BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36***(4) (2013), 855–864*

Spectral Synthesis for the Operator Space Projective Tensor Product of C^*-Algebras

[1]RANJANA JAIN AND [2]AJAY KUMAR
[1]Department of Mathematics, Lady Shri Ram College for Women, New Delhi-110024, India
[2]Department of Mathematics, University of Delhi, Delhi-110007, India
[1]ranjanaj_81@rediffmail.com, [2]akumar@maths.du.ac.in

Abstract. We study the spectral synthesis for the Banach $*$-algebra $A\widehat{\otimes}B$, the operator space projective tensor product of C^*-algebras A and B. It is shown that if A or B has finitely many closed ideals, then $A\widehat{\otimes}B$ obeys spectral synthesis. The Banach algebra $A\widehat{\otimes}A$ with the reverse involution is also studied.

2010 Mathematics Subject Classification: 46L06, 46L07, 47L25, 43A45

Keywords and phrases: C^*-algebras, operator space projective tensor norm, spectral synthesis, hull-kernel topology.

1. Introduction and notations

For operator spaces V and W, and $u \in V \otimes W$, the *operator space projective tensor norm* is defined as

$$\|u\|_\wedge = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v\otimes w)\beta\},$$

where $\alpha \in \mathbb{M}_{1,pq}, \beta \in \mathbb{M}_{pq,1}, v \in M_p(V)$ and $w \in M_q(W)$, $p,q \in \mathbb{N}$ being arbitrary, and $v \otimes w = (v_{ij} \otimes w_{kl})_{(i,k),(j,l)} \in M_{pq}(V \otimes W)$. The *operator space projective tensor product* $V\widehat{\otimes}W$ is the completion of $V \otimes W$ under $\|\cdot\|_\wedge$-norm. The algebraic tensor product $V \otimes W$ is complete with respect to $\|\cdot\|_\wedge$-norm if and only if either V or W is finite dimensional. Also, it is known that for C^*-algebras A and B, $A\widehat{\otimes}B$ is a Banach $*$-algebra under natural involution [14].

The notion of spectral synthesis has been studied extensively for commutative and unital Banach algebras, for L^1-group algebras and for Banach $*$-algebras [20, 6, 7, 13]. Spectral synthesis for Banach space projective tensor product of commutative Banach algebras and for the Haagerup tensor product of C^*-algebras has also been explored [13, 8, 1, 7]. Roughly speaking spectral synthesis holds for a Banach $*$-algebra X if every closed ideal of X is the intersection of primitive ideals containing it. Spectral synthesis for Banach space projective tensor product of commutative Banach algebras has already been explored [13]. For commutative C^*-algebras A and B, the natural contractive homomorphism of $A\widehat{\otimes}B$ into $A \otimes^h B$ is an isomorphism whose inverse has norm equal to Grothendieck constant. Thus,

Communicated by V. Ravichandran.
Received: July 26, 2011; *Revised:* January 2, 2012.

for countable locally compact Hausdorff spaces X and Y, $C_0(X)\widehat{\otimes}C_0(Y)$ has spectral synthesis. However, for Cantor set or any infinite compact group D, $C(D)\widehat{\otimes}C(D)$ does not have spectral synthesis [8, 11.2.1], [13].

In Section 2, we define the concept of spectral ideals in $A\widehat{\otimes}B$, and prove that the Banach $*$- algebra $A\widehat{\otimes}B$ satisfies spectral synthesis if and only if each closed ideal of $A\widehat{\otimes}B$ is spectral. This result is then used to produce plenty of spectral ideals in $A\widehat{\otimes}B$. We also discuss few cases where $A\widehat{\otimes}B$ obeys spectral synthesis. In particular, we prove that if A or B has finitely many closed ideals, then $A\widehat{\otimes}B$ has spectral synthesis. Thus, the Banach $*$-algebras like $C_0(X)\widehat{\otimes}\mathscr{B}(H)$, $\mathscr{B}(H)\widehat{\otimes}\mathscr{K}(H)$ and $\mathscr{B}(H)\widehat{\otimes}\mathscr{B}(H)$ all obey spectral synthesis, X being a locally compact topological space and H being an infinite dimensional separable Hilbert space. In Section 3, the algebra $A\widehat{\otimes}A$ with the reverse involution is discussed. It is shown that with this involution the algebra is symmetric and $*$-semisimple only in the trivial cases.

For a Banach algebra X, we denote the set of closed (two-sided) ideals of X by $\mathrm{Id}(X)$, the set of proper closed ideals of X by $\mathrm{Id}'(X)$ and the set of all prime ideals by $\mathrm{Prime}(X)$. If X is a Banach $*$-algebra, then $\mathrm{Prim}(X)$ stands for the set of primitive ideals of X, that is, the set of all kernels of irreducible $*$-representations of X on Hilbert space. There is a topology τ_w on $\mathrm{Id}(X)$ which is generated by the sub-basic open sets of the form

$$Z_J := \{I \in \mathrm{Id}(X) : I \not\supseteq J\},\ J \in \mathrm{Id}(X).$$

We throughout use the notation q_J for the quotient map $q_J : A \to A/J$. Recall that, for closed ideals M and N of C^*-algebras A and B, the map $q_M \otimes q_N : A \otimes B \to A/M \otimes B/N$ extends to quotient maps $q_M\widehat{\otimes}q_N : A\widehat{\otimes}B \to A/M\widehat{\otimes}B/N$ and $q_M \otimes^{\min} q_N : A \otimes^{\min} B \to A/M \otimes^{\min} B/N$.

Let A and B be C^*-algebras. Define a map $\Phi : \mathrm{Id}(A)\times\mathrm{Id}(B) \to \mathrm{Id}(A\widehat{\otimes}B)$ as

$$\Phi(M,N) = A\widehat{\otimes}N + M\widehat{\otimes}B.$$

The map Φ is well defined by [12, Proposition 3.2]. It satisfies many nice topological properties listed as below:

Proposition 1.1. *Let A and B be C^*-algebras and $\Phi : \mathrm{Id}(A)\times\mathrm{Id}(B) \to \mathrm{Id}(A\widehat{\otimes}B)$ be defined as above. Then*

(i) Φ *maps* $\mathrm{Prime}(A)\times\mathrm{Prime}(B)$ *onto* $\mathrm{Prime}(A\widehat{\otimes}B)$.
(ii) Φ *maps* $\mathrm{Prim}(A)\times\mathrm{Prim}(B)$ *into* $\mathrm{Prim}(A\widehat{\otimes}B)$. *If A and B are separable, then* Φ *maps* $\mathrm{Prim}(A)\times\mathrm{Prim}(B)$ *onto* $\mathrm{Prim}(A\widehat{\otimes}B)$.
(iii) Φ *maps* $\mathrm{Id}'(A)\times\mathrm{Id}'(B)$ *into* $\mathrm{Id}'(A\widehat{\otimes}B)$ *injectively.*
(iv) *The mapping Φ is τ_w-continuous.*
(v) *The restriction of Φ to* $\mathrm{Id}'(A)\times\mathrm{Id}'(B)$ *is a homeomorphism onto its image in* $\mathrm{Id}'(A\widehat{\otimes}B)$.
(vi) *The restriction of Φ to* $\mathrm{Prime}(A)\times\mathrm{Prime}(B)$ *is a homeomorphism onto* $\mathrm{Prime}(A\widehat{\otimes}B)$.

Proof. (i) and (ii) follow from Theorems 3.1 and 3.2 of [11], respectively.

For (iii), note that, for proper closed ideals M and N of A and B, the isomorphism of $A/M\widehat{\otimes}B/N$ onto $(A\widehat{\otimes}B)/(A\widehat{\otimes}N + M\widehat{\otimes}B)$ [11, Lemma 2.2] assures that $A\widehat{\otimes}N + M\widehat{\otimes}B$ is also proper in $A\widehat{\otimes}B$. Further, for $M_1, M_2 \in \mathrm{Id}'(A), N_1, N_2 \in \mathrm{Id}'(B)$, $A\widehat{\otimes}N_1 + M_1\widehat{\otimes}B \subseteq A\widehat{\otimes}N_2 + M_2\widehat{\otimes}B$ if and only if $M_1 \subseteq M_2, N_1 \subseteq N_2$. To see this, consider an $m \in M_1$, so that for an arbitrary $b \in B$, $m \otimes b \in \ker(q_{M_1}\widehat{\otimes}q_{N_1}) \subseteq \ker(q_{M_2}\widehat{\otimes}q_{N_2})$, giving $q_{M_2}(m) = 0$, that is $m \in M_2$, and similarly $N_1 \subseteq N_2$. Thus, Φ is injective.

(iv)–(vi) can be proved exactly on the same lines of their counterparts in Haagerup tensor product as discussed in Lemma 1.4 and Theorem 1.5 of [2]. ∎

Throughout this paper A and B represent C^*-algebras, until otherwise specified.

2. Spectral synthesis

We first give the standard definition of spectral synthesis for a Banach $*$-algebra that appear in the literature. Let X be a Banach $*$-algebra. For each $E \subseteq \mathrm{Prim}(X)$, we define a closed ideal *kernel* of E as

$$k(E) := \cap_{P \in E} P.$$

Also, for each $M \subseteq X$, *hull* of M is defined as

$$h_X(M) := \{P \in \mathrm{Prim}(X) : P \supseteq M\}.$$

We shall denote the hull of M by $h(M)$, when there is no confusion with X. Equip $\mathrm{Prim}(X)$ with the *hull-kernel topology* (or, hk-topology), where for every $E \subseteq \mathrm{Prim}(X)$, its closure is $\overline{E} = h(k(E))$. Similarly, one can talk about the hk-topology on $\mathrm{Prime}(X)$. Note that, if $E \subseteq \mathrm{Prime}(X)$, then the relative τ_w-topology on E coincides with the hull-kernel topology.

Definition 2.1. *A closed subset E of* $\mathrm{Prim}(X)$ *is called spectral if $k(E)$ is the only closed ideal in X with hull equal to E. A Banach $*$-algebra X is said to satisfy spectral synthesis if every closed subset of* $\mathrm{Prim}(X)$ *is spectral.*

A closed ideal of Banach $*$-algebra X is said to be *semisimple* if it is the intersection of all the primitive ideals of X containing it. Recall that a Banach $*$-algebra is said to have *Wiener property* if every proper closed two-sided ideal of X is annihilated by an irreducible $*$-representation [17].

Proposition 2.1. *Let X be a Banach $*$-algebra having Wiener property. Then X satisfies spectral synthesis if and only if for every $J \in \mathrm{Id}(X)$, $J = k(h(J))$, or, in other words, every closed ideal of X is semisimple.*

Proof. Let us consider a proper closed ideal J of X. Since X has Wiener property, there exists an irreducible $*$-representation, say π, of X which annihilates J, that is, $J \subseteq \ker \pi$, so that $E = h(J)$ is non empty. We claim that E is closed in the hk-topology. Let $Q \in \overline{E} = h(k(E))$, then $k(E) \subseteq Q$. Since $J \subseteq P$ for all $P \in E$ we have $J \subseteq k(E) \subseteq Q$, so that $Q \in E$. which gives that E is closed. Since X obeys spectral synthesis, and $E = h(J)$, we have $J = k(E)$, that is, J is the intersection of primitive ideals containing it. Also, note that since X has Wiener property, the empty set ϕ is spectral, so that $X = k(h(X))$.

Converse follows easily from the given condition. ∎

Corollary 2.1. *Let X be a Banach $*$-algebra having Wiener property. Then X satisfies spectral synthesis if and only if there is a one-one correspondence between the closed ideals of X and the τ_w-open subsets of* $\mathrm{Prim}(X)$ *(or,* $\mathrm{Prime}(X)$*).*

Proof. Let X satisfy spectral synthesis. For $J \in \mathrm{Id}(X)$, recall $Z_J := \{P \in \mathrm{Prim}(X) : P \not\supseteq J\} = \mathrm{Prim}(X) \setminus h(J)$ is an open subset of $\mathrm{Prim}(X)$ under the relative τ_w-topology, so that we have a well defined correspondence $J \mapsto Z_J$ between the closed ideals of X and τ_w-open subsets of $\mathrm{Prim}(X)$. For $K, L \in \mathrm{Id}(X)$, it is clear from Proposition 2.1 that $K = k(h(K))$, and $L = k(h(L))$. Thus, it can be easily seen that

$$K \subseteq L \quad \text{if and only if} \quad Z_K \subseteq Z_L,$$

which shows that the correspondence in one-one. Now consider a τ_w-open subset G of $\mathrm{Prim}(X)$, and set $J := k(\mathrm{Prim}(X) \setminus G)$. Since $\mathrm{Prim}(X) \setminus G$ is closed under the hull-kernel topology,

$$Z_J = \mathrm{Prim}(X) \setminus h(k(\mathrm{Prim}(X) \setminus G)) = \mathrm{Prim}(X) \setminus (\mathrm{Prim}(X) \setminus G) = G,$$

which proves that this correspondence is surjective.

Conversely, for every closed ideal I of X, since $h(I) = h(k(h(I)))$, we have $Z_I = Z_{k(h(I))}$. Using the given condition, this gives $I = k(h(I))$. Result now follows from Proposition 2.1. ∎

Remark 2.1. For C^*-algebras A and B, since $A\widehat{\otimes}B$ has Wiener property [12, Theorem 4.1], $A\widehat{\otimes}B$ satisfies spectral synthesis if and only if every closed ideal J of $A\widehat{\otimes}B$ is semisimple. In particular, if $A\widehat{\otimes}B$ satisfies spectral synthesis then every closed ideal J of $A\widehat{\otimes}B$ is the intersection of prime ideals containing J.

The next two results connect the spectral synthesis of a Banach $*$-algebra with that of its ideal and the corresponding quotient algebra. The first result follows on the similar lines as that in [7, Proposition 1.16]. However, we present here a proof for the sake of completion.

Proposition 2.2. *Let X be a Banach $*$-algebra with Wiener property, and J be a proper closed $*$-ideal of X having bounded approximate identity and Wiener property. If J and X/J both satisfy spectral synthesis (as Banach $*$-algebras), then X satisfies spectral synthesis.*

Proof. By Corollary 2.1, it is sufficient to show that for $I, K \in \mathrm{Id}(X)$, $I = K$, whenever $h_X(I) = h_X(K)$. Note that, since X has Wiener property, X/J also has Wiener property, so by Proposition 2.1, every closed ideal of J and X/J is semisimple. For $P \in h_{X/J}(q_J(I))$, $P = \ker\pi$ with $\pi(q_J(I)) = \{0\}$, $\pi : X/J \to \mathscr{B}(H)$ being an irreducible $*$-representation. Then $\pi_0 := \pi \circ q_J$ is an irreducible $*$-representation of X on H with $\pi_0(I) = 0$. Since $h_X(I) = h_X(K)$, $\ker\pi_0 \in h_X(K)$, which further gives $P \in h_{X/J}(q_J(K))$. Thus, $h_{X/J}(q_J(I)) = h_{X/J}(q_J(K))$. Since J has an approximate identity, by [5, Proposition 2.4], $I + J$ and $K + J$ are closed in X, so that $q_J(I)$ and $q_J(K)$ are closed ideals of X/J. Since X/J obeys spectral synthesis, by Proposition 2.1, $q_J(I) = q_J(K)$. Further, for any closed ideal L of J, it is routine to check that there is a one-one correspondence between the sets $\{P \in h_X(L) : J \nsubseteq P\}$ and $h_J(L)$ via $P \mapsto P \cap J$. Xlso, $h_X(I \cap J) = h_X(I) \cup h_X(J) = h_X(K) \cup h_X(J) = h_X(K \cap J)$. Thus, it can be easily seen that $h_J(I \cap J) = h_J(K \cap J)$. Since J satisfy spectral synthesis, this gives, $I \cap J = K \cap J$.

Now, consider $x \in I$, then $q_J(x) = q_J(y)$ for some $y \in K$, so that $a := x - y \in J$. Let J_a be the smallest closed ideal of J containing a. Since J obeys spectral synthesis, $J_a = \cap\{P \in \mathrm{Prim}(J) : J_a \subseteq P\}$. Clearly $\overline{JaJ} \subseteq J_a$. Now consider $P \in \mathrm{Prim}(J)$ such that $JaJ \subseteq P$. Since P is prime being primitive, this gives $a \in P$ which shows that $J_a \subseteq P$. Thus $\overline{JaJ} = \cap\{P \in \mathrm{Prim}(J) : JaJ \subseteq P\} = \cap\{P \in \mathrm{Prim}(J) : J_a \subseteq P\} = J_a$. So

$$x - y \in \overline{J(x-y)J} \subseteq \overline{JxJ - JyJ} \subseteq \overline{JIJ - JKJ} \subseteq \overline{I \cap J - K \cap J} = K \cap J.$$

So, $x = y - (y - x) \in K + (K \cap J) = K$, which gives $I \subseteq K$. Similarly, $K \subseteq I$, which proves the claim. ∎

In fact, the converse of the above statement is also true as presented below.

Proposition 2.3. *Let X be a Banach $*$-algebra with a closed $*$-ideal J such that X and J both possess Wiener property. If X obeys spectral synthesis, then so does J and X/J.*

Proof. By Proposition 2.1, it is enough to check that for a closed ideal L of J, $L = k(h_J(L))$. Since every closed ideal of X is semisimple, and every primitive ideal is prime, from [7, Proposition 1.14], L is also a closed ideal of X, so that by Proposition 2.1, $L = k(h_X(L))$. It can be easily verified that there is a one-one correspondence between the sets $\{P \in h_X(L) : J \nsubseteq P\}$ and $h_J(L)$ via $P \mapsto P \cap J$. So, we have

$$L = L \cap J = \bigcap_{P \in h_X(L)} (P \cap J) = \Big(\bigcap_{\substack{P \in h_X(L) \\ J \nsubseteq P}} (P \cap J) \Big) \cap \Big(\bigcap_{\substack{P \in h_X(L) \\ J \subseteq P}} (P \cap J) \Big)$$

$$= \Big(\bigcap_{P' \in h_J(L)} P' \Big) \cap J = k(h_J(L)).$$

Thus, J obeys spectral synthesis.

Next, consider a closed ideal K of X/J. Since X/J has Wiener property, it is enough to check that $K \supseteq k(h_{X/J}(K))$. Consider an element $x \in k(h_{X/J}(K))$, where $x = y + J \in X/J$. Note that $K = I/J$ for some closed ideal I of X containing J. Using the one-one correspondence between $\mathrm{Prim}(X/J)$ and $\{P \in \mathrm{Prim}(X) : J \subseteq P\}$, one can check that $y \in k(h_X(I))$. Since X obeys spectral synthesis, $I = k(h_X(I))$, so that $y \in I$, which shows that $x \in K$. Hence the result. ∎

We are now prepared to discuss spectral synthesis for operator space projective tensor product $A \widehat{\otimes} B$ of C^*-algebras A and B. Allen, Sinclair and Smith, in [1], defined the concept of spectral synthesis for the Haagerup tensor product of C^*-algebras in a somewhat different flavor. In the same spirit, using the terminologies of [1], we give another definition for the spectral synthesis of $A \widehat{\otimes} B$. It is known that for any C^*-algebras A and B, the canonical $*$-homomorphism $i : A \widehat{\otimes} B \to A \otimes^{\min} B$ is injective [10, Corollary 1], so that we can regard $A \widehat{\otimes} B$ as a $*$-subalgebra of $A \otimes^{\min} B$. Consider a closed ideal J of $A \widehat{\otimes} B$ and let $J_{\min}$ be the closure of $i(J)$ in $A \otimes^{\min} B$, in other words, $J_{\min}$ is the min-closure of J in $A \otimes^{\min} B$. Now we associate two closed ideals, namely the upper and the lower ideals, with J as:

$$J_l = \text{closure of span of all elementary tensors of} J \text{ in} A \widehat{\otimes} B,$$
$$J^u = J_{\min} \cap (A \widehat{\otimes} B).$$

Clearly $J_l \subseteq J \subseteq J^u$ for any closed ideal J of $A \widehat{\otimes} B$.

Definition 2.2. *A closed ideal J of $A \widehat{\otimes} B$ is said to be spectral if $J_l = J = J^u$.*

The main aim of this section is to show that $A \widehat{\otimes} B$ satisfies spectral synthesis if and only if its every closed ideal is spectral. We first characterize the upper ideals in terms of primitive ideals.

Lemma 2.1. *For closed ideals M and N of A and B,*

$$\ker(q_M \widehat{\otimes} q_N) = \ker(q_M \otimes^{\min} q_N) \cap A \widehat{\otimes} B.$$

Proof. For $z \in A \widehat{\otimes} B$, let $\{z_n\}$ be a sequence in $A \otimes B$ such that $\lim_n \|z_n - z\|_\wedge = 0$, then

$$\|(q_M \widehat{\otimes} q_N)(z_n) - (q_M \widehat{\otimes} q_N)(z)\|_{\min} \leq \|q_M \widehat{\otimes} q_N\| \|z_n - z\|_\wedge,$$

which shows that the sequence $\{(q_M \widehat{\otimes} q_N)(z_n)\}$ is convergent to $(q_M \widehat{\otimes} q_N)(z)$ in $A/M \otimes^{\min} B/N$. Also, $\|z_n - z\|_{\min} \leq \|z_n - z\|_\wedge$, so that $\lim_n \|z_n - z\|_{\min} = 0$, which further gives

$$(q_M \otimes^{\min} q_N)(z_n) \xrightarrow{\min} (q_M \otimes^{\min} q_N)(z).$$

Since both the mappings $q_M\widehat{\otimes}q_N$ and $q_M\otimes^{\min}q_N$ agree on $A\otimes B$, by continuity, we have $(q_M\otimes^{\min}q_N)(z)=(q_M\widehat{\otimes}q_N)(z)$, and this is true for all $z\in A\widehat{\otimes}B$, proving the given relation. ∎

Proposition 2.4. *For a closed ideal J of $A\widehat{\otimes}B$, $J=J^u$ if and only if J is semisimple.*

Proof. Let us first assume that $J=J^u$. Since, in a C^*-algebra every closed ideal is semisimple, $J_{\min}=\cap\{\ker\tilde{\pi}_\alpha : J_{\min}\subseteq\ker\tilde{\pi}_\alpha\}$, where each $\tilde{\pi}_\alpha$ is an irreducible $*$-representation of $A\otimes^{\min}B$ on some Hilbert space. Set $\pi_\alpha:=\tilde{\pi}_\alpha\circ i$, then each π_α is an irreducible $*$-representation of $A\widehat{\otimes}B$ annihilating J. Using some routine calculations, and the fact that $J=J^u$ one can prove that $J=\cap\ker\pi_\alpha$. Note that, although the collection $\{P\in\mathrm{Prim}(A\widehat{\otimes}B) : J\subseteq P\}$ is larger than $\{\ker\pi_\alpha : \pi_\alpha=\tilde{\pi}_\alpha\circ i\}$, it is easy to check that J is actually the intersection of all the primitive ideals of $A\widehat{\otimes}B$ containing J.

Conversely, let $J=\cap_{J\subseteq P_\alpha}P_\alpha$, P_α being primitive ideals of $A\widehat{\otimes}B$. Let, if possible, there exist an element $x\in J^u$ such that $x\notin J$. Then $x\notin P_\alpha$ for some α. Since P_α is primitive, by [11, Theorem 3.2], there exist closed (prime) ideals M and N in A and B, respectively, such that $P_\alpha=A\widehat{\otimes}N+M\widehat{\otimes}B$. Now, consider the bounded homomorphisms $q_M\widehat{\otimes}q_N : A\widehat{\otimes}B\to A/M\widehat{\otimes}B/N$, and $q_M\otimes^{\min}q_N : A\otimes^{\min}B\to A/M\otimes^{\min}B/N$ with $\ker(q_M\widehat{\otimes}q_N)=P_\alpha$ [12, Proposition 3.5]. By Lemma 2.1, $x\notin\ker(q_M\otimes^{\min}q_N)$, which by Hahn Banach Theorem gives a $\phi\in(A\otimes^{\min}B)^*$ such that $\phi(x)\neq 0$, and $\phi(\ker(q_M\otimes^{\min}q_N))=\{0\}$. The relation $J\subseteq P_\alpha\subseteq\ker(q_M\otimes^{\min}q_N)$ gives $J_{\min}\subseteq\ker(q_M\otimes^{\min}q_N)$, which further shows that $\phi(J_{\min})=0$. Thus $x\notin J_{\min}$, which gives a contradiction to the fact that $x\in J^u$. Hence the result. ∎

Using Propositions 2.1 and 2.4, we have a following characterization for spectral synthesis in terms of upper ideals.

Theorem 2.1. *The Banach $*$-algebra $A\widehat{\otimes}B$ satisfies spectral synthesis if and only if $J=J^u$, for every closed ideal J of $A\widehat{\otimes}B$.*

We now prove that the Banach $*$-algebra $A\widehat{\otimes}B$ satisfies spectral synthesis if and only if every closed ideal of $A\widehat{\otimes}B$ is spectral. We borrow some ideas from [15] to prove the same. We first need an elementary result.

Lemma 2.2. *Let J_i and K_i be closed ideals of C^*-algebras A_i, $i=1,2$. Then $J_1\widehat{\otimes}J_2\subseteq A_1\widehat{\otimes}K_2+K_1\widehat{\otimes}A_2$ if and only if either $J_1\subseteq K_1$ or $J_2\subseteq K_2$.*

Theorem 2.2. *For C^*-algebras A and B, the Banach $*$-algebra $A\widehat{\otimes}B$ obeys spectral synthesis if and only if every closed ideal of $A\widehat{\otimes}B$ is spectral.*

Proof. We just need to prove that for every closed ideal J of $A\widehat{\otimes}B$, $J=J_l$, if $A\widehat{\otimes}B$ satisfies spectral synthesis. Using Corollary 2.1, it is sufficient to show that $Z_J\subseteq Z_{J_l}$, where $Z_J:=\{P\in\mathrm{Prime}(A\widehat{\otimes}B) : P\not\supseteq J\}$. Set $X:=\mathrm{Prime}(A\widehat{\otimes}B)$ and consider an element P of Z_J. Since Z_J is an open subset of X and $\Phi:\mathrm{Prime}(A)\times\mathrm{Prime}(B)\to X$ is continuous, there exist open subsets U_1, U_2 of $\mathrm{Prime}(A)$ and $\mathrm{Prime}(B)$ such that $\Phi(U_1\times U_2)\subseteq Z_J$ and $P\in\Phi(U_1\times U_2)$. Let $J_1\in\mathrm{Id}(A)$, $J_2\in\mathrm{Id}(B)$ be the corresponding closed ideals such that $U_i=Z_{J_i}$, $i=1,2$. We claim that $Z_{J_1\widehat{\otimes}J_2}=\Phi(U_1\times U_2)=\Phi(Z_{J_1}\times Z_{J_2})$. For any $Q\in Z_{J_1\widehat{\otimes}J_2}$, by definition, $J_1\widehat{\otimes}J_2\not\subseteq Q$. Since $Q\in X$, and Φ is onto (Proposition 1.1), there exists $Q_1\in\mathrm{Prime}(A)$, $Q_2\in\mathrm{Prime}(B)$ such that $A\widehat{\otimes}Q_2+Q_1\widehat{\otimes}B=\Phi(Q_1,Q_2)=Q$. By Lemma 2.2, $J_1\not\subseteq Q_1$ and $J_2\not\subseteq Q_2$. This implies that $Q_i\in Z_{J_i}=U_i$, so that $Q=\Phi(Q_1,Q_2)\in\Phi(U_1\times U_2)$. Thus,

$Z_{J_1\widehat{\otimes}J_2} \subseteq \Phi(U_1 \times U_2)$. For the other containment, consider $\Phi(K_1,K_2) \in \Phi(Z_{J_1} \times Z_{J_2})$. Since $K_i \in Z_{J_i}$, we have $J_i \nsubseteq K_i$, so by Lemma 2.2, $J_1\widehat{\otimes}J_2 \nsubseteq \Phi(K_1,K_2)$. Note that $\Phi(K_1,K_2) \in X$, thus by the definition, $\Phi(K_1,K_2) \in Z_{J_1\widehat{\otimes}J_2}$. So, $Z_{J_1\widehat{\otimes}J_2} \subseteq Z_J$, which further gives, $J_1\widehat{\otimes}J_2 \subseteq J$. But, the definition of J_l says that $J_1\widehat{\otimes}J_2 \subseteq J_l$. This means that $Z_{J_1\widehat{\otimes}J_2} \subseteq Z_{J_l}$. Since, $P \in \Phi(U_1 \times U_2) = Z_{J_1\widehat{\otimes}J_2}$, this gives $P \in Z_{J_l}$. Thus $Z_J \subseteq Z_{J_l}$, which proves that $J \subseteq J_l$, and hence the result. ∎

Remark 2.2. In other words, if $A\widehat{\otimes}B$ obeys spectral synthesis, then every closed ideal J of $A\widehat{\otimes}B$ is the closure of the sum of all product ideals $J_1\widehat{\otimes}J_2 \subseteq J$, where $J_1 \in \mathrm{Id}(A)$, $J_2 \in \mathrm{Id}(B)$.

The Banach $*$-algebra $A\widehat{\otimes}B$ contains plenty of spectral ideals as demonstrated in the following and some later examples.

Proposition 2.5. *For $I \in \mathrm{Id}(A)$ and $J \in \mathrm{Id}(B)$, the closed ideal $A\widehat{\otimes}J + I\widehat{\otimes}B$ of $A\widehat{\otimes}B$ is spectral. In particular, every closed maximal ideal, primitive ideal and prime ideal of $A\widehat{\otimes}B$ is spectral.*

Proof. Set $K := A\widehat{\otimes}J + I\widehat{\otimes}B = \ker(q_I\widehat{\otimes}q_J)$, then it is clear from the definition that $K = K_l$. Consider an element $u \in K^u$. Let, if possible, $u \notin K$, then by Lemma 2.1, $u \notin \ker(q_I \otimes^{\min} q_J)$. Now, $K \subseteq \ker(q_I \otimes^{\min} q_J)$ implies $K_{\min} \subseteq \ker(q_I \otimes^{\min} q_J)$, giving $u \notin K_{\min}$, a contradiction. Thus K is spectral. Rest follows from the fact that every maximal, primitive and prime ideal can be expressed as an ideal of this form [12, Theorem 3.10], [11, Theorem 3.1, 3.2]. ∎

Next, we prepare the ingredients to prove that for an infinite dimensional separable Hilbert space H, $\mathscr{B}(H)\widehat{\otimes}\mathscr{B}(H)$ obeys spectral synthesis. We first need some elementary results regarding the lower and upper ideals of a closed ideal.

Proposition 2.6. *For closed ideals J and K in $A\widehat{\otimes}B$, we have:*

(a) *$J_l \subseteq K_l$ and $J^u \subseteq K^u$, if $J \subseteq K$;*
(b) *$(JK)_l = J_lK_l = J_l \cap K_l = (J\cap K)_l$, if J_l or K_l has a bounded approximate identity;*
(c) *$(J\cap K)^u \subseteq J^u \cap K^u$, with equality if $J = J^u, K = K^u$.*

Proof. (a) is trivial. For (b), we first show that $J_l \cap K_l \subseteq J_lK_l$. Let $x \in J_l \cap K_l$ and assume that J_l has bounded approximate identity. By Cohen's Factorization Theorem, there exist $y,z \in J_l$ such that $x = yz$ and z belongs to the closed left ideal generated by x in J_l. Clearly, $z \in J_l \cap K_l$, so that $x \in J_lK_l$. Thus, $J_l \cap K_l \subseteq J_lK_l$. Now, for an elementary tensor x in $J \cap K$, clearly $x \in J_l \cap K_l$, giving $(J\cap K)_l \subseteq J_l \cap K_l$. Also, for $a = \sum_{i=1}^n x_i \otimes y_i \in J_l$ and $b = \sum_{j=1}^m z_j \otimes w_j \in K_l$, clearly $ab \in (JK)_l$, being an elementary tensor of JK. Since J_l and K_l are both generated by elementary tensors, routine calculations show that $J_lK_l \subseteq (JK)_l$. Thus, we have

$$(J\cap K)_l \subseteq J_l \cap K_l \subseteq J_lK_l \subseteq (JK)_l \subseteq (J\cap K)_l,$$

which gives the required equality. For (c), using the fact that $(J\cap K)_{\min} \subseteq J_{\min} \cap K_{\min}$, we get

$$(J\cap K)^u \subseteq J_{\min} \cap K_{\min} \cap A\widehat{\otimes}B = J^u \cap K^u.$$ ∎

Following are some direct consequences of the above proposition.

Corollary 2.2. *If I and J are closed ideals of $A\widehat{\otimes}B$ with at least one of them having bounded approximate identity, then $I\cap J$ is spectral, whenever I and J are spectral.*

Corollary 2.3. *Every product ideal of $A\widehat{\otimes}B$ is spectral. In particular, for closed ideals I and J of A and B, $I\widehat{\otimes}J = (I\otimes^{\min} J)\cap A\widehat{\otimes}B$.*

Proof. For a product ideal $I\widehat{\otimes}J$ of $A\widehat{\otimes}B$, using [11, Proposition 2.4], we can write $I\widehat{\otimes}J = (A\widehat{\otimes}J)\cap(I\widehat{\otimes}B)$. Also, from [12, Lemma 3.1], $A\widehat{\otimes}J$ and $I\widehat{\otimes}B$ both possess bounded approximate identities. Thus, from Proposition 2.5 and Corollary 2.2, $I\widehat{\otimes}J$ is spectral. Clearly,

$$I\widehat{\otimes}J = (I\widehat{\otimes}J)^u = (I\widehat{\otimes}J)_{\min}\cap A\widehat{\otimes}B = (I\otimes^{\min} J)\cap A\widehat{\otimes}B.$$ ∎

Corollary 2.4. *If either A or B is a simple C^*-algebra, then $A\widehat{\otimes}B$ obeys spectral synthesis.*

Proof. Let A be simple. By [12, Theorem 3.8], every closed ideal of $A\widehat{\otimes}B$ is a product ideal and thus is spectral by Corollary 2.3. Using Theorem 2.2, we get $A\widehat{\otimes}B$ obeys spectral synthesis. ∎

In particular, for any C^*-algebra A, the Banach $*$-algebras $A\widehat{\otimes}C_r^*(\mathbb{F}_2)$, $A\widehat{\otimes}A_\infty$ and $A\widehat{\otimes}\mathscr{K}(H)$ obey spectral synthesis, where $C_r^*(\mathbb{F}_2)$ is the C^*-algebra associated to the left regular representations of the free group $\mathbb{F}_2$ on two generators, A_∞ is the Glimm algebra [18] and $\mathscr{K}(H)$ is the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space H.

Theorem 2.3. *For an infinite dimensional separable Hilbert space H, the Banach $*$-algebra $\mathscr{B}(H)\widehat{\otimes}\mathscr{B}(H)$ obeys spectral synthesis.*

Proof. From [12, Theorem 3.11], we know that the only non trivial closed ideals of $\mathscr{B}(H)\widehat{\otimes}\mathscr{B}(H)$ are $\mathscr{K}(H)\widehat{\otimes}\mathscr{K}(H)$, $\mathscr{B}(H)\widehat{\otimes}\mathscr{K}(H), \mathscr{K}(H)\widehat{\otimes}\mathscr{B}(H)$ and $\mathscr{B}(H)\widehat{\otimes}\mathscr{K}(H)+\mathscr{K}(H)\widehat{\otimes}\mathscr{B}(H)$. Using Proposition 2.5 and Corollary 2.3, we can see that all the proper closed ideals of $\mathscr{B}(H)\widehat{\otimes}\mathscr{B}(H)$ are spectral. The result now follows from Theorem 2.2. ∎

Proposition 2.7. *Let A and B be C^*-algebras such that A or B has finitely many closed ideals. Then $A\widehat{\otimes}B$ obeys spectral synthesis.*

Proof. Without loss of generality, we may assume that B has finitely many closed ideals say n, where $n\geq 2$. We prove the result by induction on n. For $n=2$, B is simple and the result follows from Corollary 2.4. Let the result be true for all C^*-algebras with at most $(n-1)$ ideals. Let B have $n>2$ closed ideals. Since there are finitely many closed ideals of B, there exists a minimal (non-trivial) closed ideal, say K, of B, which is clearly simple. Consider the closed $*$-ideal $J := A\widehat{\otimes}K$ of $X := A\widehat{\otimes}B$. Since K is simple, using Corollary 2.4, it is clear that J satisfies spectral synthesis. Note that, by [11, Lemma 2.2(1)], X/J is isomorphic to $A\widehat{\otimes}(B/K)$ and the latter has spectral synthesis by induction hypothesis, since B/K has at most $(n-1)$ closed ideal. So, X/J also satisfies spectral synthesis. Moreover, J and X/J both have Wiener property [11, Theorem 4.1], and J has bounded approximate identity [12, Lemma 3.1], the result now follows from Proposition 2.2. ∎

Thus, for any C^*-algebra A, $A\widehat{\otimes}\mathscr{B}(H)$ obeys spectral synthesis, where H is a separable infinite dimensional Hilbert space. In particular, $C_0(X)\widehat{\otimes}\mathscr{B}(H)$, $\mathscr{B}(H)\widehat{\otimes}\mathscr{B}(H)$ and $\mathscr{B}(H)\widehat{\otimes}\mathscr{K}(H)$ obey spectral synthesis, where X is a locally compact Hausdorff space. For more examples of C^*-algebras with finitely many closed ideals, see [16].

Corollary 2.5. *If A and B both have finite number of closed ideals, then every closed ideal of $A\widehat{\otimes}B$ is a finite sum of product ideals.*

Proof. It follows from Proposition 2.7 and Remark 2.2. ∎

Remark 2.3. Let A and B be C^*-algebras. Consider the Gelfand-Naimark semi norm on $A \otimes B$ defined as $\gamma(x) = \sup\{\|T(x)\|\}$, where the supremum runs over all the $*$-representations T of $A \otimes B$ on Hilbert spaces. Since the $\|\cdot\|_\gamma$-norm on $A \otimes B$ is continuous with respect to '$\wedge$'-norm, it can be extended to $A\widehat{\otimes}B$, and thus $A \otimes B$ is $\|\cdot\|_\gamma$-dense in $(A\widehat{\otimes}B, \|\cdot\|_\gamma)$. By [17, Proposition 10.5.20], $A\widehat{\otimes}B$ is $*$-regular, whenever $A \otimes B$ is so. Since A is nuclear, being subhomogeneous, by [9, Corollary 2.7], $A\widehat{\otimes}B$ is $*$-regular. It is also Hermitian [11, Theorem 4.6] and is $*$-semisimple (follows from [11, Theorem 4.1]). Hence, the definition of spectral synthesis in [6] is equivalent to our definition in this case.

In the case of commutative separable C^*-algebras the ideals which are not singly generated fail to be spectral. A similar result also holds true in the non-commutative situation. The following can be proved exactly on the same lines of [1, Theorem 6.12].

Proposition 2.8. *Let A and B be separable C^*-algebras, and J be a non-zero closed ideal of $A\widehat{\otimes}B$. Then J is singly generated if it is spectral.*

3. Reverse involution

Let A be a C^*-algebra. On the Banach algebra $A \otimes A$ (with usual multiplication), define the involution as $(a \otimes b)^* = b^* \otimes a^*$ for all $a, b \in A$. Then it extends to an isometric involution on $A\widehat{\otimes}A$ and $A\widehat{\otimes}A$ forms a Banach $*$-algebra with this involution, which we denote by $A\widehat{\otimes}_r A$. Regarding the closed $*$-ideals of $A\widehat{\otimes}_r A$, note that the closed ideals of $A\widehat{\otimes}_r A$ coincide with the ones in $A\widehat{\otimes}A$; however, the closed $*$-ideals differ. We do not know whether a closed ideal of $A\widehat{\otimes}A$ is a $*$-ideal or not, but in $A\widehat{\otimes}_r A$ a closed ideal need not be a $*$-ideal. For example, in the space $\mathscr{B}(H)\widehat{\otimes}_r\mathscr{B}(H)$, the closed ideals $\mathscr{K}(H)\widehat{\otimes}\mathscr{B}(H)$ and $\mathscr{B}(H)\widehat{\otimes}\mathscr{K}(H)$ are not $*$-ideals. In fact, it has only two non-trivial closed $*$-ideals, namely $\mathscr{K}(H)\widehat{\otimes}_r\mathscr{K}(H)$ and $\mathscr{B}(H)\widehat{\otimes}\mathscr{K}(H) + \mathscr{K}(H)\widehat{\otimes}\mathscr{B}(H)$.

We know that with natural involution $A\widehat{\otimes}A$ has a faithful $*$-representation and is always $*$-semisimple for any C^*-algebra A. However, we show that this is not the case with $A\widehat{\otimes}_r A$.

Proposition 3.1. *Let A be a unital C^*-algebra. Then, $A\widehat{\otimes}_r A$ has a faithful $*$-representation if and only if $A = \mathbb{C}I$, I being the unity of A.*

Proof. Let π be a faithful $*$-representation of $A\widehat{\otimes}_r A$ on a Hilbert space H. Define $\pi_1(a) := \pi(1 \otimes a)$ and $\pi_2(a) := \pi(a \otimes 1)$ for all $a \in A$. Then π_1 and π_2 are both bounded representations of A on $\mathscr{B}(H)$, with $\pi(b \otimes a) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ for all $a, b \in A$. Also

$$\pi_1(a^*) = \pi(1 \otimes a^*) = \pi((a \otimes 1)^*) = (\pi(a \otimes 1))^* = \pi_2(a)^* \tag{3.1}$$

for all $a \in A$. It is known that an element $h \in A$ is self adjoint if and only if $\|\exp ith\| = 1$ for all $t \in \mathbb{R}$. For a self adjoint element $h \in A$, using the facts that π is contractive and that $\|\cdot\|_\wedge$-norm is a cross norm, we have

$$\begin{aligned}\|\exp it\pi_1(h)\| = \|\pi(\exp it(h \otimes 1))\| \leq \|\exp it(h \otimes 1)\|_\wedge &= \lim_m \Big\| \sum_{n=1}^m \frac{i^n t^n (h^n \otimes 1)}{n!} \Big\|_\wedge \\ &= \lim_m \Big\| \Big(\sum_{n=1}^m \frac{i^n t^n h^n}{n!} \Big) \otimes 1 \Big\|_\wedge = \lim_m \Big\| \sum_{n=1}^m \frac{i^n t^n h^n}{n!} \Big\| = \|\exp ith\| = 1,\end{aligned}$$

and this is true for all $t \in \mathbb{R}$. Thus, $\|\exp it\pi_1(h)\| = 1$ for all $t \in \mathbb{R}$, which shows that $\pi_1(h)$ is a self adjoint element of $\mathscr{B}(H)$. This, combined with equation (3.1), gives $\pi_1(h) = \pi_2(h)$, that is $\pi(1 \otimes h) = \pi(h \otimes 1)$. Since π is faithful, $1 \otimes h = h \otimes 1$. So, for any $\phi \in A^*$, $\phi(1)h =$

$\phi(h)1$, which further gives $h \in \mathbb{C}I$, and this is true for any self adjoint element h of A. Since any $a \in A$ can be written as $a = h + ik$, h and k being self adjoint elements of A, we obtain the required result. ∎

Corollary 3.1.
(i) *$A\widehat{\otimes}_r A$ is $*$-semisimple if and only if $A = \mathbb{C}I$.*
(ii) *$A\widehat{\otimes}_r A$ is symmetric if and only if $A = \mathbb{C}I$*

Proof. (i) Follows easily from the fact that a semisimple Banach $*$-algebra possesses a faithful $*$-representation [19, Corollary 4.7.16]. (ii) Let $A\widehat{\otimes}_r A$ be symmetric. Using the same argument as in [1, Proposition 5.16], one can show that the radical of $A\widehat{\otimes}_r A$ is $\{0\}$. By [19, Theorem 4.7.15], $*$-radical of $A\widehat{\otimes}_r A$ coincides with its radical. Thus $A\widehat{\otimes}_r A$ is $*$-semisimple, which using above part implies $A = \mathbb{C}I$. ∎

References

[1] S. D. Allen, A. M. Sinclair and R. R. Smith, The ideal structure of the Haagerup tensor product of C^*-algebras, *J. Reine Angew. Math.* **442** (1993), 111–148.

[2] R. J. Archbold, E. Kaniuth, G. Schlichting and D. W. B. Somerset, Ideal spaces of the Haagerup tensor product of C^*-algebras, *Internat. J. Math.* **8** (1997), no. 1, 1–29.

[3] B. Blackadar, *Operator Algebras*, Encyclopaedia of Mathematical Sciences, 122, Springer, Berlin, 2006.

[4] P. J. Cohen, Factorization in group algebras, *Duke Math. J* **26** (1959), 199–205.

[5] P. G. Dixon, Non-closed sums of closed ideals in Banach algebras, *Proc. Amer. Math. Soc.* **128** (2000), no. 12, 3647–3654 (electronic).

[6] J. F. Feinstein, E. Kaniuth and D. W. B. Somerset, Spectral synthesis and topologies on ideal spaces for Banach *-algebras, *J. Funct. Anal.* **196** (2002), no. 1, 19–39.

[7] J.-F. Feinstein and D. W. B. Somerset, Spectral synthesis for Banach algebras. II, *Math. Z.* **239** (2002), no. 1, 183–213.

[8] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Grundlehren der Mathematischen Wissenschaften, 238, Springer, New York, 1979.

[9] W. Hauenschild, E. Kaniuth and A. Voigt, $*$-regularity and uniqueness of C^*-norm for tensor products of $*$-algebras, *J. Funct. Anal.* **89** (1990), no. 1, 137–149.

[10] R. Jain and A. Kumar, Operator space tensor products of C^*-algebras, *Math. Z.* **260** (2008), no. 4, 805–811.

[11] R. Jain and A. Kumar, Ideals of operator space projective tensor product of C^*-algebras, *J. Aust. Math. Soc.* **91** (2011), 275–288.

[12] R. Jain and A. Kumar, The operator space projective tensor product: Embedding into the second dual and ideal structure, *Proc. Edinburgh Math. Soc.*, to appear.

[13] E. Kaniuth, *A course in Commutative Banach Algebras*, Graduate Texts in Mathematics, 246, Springer, New York, 2009.

[14] A. Kumar, Operator space projective tensor product of C^*-algebras, *Math. Z.* **237** (2001), no. 2, 211–217.

[15] A. J. Lazar, The space of ideals in the minimal tensor product of C^*-algebras, *Math. Proc. Cambridge Philos. Soc.* **148** (2010), no. 2, 243–252.

[16] H. X. Lin, Ideals of multiplier algebras of simple AF C^*-algebras, *Proc. Amer. Math. Soc.* **104** (1988), no. 1, 239–244.

[17] T. W. Palmer, *Banach Algebras and the General Theory of $*$-Algebras. Vol. 2*, Encyclopedia of Mathematics and its Applications, 79, Cambridge Univ. Press, Cambridge, 2001.

[18] G. K. Pedersen, *C^*-Algebras and Their Automorphism Groups*, London Mathematical Society Monographs, 14, Academic Press, London, 1979.

[19] C. E. Rickart, *General Theory of Banach Algebras*, The University Series in Higher Mathematics D. van Nostrand Co., Inc., Princeton, NJ, 1960.

[20] D. W. B. Somerset, Spectral synthesis for Banach algebras, *Quart. J. Math. Oxford Ser. (2)* **49** (1998), no. 196, 501–521.

[21] M. Takesaki, *Theory of Operator Algebras. I*, Springer, New York, 1979.