = d_n,\ D_n = d_n, \end{cases} \tag{3.1}$$

where $R^\star_n = c_n^{-1}(R_n - d_n)$. The implication (i) follows from the first part of (3.1), Theorem 3.1, Lemma 3.1 in Barakat [5] and from the independency between R_r and R_s, if $s - r \to \infty$, as $n \to \infty$ (see Barakat [4]). The implication (ii) follows from the second part of (3.1) and Theorem 3.1 (note that Theorem 3.1 implies $c_n\ d_n^{-1} \to 0$, as $n \to \infty$, in Parts (a) and (b), while the condition $(\Psi_F(n+\sqrt{n}))/\Psi_F(n) \to 1$, as $n \to \infty$, implies $c_n\ d_n^{-1} \to 0$, as $n \to \infty$, in Part (c)). ∎

Example 3.1. For the Weibull, $F_1(x) = P(X_1 \le x) = 1 - e^{-x^c}$, $x > 0$, and the Logistic $F_2(x) = P(X_2 \le x) = e^x/(1+e^x)$, $\forall x$, distributions, we can easily show that $\Psi_{F_1}(u) = u^{1/c}$ and $\Psi_{F_2}(u) = \log(e^u - 1)$, respectively. Therefore $(\Psi_{F_1}(n+\sqrt{n}))/\Psi_{F_1}(n) = (1 + 1/\sqrt{n})^{1/c} \to 1$, as $n \to \infty$, and $(\Psi_{F_2}(n+\sqrt{n}))/\Psi_{F_2}(n) = (\log(e^{n+\sqrt{n}} - 1))/(\log(e^n - 1)) \to 1$, as $n \to \infty$. Thus, for both distributions, we get $P(q^\star_n(-1,1) \le q) \ \underset{n}{\overset{w}{\to}} \ P(W_i \le q+1)$, where $W_i = 1/X_i$, $i = 1,2$.

4. Applications

In this section, within some illustrative examples, we show that the domains of attraction of the non-trivial types of the generalized extremal quotient are non-empty. Some of these examples individually express intersecting facts. In all the following examples, the normalizing constants can be found in Table 4.1.

Example 4.1 (Standard Cauchy Distribution)**.** It can be shown that $a_n^{-1}(M_n - b_n)$ and $\alpha_n^{-1}(L_n - \beta_n)$ weakly converge to $\overline{H}^{(0,1)}_{1,1}$ and $\underline{H}^{(0,1)}_{1,1}$, respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.1, Part 1, we get

$$P(q^*_n(m,k) \le q) \ \underset{n}{\overset{w}{\longrightarrow}} \ Q^{(m,k)}_{1,1,1,1}(q) = \begin{cases} 1, & q \ge 0, \\ 1 - \int_0^\infty \Gamma_\ell((\frac{y}{|q|})^{m+1}) e^{-y}\, dy, & q < 0. \end{cases}$$

Example 4.2 (Pareto Distribution)**.** It can be shown that, for the Pareto distribution $F(x) = (1 - x^{-\sigma}) I_{[1,\infty)}(x), \sigma > 0$, $a_n^{-1}(M_n - b_n)$ and $\alpha_n^{-1}(L_n - \beta_n)$ weakly converge to $\overline{H}^{(0,1)}_{1,\sigma}$ and $\underline{H}^{(0,1)}_{1,1}$, respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.1, Part 1, we get

$$P(q^*_n(m,k) \le q) \ \underset{n}{\overset{w}{\longrightarrow}} \ Q^{(m,k)}_{1,1,\sigma,1}(q) = \begin{cases} 1, & q \ge 0, \\ 1 - \int_0^\infty \Gamma_\ell((\frac{y}{|q|})^{(m+1)\sigma}) e^{-y}\, dy, & q < 0. \end{cases}$$

Example 4.3 (Uniform Distribution)**.** For the uniform $(-\theta,\theta)$, $(-\theta,0)$ and $(0,\theta)$ distributions, it can be shown that $a_n^{-1}(M_n-b_n)$ and $\alpha_n^{-1}(L_n-\beta_n)$ weakly converge to $\overline{H}_{2,1}^{(0,1)}$ and $\underline{H}_{2,1}^{(0,1)}$, respectively. Therefore, for the uniform $(-\theta,\theta)$ distribution, in view of Theorems 1.1, 1.2, we can easily show that

$$\eta=\lim_{n\to\infty}\frac{a_{n,m}}{\alpha_{n,m}}=\begin{cases}1, & \text{if } m=0,\\ \infty, & \text{if } m>0.\end{cases}$$

Thus, by using Theorem 2.1, Part 2, we get the non-trivial convergence $P(q_n^*(0,k)\le q)\xrightarrow[n]{w}Q_{2,2,1,1}^{(0,k):3}(q)=1-Q_{2,2,1,1}^{(0,k):2}(-q)$, where $Q_{2,2,1,1}^{(0,k):2}(q)=\int_{-\infty}^{\infty}(\underline{H}_{2,1}^{(0,k)}(q-z)d\overline{H}_{2,1}^{(0,k)}(z)$. Moreover, for the uniform $(-\theta,0)$ and $(0,\theta)$ distributions, by using Theorem 2.2, Part 5, the df of the statistic $q_n^*(m,k)$ weakly converges to the trivial types $1-\overline{H}_{2,1}^{(m,k)}(-q)$ and $e^{-q^{-1}}I_{[0,\infty)}(q)$, respectively.

Example 4.4 (Beta(α,β) Distribution)**.** For the beta distribution $F(x;\alpha,\beta), 0\le x\le 1, \alpha,\beta>0$, it can be shown that $a_n^{-1}(M_n-b_n)$ and $\alpha_n^{-1}(L_n-\beta_n)$, weakly converge to $\overline{H}_{2,\beta}^{(0,1)}$ and $\underline{H}_{2,\alpha}^{(0,1)}$, respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.2, Part 5 (i), we have the trivial convergence (since $x_0=0$) $P(q_n^*(m,k)\le q)\xrightarrow[n]{w}e^{-q^{-\alpha}}I_{[0,\infty)(q)}$. Clearly, the same result holds for the power distribution $F(x;\alpha,1)$.

Example 4.5 (Standard Normal, Logistic, Laplace and Log-Normal Distributions)**.** It is known that (see Ahsanullah and Nevzorov [1]), for the above four distributions, we have $a_n^{-1}(M_n-b_n)$ and $\alpha_n^{-1}(L_n-\beta_n)$ weakly converge to $\overline{H}_{3,0}^{(0,1)}$ and $\underline{H}_{3,0}^{(0,1)}$, respectively. Therefore, in view of Theorems 1.1 and 1.2, and after some algebra, we get
(4.1)

$$\zeta=\lim_{n\to\infty}a_{n,m}^{-1}\ \alpha_{n,m}\mid b_{n,m}\beta_{n,m}^{-1}\mid=\begin{cases}\frac{4+M\log M}{4M}, & \text{for the normal distribution,}\\ \frac{1}{M}, & \text{for the logistic and Laplace distributions,}\\ \frac{1}{\sqrt{M}}, & \text{for the log-normal distribution,}\end{cases}$$

where $M=m+1$. Thus, in view of Theorem 2.1, Part 3, we get $P(q_n^*(m,k)\le q)\xrightarrow[n]{w}(1-\underline{H}_3^{(m,k)}(-\zeta^{-1}q))*(1-\overline{H}_3^{(m,k)}(-q))$, where ζ is given by (4.1). This example reveals the following interesting facts:
1. The extremal quotient for the logistic and Laplace distributions weakly converges to the same limit df.
2. In the case of ordinary order statistics, i.e., $m=0,k=1$, the extremal quotient, for the normal, logistic, Laplace and log-normal distributions, weakly converges to the same limit df, as the limit df of the sample range, i.e., $\overline{H}_3^{(0,1)}(q)*\underline{H}_3^{(0,1)}(q)$.

Example 4.6 (Exponential(σ))**.** It can be shown that $a_n^{-1}(M_n-b_n)$ and $\alpha_n^{-1}(L_n-\beta_n)$ weakly converge to $\overline{H}_{3,0}^{(0,1)}$ and $\underline{H}_{2,1}^{(0,1)}$, respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.2, Part 4, we get the trivial convergence (since $x_0=0$) $P(q_n^*(m,k)\le q)\xrightarrow[n]{w}e^{-q^{-1}}I_{[0,\infty)(q)}$.

Example 4.7 (Rayleigh(σ))**.** For the Rayleigh distribution $F(x)=(1-e^{-x^2/\sigma^2})I_{[0,\infty)}(x)$, $\sigma>0$, it can be shown that $a_n^{-1}(M_n-b_n)$ and $\alpha_n^{-1}(L_n-\beta_n)$ weakly converge to $\overline{H}_{3,0}^{(0,1)}$ and

$\underline{H}_{2,2}^{(0,1)}$, respectively. Therefore, in view of Theorems 1.1, 1.2 and 2.2, Part 4, we get the trivial convergence (since $x_0 = 0$) $P(q_n^*(m,k) \le q) \xrightarrow[n]{w} e^{-q^{-2}} I_{[0,\infty)(q)}$.

Example 4.8 (The sos Model)**.** Consider a sos $X(r,n,1,1)$ (in this case $m = k = 1, \gamma_i = 1+2(n-i), i \in \{1,2,...,n-1\}, \gamma_n = k = 1$ and $\ell = 1/2$), with $\alpha_i = 2 - 1/(n-i+1), i \in \{1,2,...,n-1\}$. Thus, Theorems 2.1, Part 1, and 3.1, Part 1 yield, the implications:
If $F \in D(\underline{H}_{1,\alpha}^{(1,1)}), D(\overline{H}_{1,\beta}^{(1,1)})$, and $F \in D(\underline{H}_{1,\beta}^{d(1,1)}), D(\overline{H}_{1,\alpha}^{(1,1)})$, then

$$P(q_n^*(1,1) \le q) \xrightarrow[n]{w} Q_{1,1,\beta,\alpha}^{(1,1)}(q) = \begin{cases} 1, & q \ge 0, \\ -1 + 2\int_0^\infty \mathcal{N}(\sqrt{2}\,|q|^{-\beta} y^{\beta/\alpha})\, e^{-y} dy, & q < 0, \end{cases}$$

and

$$P(q_{d,n}^*(1,1) \le q) \xrightarrow[n]{w} Q_{1,1,\beta,\alpha}^{d(1,1)}(q) = \begin{cases} 1, & q \ge 0, \\ -1 + 2\int_0^\infty \mathcal{N}(\sqrt{2}|q|^{-\alpha} y^{\alpha/\beta}) e^{-y} dy, & q < 0, \end{cases}$$

respectively. Moreover, if $x^0 = 0$, $F \in D(\underline{H}_{1,\alpha}^{(1,1)}), D(\overline{H}_{2,\beta}^{(1,1)})$ and $F \in D(\underline{H}_{1,\beta}^{d(1,1)}), D(\overline{H}_{2,\alpha}^{d(1,1)})$, $x^0 = 0$, then

$$P(q_n^*(1,1) \le q) \xrightarrow[n]{w} Q_{2,1,\beta,\alpha}^{(1,1)}(q) = \begin{cases} 0, & q < 0, \\ -1 + 2\int_0^\infty \mathcal{N}(\sqrt{2}|q|^{\beta} y^{-\beta/\alpha}) e^{-y} dy, & q \ge 0, \end{cases}$$

and

$$P(q_{d,n}^*(1,1) \le q) \xrightarrow[n]{w} Q_{1,2,\beta,\alpha}^{d(1,1)}(q) = \begin{cases} 0, & q < 0, \\ -1 + 2\int_0^\infty \mathcal{N}\,(\sqrt{2}\,|q|^{\alpha}\, y^{-\alpha/\beta})\, e^{-y}\, dy\,, & q \ge 0, \end{cases}$$

respectively. Finally, if $x_0 = 0$, $F \in D(\underline{H}_{2,\alpha}^{(1,1)}), D(\overline{H}_{1,\beta}^{(1,1)})$ and $F \in D(\underline{H}_{2,\beta}^{d(1,1)}), D(\overline{H}_{1,\alpha}^{d(1,1)})$, $x_0 = 0$, then

$$P(q_n^*(1,1) \le q) \xrightarrow[n]{w} Q_{1,2,\beta,\alpha}^{(1,1)}(q) = \begin{cases} 0, & q < 0, \\ 2 - 2\int_0^\infty \mathcal{N}(\sqrt{2}\,|q|^{-\beta} y^{-\beta/\alpha}) e^{-y} dy, & q \ge 0, \end{cases}$$

and

$$P(q_{d,n}^*(1,1) \le q) \xrightarrow[n]{w} Q_{2,1,\beta,\alpha}^{d(1,1)}(q) = \begin{cases} 0, & q < 0, \\ 2 - 2\int_0^\infty \mathcal{N}(\sqrt{2}|q|^{-\alpha} y^{-\alpha/\beta}) e^{-y} dy, & q \ge 0, \end{cases}$$

respectively.

Table 1. The normalizing constants of some popular distributions.

Distribution	$(a_n,\, b_n)$	$(\alpha_n,\, \beta_n)$	$(A_{n,m},\, B_{n,m})$
Standard Cauchy	$(n/\pi,\, 0)$	$(\cot \pi/n,\, 0)$	$(a_{n,m}/b_{n,m},\, 0)$
Pareto	$\left(n^{1/\sigma},\, 0\right)$	$\left((n/(n+1))^{1/\sigma},\, 0\right)$	$(a_{n,m}/b_{n,m},\, 0)$
Beta(α,β)	$\left((\beta/(cn))^{1/\beta},\, 1\right)$	$\left(((c\alpha)/n)^{1/\alpha},\, 0\right)$	$(\lvert b_{n,m}\rvert/a_{n,m},\, 0)$
Power(α)	$\left(n^{-1/\alpha},\, 1\right)$	$(1/(\alpha n),\, 0)$	$(\lvert b_{n,m}\rvert/a_{n,m},\, 0)$
Standard Normal	$(\tau_n,\, \rho_n)$ $\tau_n = 1/\sqrt{2\log n}$, $\rho_n = 1/\tau_n - (\tau_n D_n)/2$, $D_n = \log\log n + \log 4\pi$	$(\tau_n,\, -\rho_n)$	$(a_{n,m}/\lvert\beta_{n,m}\rvert,\, b_{n,m}/\beta_{n,m})$
Logistic	$(1,\, \log n)$	$(1,\, -\log n)$	$(a_{n,m}/\lvert\beta_{n,m}\rvert,\, b_{n,m}/\beta_{n,m})$
Laplace	$(1,\, \log n/2)$	$(1,\, -\log n/2)$	$(a_{n,m}/\lvert\beta_{n,m}\rvert,\, b_{n,m}/\beta_{n,m})$
log-Normal	$(\tau_n e^{\rho_n},\, e^{\rho_n})$	$\left(\tau_n e^{-\rho_n},\, e^{-\rho_n}\right)$	$(a_{n,m}/\lvert\beta_{n,m}\rvert,\, b_{n,m}/\beta_{n,m})$
Exponential(σ)	$(1/\sigma,\, 1/\sigma \log n)$	$(1/n\sigma,\, 0)$	$(b_{n,m}/a_{n,m},\, 0)$
Rayleigh(σ)	$\left(\sigma/2(\log n)^{-1/2},\, \sigma(\log n)^{1/2}\right)$	$(\sigma/\sqrt{n},\, 0)$	$(b_{n,m}/a_{n,m},\, 0)$
Uniform $(-\theta,\theta)$	$(2\theta/n,\, \theta)$	$(2\theta/n,\, -\theta)$	$(a_{n,m}/\lvert\beta_{n,m}\rvert,\, b_{n,m}/\beta_{n,m})$
Uniform $(0,\theta)$	$(\theta/n,\, \theta)$	$(\theta/n,\, 0)$	$(\lvert b_{n,m}\rvert/\alpha_{n,m},\, 0)$
Uniform $(-\theta,0)$	$(\theta/n,\, 0)$	$(\theta/n,\, -\theta)$	$(a_{n,m}/\lvert\beta_{n,m}\rvert,\, 0)$

Acknowledgement. The authors would like to thank the anonymous referees for constructive suggestions leading to improvements in the presentation of the paper.

References

[1] M. Ahsanullah and V. B. Nevzorov, *Ordered Random Variables*, Nova Sci. Publ., Huntington, NY, 2001.

[2] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja, *Records*, Wiley Series in Probability and Statistics: Probability and Statistics, Wiley, New York, 1998.

[3] H. M. Barakat, Limit theory of generalized order statistics, *J. Statist. Plann. Inference* **137** (2007), no. 1, 1–11.

[4] H. M. Barakat, Measuring the asymptotic dependence between generalized order statistics, *J. Stat. Theory Appl.* **6** (2007), no. 2, 106–117.

[5] H. M. Barakat, Weak limit of the sample extremal quotient, *Aust. N. Z. J. Stat.* **40** (1998), no. 1, 83–93.

[6] M. Burkschat, E. Cramer and U. Kamps, Dual generalized order statistics, *Metron* **61** (2003), no. 1, 13–26.

[7] V. A. Canard, *Methods in Climatology*, Cambridge, Mass.: Harvard University, 1946.

[8] E. Cramer and U. Kamps, Sequential *k*-out-of-*n* systems, in *Advances in Reliability*, 301–372, Handbook of Statist., 20 North-Holland, Amsterdam, 2001.

[9] L. de Haan, Weak limits of sample range, *J. Appl. Probability* **11** (1974), 836–841.

[10] J. Galambos and I. Simonelli, *Products of Random Variables*, Monographs and Textbooks in Pure and Applied Mathematics, 268, Dekker, New York, 2004.

[11] B. Gnedenko, Sur la distribution limite du terme maximum d'une série aléatoire, *Ann. of Math. (2)* **44** (1943), 423–453.

[12] U. Kamps, *A Concept of Generalized Order Statistics*, Teubner Skripten zur Mathematischen Stochastik. Teubner, Stuttgart, 1995.

[13] D. Nasri-Roudsari, Extreme value theory of generalized order statistics, *J. Statist. Plann. Inference* **55** (1996), no. 3, 281–297.

[14] D. Nasri-Roudsari and E. Cramer, On the convergence rates of extreme generalized order statistics, *Extremes* **2** (1999), no. 4, 421–447 (2000).

[15] S. I. Resnick, Limit laws for record values, *Stochastic Processes Appl.* **1** (1973), 67–82.

[16] P. G. Wong and S. P. Wong, A curtailed test for the shape parameter of the Weibull distribution, *Metrika* **29** (1982), no. 3, 203–209.
[17] P. G. Wong and S. P. Wong, An extremal quotient test for exponential distributions, *Metrika* **26** (1979), 1–4.