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Existence of Homoclinic Travelling Waves in Infinite Lattices

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Abstract. By using critical point theory, we investigate the existence of homoclinic travelling waves in an one-dimensional infinite lattice with nearest-neighbor interactions and a on-site potential (density) f. The system is described by the infinite system of second-order differential equations:

$$\ddot{q}_j + f'(q_j(t)) = V'(q_{j+1}(t) - q_j(t)) - V'(q_j(t) - q_{j-1}(t)), \quad t \in \mathbb{R},\ j \in \mathbb{Z},$$

where $f, V \in C^1(\mathbb{R},\mathbb{R})$. We establish three new criteria ensuring the existence of non-trivial homoclinic travelling wave solutions, for any given speed c bigger (or smaller) than some constant depending on f and V. Relevant results in the literatures are extended.

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1. Introduction

In this paper, we consider an one-dimensional infinite lattice with nearest-neighbor interactions and a potential f:

$$\ddot{q}_j + f'(q_j(t)) = V'(q_{j+1}(t) - q_j(t)) - V'(q_j(t) - q_{j-1}(t)), \quad t \in \mathbb{R},\ j \in \mathbb{Z}, \tag{1.1}$$

where $f, V \in C^1(\mathbb{R},\mathbb{R})$. The above lattice system with the on-site potential $f(x) = K(1 - \cos x), K > 0$ is sometimes called the Frenkel-Kontorova model, even if V is not harmonic, i.e. $V(x) \neq \varepsilon x^2/2$. If $f \equiv 0$, (1.1) becomes the usual lattice equation which is called the Fermi-Pasta-Ulam (FPU) lattice. We are interested in travelling wave solutions to (1.1), that is, solutions of the form

$$q_j(t) = u(j - ct), \quad j \in \mathbb{Z}, \tag{1.2}$$

where u: $\mathbb{R} \to \mathbb{R}$ is the wave profile and $c > 0$ is the wave speed. For this ansatz (1.2), (1.1) becomes the following second-order forward-backward differential equation:

$$c^2\ddot{u} - V'(u(t+1) - u(t)) + V'(u(t) - u(t-1)) + f'(u) = 0, \tag{1.3}$$

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where $t \in \mathbb{R}, f, V \in C^1(\mathbb{R},\mathbb{R})$. We say that a nontrivial solution u of (1.3) is homoclinic to zero if $u \in C^2(\mathbb{R},\mathbb{R}), u(t) \not\equiv 0, u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$. In a suitable setting, equation (1.3) is the Euler-Lagrange equation of the action functional

$$I(u) = \int_{\mathbb{R}} \left[\frac{c^2}{2} |\dot{u}(t)|^2 - V(u(t+1) - u(t)) - f(u(t)) \right] dt. \tag{1.4}$$

There have been many papers on the study of the existence of travelling waves, periodic motions and chains of oscillators for FPU by using all kinds of methods, such as bifurcation theory, numerical methods, the exp-function method, variational techniques, and so on. We refer the readers to see [1, 2, 6, 13–15]. Many authors have also studied the Frenkel-Kontorova model (see [3, 5, 7, 9]). However, the use of variational methods is quite recent on the study of homoclinic travelling waves of these models. By using the mountain pass theorem, Makita [10] investigated the existence of nonconstant periodic travelling waves and homoclinic travelling waves for (1.1) under growth conditions $V(u) = \alpha u^2/2 + W(u)$ and $f(u) = -\omega u^2/2 + g(u)$. In particular, if g and W satisfy the following so-called global Ambrosetti-Rabinowitz condition: there is a constant $\mu > 2$ such that

$$0 < \mu W(u) \le (u, W'(u)) \tag{1.5}$$

and

$$0 < \mu g(u) \le (u, g'(u)) \tag{1.6}$$

for $u \in \mathbb{R} \setminus \{0\}$, where $(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denotes the standard inner product and $|\cdot|$ is the induced norm in $\mathbb{R}$, existence of homoclinic travelling waves is obtained as a limit of periodic waves by letting the period go to infinity. This method is very classical on the study of homoclinic orbits in Hamiltonian systems (see [8, 12, 17, 20–22]).

The aim of this paper is to investigate the existence of homoclinic travelling waves for (1.1), which is equivalent to the study of homoclinic solutions of (1.3). With the lack of global compactness due to unboundedness of domain, the Sobolev compact embedding of $H^1(\mathbb{R},\mathbb{R})$ in $L^2(\mathbb{R},\mathbb{R})$ is unreasonable. To overcome this difficulty, we use a variant version of the mountain pass theorem without (PS) condition. Under some reasonable assumptions about V and f, by using variable structure, which is different from [10] (see also [16, 18, 19, 23, 24]), we obtain three results on the existence of homoclinic solutions to equation (1.3) (namely, Theorems 1.1, 1.2, and 1.3). Firstly, in Theorem 1.1, we give negative sign on g, which is very different from [10, Theorem 1.2]. Secondly, if V is asymptotically quadratic, we establish Theorem 1.2 to guarantee the existence of of homoclinic solutions for (1.3). Thirdly, under more relaxed assumptions on g and V, Theorem 1.3 generalizes [10, Theorem 1.2].

We now state our main results.

Theorem 1.1. *Assume that* $V(u) = c_0^2 u^2/2 + W(u)$ *and* $f(u) = -d_1 u^2/2 - g(u)$*, where* $c_0 \ge 0$*,* $d_1 > 0$*,* W *and* g *satisfy the following conditions:*

- **($\mathscr{H}_1$):** *There exists a constant* $\mu > 2$ *such that* (1.5) *holds for all* $x \in \mathbb{R} \setminus \{0\}$.
- **($\mathscr{H}_2$):** $g(0) = 0$*,* $\mu g(u) \ge (u, g'(u))$ *and* $g(u) \ge 0$ *for all* $u \in \mathbb{R}$*. Moreover, there exist positive constants* T *and* $\theta \in [1, \mu - 1)$ *such that* $g'(u) \le T|u|^\theta$ *for all* $u \in \mathbb{R}$.

If

$$\min\{c^2, d_1\} - c_0^2 - 2M \ge \frac{1}{2}, \quad M := \max\{W(u) : |u| = 1\}$$

and d_1 is large enough, then problem (1.3) *possesses a nontrivial homoclinic solution.*

In the assumptions of Theorem 1.1, inequality (1.5) holds but (1.6) does not hold. Hence, this theorem is a complement of the results of Makita [10].

Theorem 1.2. *Assume that $f(u) = -d_1u^2/2$ and $V(u)$ satisfy the following conditions:*

($\mathscr{H}_1'$): *$V(0) = 0$, $V(u) > 0$, and if $u \neq 0$, $V'(u) = o(|u|)$ as $|u| \to 0$, then there exists some constant $C > 0$ such that*

$$\frac{|V'(u)|}{|u|} \le C, \quad u \in \mathbb{R}.$$

($\mathscr{H}_3$): *There exists some constant $d > e\max\{c^2, d_1\}/(e-2)$ such that*

$$\frac{|V'(u) - du|}{|u|} \to 0, \quad \text{as } |u| \to +\infty, \quad \textit{where} \quad \ln e = 1.$$

($\mathscr{H}_4$): *Let $K(u) = (V'(u), u)/2 - V(u)$ and assume that $K(u) > 0$, and if $u \neq 0$, $K(u) \to +\infty$ as $|u| \to +\infty$, and for any fixed $0 < b_1 < b_2 < +\infty$,*

$$\inf_{b_1 \le |u| \le b_2} \frac{K(u)}{|u|^2} > 0,$$

then problem (1.3) *has at least a nontrivial homoclinic solution.*

Theorem 1.3. *Assume that $V(u) = c_0^2u^2/2 + W(u)$ and $f(u) = -d_1u^2/2 + g(u)$, where $c_0 \ge 0$, $d_1 > 0$, $W'(u) = o(u)$ as $|u| \to 0$ and for any $0 < r_0 \le 1$,* (1.5) *holds for $|u| \ge r_0$, Besides, g satisfies the following condition:*

($\mathscr{H}_5$): *$g(0) = 0, g(u) \ge 0, u \in \mathbb{R}$ and $g'(u) = o(|u|)$ as $u \to 0$. Moreover, for any $a_0 > 0$, there exist $\kappa, \gamma > 0$ and $\gamma_1 < 2$ such that*

$$0 \le \left(2 + \frac{1}{\kappa + \gamma|u|^{\gamma_1}}\right) g(u) \le (u, g'(u)), \quad |u| \ge a_0.$$

If

$$\min\{c^2, d_1\} - c_0^2 > 0$$

and d_1 is large enough, then problem (1.3) *possesses a nontrivial homoclinic solution.*

Remark 1.1. Indeed, respectively, in the proof of nontrivial solutions in Theorems 1.1 and 1.3, we can deduce that d_1 in Theorems 1.1 and 1.3 is bounded below.

Remark 1.2. Assumption ($\mathscr{H}_1$) implies that $W(u) = o(|u|^2)$ as $u \to 0$, and $W(0) = 0$. Moreover, by ($\mathscr{H}_1'$), we have

$$0 \le V(u) = \int_0^1 (V'(\zeta u), u)d\zeta \le C|u|^2, u \in \mathbb{R} \tag{1.7}$$

and, for any given $\xi > 0$, there exists some $r_\xi > 0$ such that

$$0 \le V(u) \le \xi|u|^2, \quad |u| \le r_\xi. \tag{1.8}$$

Remark 1.3. If (1.6) holds, then $g(u) = o(|u|^2)$ as $u \to 0$ and by choosing $\kappa > 1/(\mu-2), \gamma > 0, 0 < \gamma_1 < 2$,

$$0 \le \left(2 + \frac{1}{\kappa + \gamma|u|^{\gamma_1}}\right) g(u) \le \mu g(u) \le (u, g'(u)), \quad u \in \mathbb{R}.$$

This shows that (1.6) implies $(\mathscr{H}_5)$. In addition, (1.5) implies the assumptions of W in Theorem 1.3. Therefore, Theorem 1.3 generalizes [10, Theorem 1.2] by relaxing (1.5) and (1.6).

The following three examples is to illustrate our results.

Example 1.1. In (1.3), one can easily check that if

$$W(u)=\begin{cases} u^4, & |u|\le 1,\\ \frac{6}{7}u^{\frac{14}{3}}+\frac{1}{7}, & |u|>1,\end{cases}$$

and

$$g(u)=\begin{cases} u^4, & |u|\le 1,\\ \frac{6}{5}u^{\frac{10}{3}}-\frac{1}{5}, & |u|>1,\end{cases}$$

for $T=\mu=4$, $\theta=7/3$, then all the assumptions of Theorem 1.1 hold.

Example 1.2. In (1.3), let

$$V(u)=d'|u|^2\left(1-\frac{1}{\ln(e+|u|)}\right),$$

where $d'>e\max\{c^2,d_1\}/2(e-2)$. By an easy calculation, we have

$$V'(u)=2d'\left(1-\frac{1}{\ln(e+|u|)}\right)u+\frac{d'|u|u}{(e+|u|)\ln^2(e+|u|)}$$

and

$$K(u)=\frac{d'|u|^3}{2(e+|u|)\ln^2(e+|u|)}.$$

Let $d=2d'$, then it is clear that all the assumptions of Theorem 1.2 hold.

Example 1.3. In (1.3), take

$$g(u)=|u|^2\ln(1+|u|).$$

By an easy calculation, we have

$$(g'(u),u)=2|u|^2\ln(1+|u|)+\frac{|u|^3}{1+|u|}\ge\left(2+\frac{1}{1+|u|}\right)g(u).$$

This shows that $(\mathscr{H}_5)$ holds with $\kappa=\gamma=\gamma_1=1$. It is clear that the assumptions of Theorem 1.3 hold when $W(u)=u^4$.

This paper is organized as follows. In Section 2, we present some preliminaries. Section 3 is devoted to the proof of our main results.

2. Preliminaries

In this section, we present some definitions and lemmas that will be used in the proof of our main results. Let

$$E=\{u\in H^1(\mathbb{R},\mathbb{R}):\int_{\mathbb{R}}(|\dot{u}|^2+|u|^2)dt<+\infty\}.$$

Then the space E is a Hilbert space with the inner product

$$\langle u,v\rangle=\int_{\mathbb{R}}[(\dot{u}(t),\dot{v}(t))+(u(t),v(t))]dt$$

and the corresponding norm $\|u\|^2 = \langle u,u\rangle$. Note that E is continuously embedded in $L^k(\mathbb{R},\mathbb{R})$ for all $k \in [2,+\infty]$. Therefore, there exists a constant $D_k > 0$ such that

$$\|u\|_k \le D_k\|u\|, \quad \forall u \in E. \tag{2.1}$$

Here, $L^k(\mathbb{R},\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}$ under the norm

$$\|u\|_k := \left(\int_{\mathbb{R}} |u(t)|^k dt\right)^{\frac{1}{k}}$$

and $L^\infty(\mathbb{R},\mathbb{R})$ is the Banach space of essentially bounded functions equipped with the norm

$$\|u\|_\infty := \operatorname{esssup}\{|u(t)| : t \in \mathbb{R}\}.$$

On E, we define the linear operator

$$A(u(t)) := u(t+1) - u(t).$$

Lemma 2.1. *The operator A is continuous from E to $L^2(\mathbb{R},\mathbb{R})\bigcap L^\infty(\mathbb{R},\mathbb{R})$ and $\|A(u)\|_\infty \le \|u\|, \|A(u)\|_2 \le (\int_{\mathbb{R}} |\dot{u}|^2 dt)^2 \le \|u\|$.*

The proof of Lemma 2.1 is similar to [14, Proposition 1] and so omitted.

Lemma 2.2. *Under the conditions of Theorem 1.1, if $u_k \to u$ in E, then $g'(u_k) \to g'(u)$ in $L^2(\mathbb{R},\mathbb{R})$.*

Proof. Assume that $u_k \to u$ in E. It follows from $(\mathscr{H}_2)$ that

$$\begin{aligned} |g'(u_k(t)) - g'(u(t))| &\le T(|u_k(t)|^\theta + |u(t)|^\theta) \\ &\le T[2^{\theta-1}|u_k(t) - u(t))|^\theta + (2^{\theta-1}+1)|u(t)|^\theta]. \end{aligned} \tag{2.2}$$

In view of the Banach-Steinhaus Theorem and (2.1), we have

$$\sup_{k\in\mathbb{N}} \|u_k\|_\infty \le D, \quad \|u\|_\infty \le D, \tag{2.3}$$

where $D > 0$ is a constant. Since $u_k \to u$ in $L^2(\mathbb{R},\mathbb{R})$, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_2 < +\infty.$$

But this implies that $u_k \to u$ for almost every $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} |u_k(t) - u(t)| := v(t) \in L^2(\mathbb{R},\mathbb{R}). \tag{2.4}$$

By (2.2), (2.3) and (2.4), we obtain

$$|g'(u_k(t)) - g'(u(t))| \le T[2^{\theta-1}(2D)^{\theta-1}|v(t)| + (2^{\theta-1}+1)D^{\theta-1}|u(t)|],$$

which yields that

$$|g'(u_k(t)) - g'(u)(t)|^2 \le 2T^2[2^{2\theta-2}(2D)^{2\theta-2}|v(t)|^2 + (2^{\theta-1}+1)^2 D^{2\theta-2}|u(t)|^2].$$

From the above inequality, we have

$$\begin{aligned} \int_{\mathbb{R}} |g'(u_k(t)) - g'(u(t))|^2 dt &\le \int_{\mathbb{R}} 2T^2[2^{2\theta-2}(2D)^{2\theta-2}|v(t)|^2 + (2^{\theta-1}+1)^2 D^{2\theta-2}|u(t)|^2]dt \\ &\le T^2 2^{2\theta-1}(2D)^{2\theta-2}\|v\|_2^2 + 2T^2(2^{\theta-1}+1)^2 D^{2\theta-2} D_2^2 \|u\|^2. \end{aligned}$$

Using the Lebesgue dominated convergence theorem, $g'(u_k) \to g'(u)$ in $L^2(\mathbb{R},\mathbb{R})$. This completes the proof of Lemma 2.2. ∎

Under the conditions of Theorem 1.1, the action functional I in (1.4) becomes

$$I(u)=\int_{\mathbb{R}}\left[\frac{c^2}{2}|\dot{u}(t)|^2+\frac{d_1}{2}|u(t)|^2-V(Au(t))+g(u(t))\right]dt. \tag{2.5}$$

Lemma 2.3. *Under the conditions of Theorem 1.1, $I\in C^1(E,\mathbb{R})$ and*

$$\begin{aligned} &I'(u)v \\ &=\int_{\mathbb{R}}[c^2(\dot{u}(t),\dot{v}(t))+d_1(u(t),v(t))-(V'(Au(t)),Av(t))]dt+\int_{\mathbb{R}}(g'(u(t)),v(t))dt \end{aligned} \tag{2.6}$$

for any $u,v\in E$, which yields

$$I'(u)u=\int_{\mathbb{R}}[c^2(\dot{u}(t),\dot{u}(t))+d_1(u(t),u(t))-(V'(Au(t)),Au(t))]dt+\int_{\mathbb{R}}(g'(u(t)),u(t))dt.$$

Moreover, any critical point u of I on E is a classical solution for (1.3) satisfying $u\in C^2(\mathbb{R},\mathbb{R}),u(t)\to 0$ and $\dot{u}(t)\to 0$ as $|t|\to\infty$.

Proof. We first show that $I\in C^1(E,\mathbb{R})$. Rewrite I as

$$I=I_1-I_2+I_3,$$

where

$$I_1:=\int_{\mathbb{R}}\left[\frac{c^2}{2}|\dot{u}(t)|^2+\frac{d_1}{2}|u(t)|^2\right]dt,\quad I_2:=\int_{\mathbb{R}}V(Au(t))dt,\quad I_3:=\int_{\mathbb{R}}g(u(t))dt.$$

It is easy to check that $I_1\in C^1(E,\mathbb{R})$ and

$$I_1'(u)v=\int_{\mathbb{R}}[c^2(\dot{u}(t),\dot{v}(t))+d_1(u(t),v(t))]dt,\quad v\in E.$$

Moreover, it follows from $(\mathscr{H}_1)$ and [10, Proposition 4.1] that $I_2\in C^1(E,\mathbb{R})$ and

$$I_2'(u)v=\int_{\mathbb{R}}(V'(Au(t)),Av(t))dt,\quad v\in E.$$

Therefore, it suffices to show that $I_3\in C^1(E,\mathbb{R})$. At first, we will see that

$$I_3'(u)v=\int_{\mathbb{R}}(g'(u(t)),v(t)))dt,\quad u,v\in E. \tag{2.7}$$

For any given $u\in E$, we define $J(u):E\to\mathbb{R}$ as follows

$$J(u)v=\int_{\mathbb{R}}(g'(u(t)),v(t))dt,\quad v\in E.$$

It is obvious that $J(u)$ is linear. Now we show that $J(u)$ is bounded. Indeed, for any given $u\in E$, by (2.1) and the Hölder inequality, one gets

$$\begin{aligned} |J(u)v|=|\int_{\mathbb{R}}(g'(u(t)),v(t))dt|&\le\int_{\mathbb{R}}|g'(u(t))||v(t)|dt\le\int_{\mathbb{R}}T|u(t)|^{\theta}|v(t)|dt \\ &\le T\|u\|_{2\theta}^{\theta}\|v\|_2\le TD_2D_{2\theta}^{\theta}\|u\|^{\theta}\|v\|. \end{aligned} \tag{2.8}$$

Moreover, for $u,v\in E$, by the mean value theorem, we have

$$\int_{\mathbb{R}}g(u(t)+v(t))dt-\int_{\mathbb{R}}g(u(t))dt=\int_{\mathbb{R}}(g'(u(t)+\psi_t v(t)),v(t))dt,$$

where $\psi_t \in (0,1)$. Therefore, by Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned}(2.9)\qquad &\int_{\mathbb{R}}(g'(u(t)+\psi_t v(t)),v(t))dt-\int_{\mathbb{R}}(g'(u(t)),v(t))dt\\ &=\int_{\mathbb{R}}(g'(u(t)+\psi_t(t)v(t))-g'(u(t)),v(t))dt\to 0,\end{aligned}$$

as $v\to 0$ in E. Combining (2.8) and (2.9), we see that (2.7) holds.

It remains to prove that I_3' is continuous. Suppose that $u\to u_0$ in E and note that

$$\begin{aligned}\sup_{\|v\|=1}|I_3'(u)v-I_3'(u_0)v| &= \sup_{\|v\|=1}|\int_{\mathbb{R}}(g'(u(t))-g'(u_0(t)),v(t))dt|\\ &\le \sup_{\|v\|=1}\|g'(u(\cdot))-g'(u_0(\cdot))\|_2\|v\|_2\le D_2\|g'(u(\cdot))-g'(u_0(\cdot))\|_2.\end{aligned}$$

By Lemma 2.2, we have $I_3'(u)v-I_3'(u_0)v\to 0$ as $u\to u_0$ uniformly with respect to v, which implies I_3' is continuous and $I\in C^1(E,\mathbb{R})$. Finally, similar to the discussion in the proof of [24, Lemma 3.1], we see that the critical points of I on E are classical solutions to equation (1.3) with $u\in C^2(\mathbb{R},\mathbb{R}), u(t)\to 0$ and $\dot{u}(t)\to 0$ as $|t|\to\infty$. This completes the proof of Lemma 2.3. ∎

Obviously, under the conditions of Theorem 1.2, the action functional I in (1.4) becomes

$$(2.10)\qquad I(u)=\int_{\mathbb{R}}\left[\frac{c^2}{2}|\dot{u}(t)|^2+\frac{d_1}{2}|u(t)|^2-V(Au(t))\right]dt.$$

Lemma 2.4. *Under the conditions of Theorem 1.2, $I\in C^1(E,\mathbb{R})$ and*

$$(2.11)\qquad I'(u)v=\int_{\mathbb{R}}[c^2(\dot{u}(t),\dot{v}(t))+d_1(u(t),v(t))-(V'(Au(t)),Av(t))]dt$$

for any $u,v\in E$, which yields

$$I'(u)u=\int_{\mathbb{R}}[c^2|\dot{u}(t)|^2+d_1|u(t)|^2]dt-\int_{\mathbb{R}}V'(Au(t)),Au(t))dt.$$

Moreover, any critical point u of I on E is a classical solution for (1.3) *satisfying $u\in C^2(\mathbb{R},\mathbb{R})$, $u(t)\to 0$ and $\dot{u}(t)\to 0$ as $|t|\to\infty$.*

Proof. By Remark 1.1, we have

$$V(u)=o(|u|^2),\quad u\to 0,$$

and A is a bounded operator. By [10, Proposition 4.1], we know that $I\in C^1(E,\mathbb{R})$ and (2.11) holds. Moreover, by [10, Lemma 4.2], then any critical point u of I on E is a classical solution for (1.3)) satisfying $u\in C^2(\mathbb{R},\mathbb{R})$, $u(t)\to 0$ and $\dot{u}(t)\to 0$ as $|t|\to\infty$. ∎

In the following, under the conditions of Theorem 1.3, the action functional I in (1.4) becomes

$$(2.12)\qquad I(u)=\int_{\mathbb{R}}\left[\frac{c^2}{2}|\dot{u}(t)|^2+\frac{d_1}{2}|u(t)|^2-V(Au(t))-g(u(t))\right]dt.$$

Lemma 2.5. *Under the conditions of Theorem 1.3, $I\in C^1(E,\mathbb{R})$ and*

$$\begin{aligned}(2.13)\qquad &I'(u)v\\ &=\int_{\mathbb{R}}[c^2(\dot{u}(t),\dot{v}(t))+d_1(u(t),v(t))-(V'(Au(t)),Av(t))]dt-\int_{\mathbb{R}}(g'(u(t)),v(t))dt\end{aligned}$$

for any $u, v \in E$. *which yields*

$$I'(u)u = \int_{\mathbb{R}} [c^2(\dot{u}(t), \dot{u}(t)) + d_1(u(t), u(t)) - (V'(Au(t)), Au(t))]dt - \int_{\mathbb{R}} (g'(u(t)), u(t))dt.$$

Moreover, and any critical point of I *on* E *is a classical solution for* (1.3) *satisfying* $u \in C^2(\mathbb{R}, \mathbb{R})$, $u(t) \to 0$ *and* $\dot{u}(t) \to 0$ *as* $|t| \to \infty$.

Proof. By Remark 1.1 and $(\mathscr{H}_5)$, we have

$$W(u) = o(|u|^2), \quad g(u) = o(|u|^2), \quad u \to 0,$$

and A is a bounded operator. By [10, Proposition 4.1], we know that $I \in C^1(E, \mathbb{R})$ and (2.13) holds. Moreover, by [10, Lemma 4.2], then any critical point u of I on E is a classical solution for (1.3) satisfying $u \in C^2(\mathbb{R}, \mathbb{R}), u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$. ∎

By the assumption $(\mathscr{H}_1)$, we will give the following lemma, which is similar to the proof of [8, Fact (2.1)].

Lemma 2.6. *The following inequalities hold for assumption* $(\mathscr{H}_1)$:

$$W(u) \le W\left(\frac{u}{|u|}\right)|u|^\mu \quad \text{if} \quad 0 < |u| \le 1, \tag{2.14}$$

$$W(u) \ge W\left(\frac{u}{|u|}\right)|u|^\mu \quad \text{if} \quad |u| \ge 1. \tag{2.15}$$

To prove this fact, it suffices to show that for every $u \neq 0$, *the map* $(0, +\infty) \ni \zeta \to W(\zeta^{-1}u)\zeta^\mu$ *is nonincreasing. It is an immediate consequence of* $(\mathscr{H}_1)$.

In order to obtain the nontrivial critical points of the functional corresponding to (1.3), Cerami sequence is employed to instead of (PS) sequence in our situation. A sequence $\{u_j\}_{j \in \mathbb{N}} \subset E$ is called a Cerami sequence at level a if $I(u_j) \to a$ and

$$(1 + \|u_j\|_E)\|I'(u_j)\|_{E^*} \to 0, \quad j \to +\infty,$$

where E^* is the dual space of E.

Theorem 2.1 (Mountain Pass Lemma)**.** [4] *Let* E *be a real Banach space with its dual space* E^* *and suppose that* $I \in C^1(E, \mathbb{R})$ *satisfies*

$$\max\{I(0), I(\mathrm{e})\} \le \eta < \alpha \le \inf_{\|u\| = \rho} I(u),$$

for some $\eta < \alpha, \rho > 0$ *and* $\mathrm{e} \in E$ *with* $\|\mathrm{e}\| > \rho$. *Let* $a \ge \alpha$ *be characterized by*

$$a = \inf_{p \in \Gamma} \max_{s \in [0,1]} I(p(s)),$$

where $\Gamma = \{p \in C([0,1], E) | p(0) = 0, p(1) = \mathrm{e}\}$ *is the set of continuous paths joining* 0 *and* e, *then there exists* $\{u_k\} \subset E$ *such that* $I(u_k) \to a$ *and*

$$(1 + \|u_k\|_E)\|I'(u_k)\|_{E^*} \to 0, \quad k \to +\infty.$$

3. Proof of main results

Proof of Theorem 1.1. Under the conditions of Theorem 1.1, we are going to complete the proof for the result in four steps.

Step 1. We show that there exist constants $\rho, \alpha > 0$ such that I satisfies the assumptions of Theorem 2.1 with these constants. In view of Lemma 2.1, if $0 < \|u\| \leq 1$, then $0 < \|Au\|_\infty \leq 1$. It follows from (2.14) that

$$\int_{\mathbb{R}} W(Au(t))dt \leq \int_{\{t\in\mathbb{R}:|Au(t)|\neq 0\}} W\left(\frac{Au(t)}{|Au(t)|}\right)|Au(t)|^\mu dt \leq M\int_{\mathbb{R}} |Au(t)|^2 dt \leq M\|u\|^2.$$

This, together with (2.5), implies

$$\begin{aligned} I(u) &\geq \int_{\mathbb{R}} \left(\frac{c^2}{2}|\dot{u}(t)|^2 + \frac{d_1}{2}|u(t)|^2\right)dt - M\|u\|^2 - \int_{\mathbb{R}} \frac{c_0^2}{2}|Au(t)|^2 dt \\ &\geq \frac{1}{2}(\min\{c^2, d_1\} - c_0^2 - 2M)\|u\|^2 \geq \frac{1}{4}\|u\|^2. \end{aligned} \tag{3.1}$$

Set $\rho = 1$, thus, it follows from (3.1) that

$$\inf_{\|u\|=\rho} I(u) \geq \alpha := \frac{1}{4}\rho^2.$$

Step 2. We shall show that there exists $\mathrm{e} \in E$ such that $\|\mathrm{e}\| > \rho$ and $I(\mathrm{e}) \leq 0$. In view of Lemma 2.1, if $\|Au\|_\infty \geq 1$ then $\|u\| \geq 1$. Choosing $v \in E\setminus\{0\}$ such that $\|Av\|_\infty > 0$ and $\text{meas}\{t \in \mathbb{R} : |Av(t)| \geq b_1\} \geq b_2$, which b_1 and b_2 are positive constants. By $(\mathscr{H}_2)$, (2.15) and (2.5), we have

$$\begin{aligned} &I(\lambda v) \\ &\leq \frac{\lambda^2}{2}\max\{c^2, d_1\}\|v\|^2 - \int_{\mathbb{R}} W(A\lambda v(t))dt - \int_{\mathbb{R}} \frac{c_0^2}{2}(A\lambda v(t))^2 dt + \int_{\mathbb{R}} g(\lambda v(t))dt \\ &\leq \frac{\lambda^2}{2}\max\{c^2, d_1\}\|v\|^2 - \lambda^\mu \int_{\{t\in\mathbb{R}:|Av(t)|\geq b_1\}} W\left(\frac{A\lambda v(t)}{|A\lambda v(t)|}\right)|Av(t)|^\mu dt + \int_{\mathbb{R}} \frac{T}{2}|\lambda v(t)|^{\theta+1}dt \\ &\leq \frac{\lambda^2}{2}\max\{c^2, d_1\}\|v\|^2 - \lambda^\mu M' b_2 b_1^\mu + \frac{T\lambda^{\theta+1}}{2}D_{\theta+1}^{\theta+1}\|v\|^{\theta+1} \end{aligned}$$

for $\lambda \geq 1/b_1$, $M' := \min\{W(u) : |u| = 1\}$. Since $\mu > 2$ and $\mu > \theta + 1$, there exists $\mathrm{e} := \lambda_0 v \in E$ with $\lambda_0 \geq 1/b_1$ such that $\|\mathrm{e}\| > \rho$ and $I(\mathrm{e}) \leq 0$. In addition, it is clear that $I(0) = 0$.

Step 3. Based on Steps 1 and 2, Theorem 2.1 implies that there is a sequence $\{u_k\} \subset E$ such that $I(u_k) \to a \geq \alpha > 0$ and

$$(1 + \|u_k\|)\|I'(u_k)\|_{E^*} \to 0, \quad k \to +\infty. \tag{3.2}$$

We now prove the sequence $\{u_k\}$ is bounded. It follows from (3.2), (2.5) and (2.6) that there exist positive constants C_1 and C_2 such that

$$\begin{aligned} C_1 &\geq 2I(u_k) - (I'(u_k), u_k) \\ &= \int_{\mathbb{R}} [(W'(Au_k(t)), Au_k(t)) - 2W(Au_k(t))]dt + \int_{\mathbb{R}} [2g(u_k(t)) - (g'(u_k(t)), u_k(t))]dt \end{aligned} \tag{3.3}$$

and

$$I(u_k(t)) \leq C_2. \tag{3.4}$$

Then it follows from (2.5), (3.3), (3.4), Lemma 2.1, $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$ that

$$\begin{aligned}\frac{1}{2}(\min\{c^2,d_1\}-c_0^2)\|u_k\|^2 &\le I(u_k(t))+\int_{\mathbb{R}}W(Au_k(t))dt-\int_{\mathbb{R}}g(u_k(t))dt\\ &\le C_2+\frac{1}{\mu-2}\int_{\mathbb{R}}[(W'(Au_k(t)),Au_k(t))-2W(Au_k(t))]dt\\ &\quad+\frac{1}{\mu-2}\int_{\mathbb{R}}[2g(u_k(t))-(g'(u_k(t)),u_k(t))]dt\\ &\le C_2+\frac{C_1}{\mu-2}.\end{aligned}$$

Hence, it is easy to see that $\{\|u_k\|\}$ is bounded. So we may assume that, up to a subsequence, $u_k \rightharpoonup u$ weakly in E as $k\to+\infty$ for some $u\in E$.

Step 4. For any fixed $v\in C_0^\infty(\mathbb{R},\mathbb{R})$, assume that some $R>0$ such that $\operatorname{supp}(v)\subset[-R,R]$. It follows that

$$I'(u_k)v=\int_{-R}^{R}[c^2(\dot{u_k}(t),\dot{v}(t))+d_1(u_k(t),v(t))-(V'(Au_k(t)),Av(t))+(g'(u_k(t)),v(t))]dt.$$

It is obvious that the operator defined by $S:E\to H^1([-R-1,R+1],\mathbb{R}):u\to u|_{[-R-1,R+1]}$ is a linear continuous map. Therefore, $u_k\rightharpoonup u$ in $H^1([-R-1,R+1],\mathbb{R})$. Sobolev's theorem (see [11]) implies that $u_k\to u$ uniformly on $[-R-1,R+1]$. So, we have

$$\int_{-R}^{R}[c^2(\dot{u_k}(t),\dot{v}(t))+d_1(u_k(t),v(t))dt \;\to\; \int_{-R}^{R}[c^2(\dot{u}(t),\dot{v}(t))+d_1(u(t),v(t))dt$$

$$\int_{-R}^{R}[-(V'(Au_k(t)),Av(t))+(g'(u_k(t)),v(t))]dt \;\to\; \int_{-R}^{R}[-V'(Au(t)),Av(t))+(g'(u(t)),v(t))]dt$$

as $k\to+\infty$. Thus $I'(u_k)v\to I'(u)v$ as $k\to+\infty$, and, in consequence, $I'(u)v=0$. Since $C_0^\infty(\mathbb{R},\mathbb{R})$ is dense in E, we get $I'(u)=0$.

We are now in the position to prove that u is a nontrivial solution. Since $u_k\to u$ in $L^\infty_{loc}(\mathbb{R},\mathbb{R})$, $u_k\to u$ in $L^2([-R',R'],\mathbb{R})$ for all $0<R'<+\infty$. Hence, it suffices to show there is a $R'>0$ such that $u_k\nrightarrow 0$ in $L^2([-R',R'],\mathbb{R})$. We proceed arguing by contradiction. Assuming that $u_k\to 0$ in $L^2([-R',R'],\mathbb{R})$ for all $R'>0$. Then there exists a constant $m>0$, such that

$$\limsup_{k\to\infty}\|u_k\|^2_{L^2(\mathbb{R},\mathbb{R})}\le\frac{m}{d_1}. \tag{3.5}$$

Indeed, we let

$$\eta(u_k)=\left(\int_{\mathbb{R}}[c^2|\dot{u}_k(t)|^2+d_1|u_k(t)|^2]dt\right)^{\frac{1}{2}}.$$

Then it follows from $(\mathscr{H}_1)$, $(\mathscr{H}_2)$, Lemma 2.1, (2.5) and (2.6) that

$$\left(1-\frac{2}{\mu}\right)\int_{\mathbb{R}}d_1u_k^2dt\le 2I(u_k)-\frac{2}{\mu}(I'(u_k),u_k).$$

Since (3.4) and (3.2) are satisfied, the above inequality implies that $\{\eta(u_k)\}$ is bounded independently of d_1 . Set $m=\sup_k\eta^2(u_k)$. Consequently,

$$\|u_k\|^2_{L^2(\mathbb{R},\mathbb{R})}=\int_{-R'}^{R'}|u_k(t)|^2dt+\int_{\mathbb{R}\setminus[-R',R']}|u_k(t)|^2dt$$

$$= \int_{-R'}^{R'} |u_k(t)|^2 dt + \frac{1}{d_1} \int_{\mathbb{R}\setminus[-R',R']} d_1 |u_k(t)|^2 dt \le \int_{-R'}^{R'} |u_k(t)|^2 dt + \frac{m}{d_1}.$$

By letting $k \to \infty$, (3.5) holds. Set

$$\varpi = \sup_k \|u_k\|_\infty.$$

It follows from $(\mathscr{H}_2)$ that $|g'(u)| \le T|u|$ for all $|u| \le 1$. Moreover, we know that $|g(u)| \le T|u|^2, \forall |u| \le 1$. Therefore,

$$|2g(u) - (g'(u),u)| \le 2|g(u)| + |g'(u)||u| \le 3T|u|^2, \quad \forall |u| \le 1.$$

If $\varpi > 1$, then

$$|2g(u) - (g'(u),u)| \le 2|g(u)| + |g'(u)||u| \le (2+\varpi)M_2|u|^2$$

for $1 < |u| \le \varpi$, where $M_2 = \max_{|u|\le\varpi}(|g(u)| + |g'(u)|)$. These lead to

$$|2g(u) - (g'(u),u)| \le M_3|u|^2 \tag{3.6}$$

for $|u| \le \varpi$, where $M_3 = 3T + (2+\varpi)M_2$.

Since $W'(u) = o(u)$ as $|u| \to 0$, there is $\rho_0 > 0$ such that

$$|(W'(u),u) - 2W(u)| \le |W'(u)||u| + |2W(u)| \le 3|u|^2$$

for $|u| \le \rho_0$. If $2\varpi > \rho_0$, we let $M_4 = \max_{|u|\le 2\varpi} |W'(u)| + |W(u)|$. Hence, one has

$$|(W'(u),u) - 2W(u)| \le |W'(u)||u| + |2W(u)| \le 2M_4(\varpi+1)\rho_0^{-2}|u|^2$$

for $\rho_0 < |u| \le 2\varpi$. Therefore, there exists $M_5 > 0$ such that

$$|(W'(u),u) - 2W(u)| \le M_5|u|^2, \quad |u| \le 2\varpi, \tag{3.7}$$

where $M_5 = 3 + 2M_4(\varpi+1)$. By (3.2), (3.3), (3.5), (3.6) and (3.7), one has

$$\begin{aligned} 2a &= \lim_{k\to\infty} 2I(u_k) - (I'(u_k),u_k) \\ &= \lim_{k\to\infty} \int_{\mathbb{R}} [(W'(Au_k(t)),Au_k(t)) - 2W(Au_k(t))]dt + \int_{\mathbb{R}} [2g(u_k(t)) - (g'(u_k(t)),u_k(t))]dt \\ &\le (M_3 + 4M_5) \limsup_{k\to\infty} \|u_k\|^2_{L^2(\mathbb{R},\mathbb{R})} \le \frac{m(M_3+4M_5)}{d_1}. \end{aligned}$$

This contradicts d_1 is large enough. Hence, there exists a $R' > 0$ such that $u_k \nrightarrow 0$ in $L^2([-R',R'],\mathbb{R})$, that is, $u \ne 0$.

So we know that I possesses at least one nontrivial critical point. Therefore, (1.3) possesses at least one nontrivial homoclinic solution. The proof of Theorem 1.1 is completed. ∎

In order to prove Theorem 1.2, we firstly give Lemmas 3.1 and 3.2, which ensure that the functional I has what is called the mountain pass geometry.

Lemma 3.1. *Under the conditions of Theorem 1.2, there exist* $\rho > 0$ *and* $\alpha > 0$ *such that*

$$\inf\{I(u) : u \in E \text{ with } \|u\| = \rho\} > \alpha.$$

Proof. Choose $\|u\| = r_\xi$, where $r_\xi > 0$ is defined in (1.8). Then, by Lemma 2.1, we have $\|Au\|_\infty \le r_\xi$. It follows from Lemma 2.1, (2.10) and (1.8) that

$$\begin{aligned}
I(u) &= \int_{\mathbb{R}} \left[\frac{c^2}{2}|\dot{u}(t)|^2 + \frac{d_1}{2}|u(t)|^2\right] dt - \int_{\mathbb{R}} V(Au(t))dt \ge \frac{1}{2}\min\{c^2, d_1\}\|u\|^2 - \int_{\mathbb{R}} V(Au(t))dt \\
&\ge \frac{1}{2}\min\{c^2, d_1\}\|u\|^2 - \xi\|Au\|_2^2 \ge \frac{1}{2}\min\{c^2, d_1\}\|u\|^2 - \xi\|u\|^2 \\
&= [\frac{1}{2}\min\{c^2, d_1\} - \xi]\|u\|^2.
\end{aligned}$$

Hence, by fixing $\xi \in (0, (1/2)\min\{c^2, d_1\})$ and letting $\|u\| = \rho := r_\xi > 0$ small enough, it is easy to see that there is $\alpha > 0$ such that the conclusion of this lemma holds. The proof is completed. ∎

Lemma 3.2. *Under the conditions of Theorem 1.2, then there exists some $\|\mathrm{e}\| > \rho$ such that $I(\mathrm{e}) \le 0$.*

Proof. Define

$$R(u) = V(u) - \frac{1}{2}d|u|^2.$$

It follows from (1.7) and ($\mathscr{H}_3$) that

$$R(u) \le (C+d)|u|^2, \quad \lim_{|u|\to+\infty} \frac{R(u)}{|u|^2} = 0. \tag{3.8}$$

Define $E_1 := \{ae^{-|t|} : a \in \mathbb{R}\} \subset E$, which is a finite-dimensional subspace of E. An easy computation shows that

$$\|u\|^2 \le \frac{e}{e-2}\|Au\|_2^2, \quad u \in E_1. \tag{3.9}$$

In the following, we show that, for fixed $u \in E_1$ with $\|u\| = 1, I(hu) \to -\infty$ as $h \to +\infty$. Assume on the contrary that for some sequence $\{h_k\}$ with $h_k \to +\infty$ as $k \to +\infty$, there exists $M > 0$ such that $I(h_k u) \ge -M$ for all k. By (2.10), we have

$$\frac{-M}{h_k^2} \le \frac{I(h_k u)}{h_k^2} = \int_{\mathbb{R}} \left[\frac{c^2}{2}|\dot{u}(t)|^2 + \frac{d_1}{2}u^2(t)\right] dt - \int_{\mathbb{R}} \frac{V(A(h_k u(t)))}{h_k^2} dt. \tag{3.10}$$

It is clear that there exists some constant $\rho \in [\min\{c^2, d_1\}, \max\{c^2, d_1\}]$ such that

$$\int_{\mathbb{R}} \left[\frac{c^2}{2}|\dot{u}(t)|^2 + \frac{d_1}{2}u^2(t)\right] dt = \frac{1}{2}\rho\|u\|^2.$$

Combining the above equality with (3.10), we get

$$\frac{-M}{h_k^2} \le \frac{I(h_k u)}{h_k^2} = \frac{\rho}{2} - \frac{1}{2}\int_{\mathbb{R}} d|Au(t)|^2 dt - \int_{\mathbb{R}} \frac{R(A(h_k u(t)))}{h_k^2} dt. \tag{3.11}$$

Since $u \in C(\mathbb{R}, \mathbb{R})$, by (3.8) and Lemma 2.1 we have

$$\frac{R(A(h_k u(t)))}{h_k^2} \le (C+d)|Au(t)|^2$$

and

$$\frac{|R(A(h_k u(t)))|}{4h_k^2} \le \frac{|R(A(h_k u(t)))|}{h_k^2\|Au\|^2} \le \frac{D_\infty^2|R(A(h_k u(t)))|}{h_k^2\|Au\|_\infty^2} \le \frac{D_\infty^2|R(A(h_k u(t)))|}{|A(h_k u(t))|^2} \to 0$$

as $k \to +\infty$, which yields that

$$\frac{R(A(h_k u(t)))}{h_k^2} \to 0, \quad k \to +\infty.$$

It follows from the Lebesgue dominated convergence theorem that

$$\int_{\mathbb{R}} \frac{R(A(h_k u(t)))}{h_k^2} dt \to 0, \quad k \to +\infty.$$

Hence, by (3.9), (3.11) and $(\mathscr{H}_3)$, we have

$$o(1) = \frac{-M}{h_k^2} \le \frac{\rho}{2} - d\frac{e-2}{2e} < 0, \quad k \to +\infty,$$

which is a contradiction. The proof is completed. ∎

Based on Lemma 3.1 and 3.2, Theorem 2.1 implies that, under the conditions of Theorem 1.2, there is a sequence $\{u_k\} \subset E$ such that $I(u_k) \to a \ge \alpha > 0$ and

$$(1 + \|u_k\|)\|I'(u_k)\|_{E^*} \to 0, \quad k \to +\infty. \tag{3.12}$$

Lemma 3.3. *Assume that the conditions of Theorem 1.2 are satisfied, Then the sequence* $\{u_k\}$ *given in* (3.12) *is bounded.*

Proof. It follows from (2.10), (2.11) and (3.12) that there exists $C_3 > 0$ such that

$$C_3 \ge I(u_k) - \frac{1}{2}I'(u_k)u_k = \int_{\mathbb{R}} K(Au_k(t))dt.$$

This implies that

$$C_3 \ge \int_{\Psi_k(0,c_1)} K(Au_k(t))dt + \int_{\Psi_k(c_1,c_2)} K(Au_k(t))dt + \int_{\Psi_k(c_2,+\infty)} K(Au_k(t))dt, \tag{3.13}$$

where $\Psi_k(c_1,c_2) = \{t \in \mathbb{R} : c_1 \le |Au_k(t)| \le c_2\}$ for $0 < c_1 < c_2$. In fact, assume on the contrary that $\|u_k\| \to +\infty$ as $k \to +\infty$. Obviously, $\eta(u_k) \to +\infty$ as $k \to +\infty$. Set $v_k = u_k/\eta(u_k)$, then $\|v_k\| \in [1/\sqrt{\max\{c^2,d_1\}}, 1/\sqrt{\min\{c^2,d_1\}}]$. Note that, by (2.11) and (3.12),

$$(1+\|u_k\|)\|I'(u_k)\|_{E^*} \ge |I'(u_k)u_k| = \eta^2(u_k)|1 - \int_{\mathbb{R}} \frac{(V'(Au_k(t)), Au_k(t))}{\eta^2(u_k)}dt| \ge 0,$$

which yields that

$$\int_{\mathbb{R}} \frac{(V'(Au_k(t)), Av_k(t))}{|Au_k(t)|}|Av_k(t)|dt = \int_{\mathbb{R}} \frac{(V'(Au_k(t)), Au_k(t))}{\eta^2(u_k)}dt \to 1, \quad k \to +\infty. \tag{3.14}$$

Let $0 < \varepsilon < 1/3$. By $\mathscr{H}_1'$, there exists$\alpha_\varepsilon > 0$ such that

$$|V'(Au)| < \varepsilon \min\{c^2, d_1\}|Au|, \quad |Au| \le \alpha_\varepsilon.$$

Consequently, since $\|v_k\| \le 1/\sqrt{\min\{c^2,d_1\}}$, by Lemma 2.1 we have

$$\int_{\Psi_k(0,\alpha_\varepsilon)} \frac{|V'(Au_k(t))|}{|Au_k(t)|}|Av_k(t)|^2 dt \le \int_{\Psi_k(0,\alpha_\varepsilon)} \varepsilon \min\{c^2,d_1\}|Av_k(t)|^2 dt \le \varepsilon. \tag{3.15}$$

For $r \ge 0$, set

$$\phi(r) := \inf\{K(Au(t)) | t \in \mathbb{R} \text{ and } |Au(t)| \ge r\}.$$

By $(\mathscr{H}_4)$, $\phi(r) \to +\infty$ as $r \to +\infty$. According to (3.13), we have

$$\text{meas}(\Psi_k(\beta,+\infty)) \le \frac{C_3}{\phi(\beta)} \to 0, \quad \beta \to +\infty. \tag{3.16}$$

Therefore, we can take β_ε large enough such that

$$\int_{\Psi_k(\beta_\varepsilon,+\infty)} |Av_k(t)|^2 dt < \frac{\varepsilon}{C}.$$

Hence, by $(\mathscr{H}_1')$ we get

$$\int_{\Psi_k(\beta_\varepsilon,+\infty)} \frac{|V'(Au_k(t))|}{|Au_k(t)|} |Av_k(t)|^2 dt \le C \int_{\Psi_k(\beta_\varepsilon,+\infty)} |Av_k(t)|^2 dt < \varepsilon. \tag{3.17}$$

Next, we set

$$c_\varepsilon := \inf\left\{ \frac{K(Au(t))}{|Au(t)|^2} : t \in \mathbb{R} \text{ with } \alpha_\varepsilon \le |Au(t)| \le \beta_\varepsilon \right\}$$

and

$$d_\varepsilon := \max\left\{ \frac{V'(Au(t))}{|Au(t)|} : t \in \mathbb{R} \text{ with } \alpha_\varepsilon \le |Au(t)| \le \beta_\varepsilon \right\}.$$

By $(\mathscr{H}_4)$, we know that $c_\varepsilon > 0$ and $K(Au_k(t)) \ge c_\varepsilon |Au_k(t)|^2$ for all $t \in \Psi_k(\alpha_\varepsilon, \beta_\varepsilon)$. It follows from (3.13) that

$$\int_{\Psi_k(\alpha_\varepsilon,\beta_\varepsilon)} |Av_k(t)|^2 dt = \frac{1}{\eta^2(u_k)} \int_{\Psi_k(\alpha_\varepsilon,\beta_\varepsilon)} |Au_k(t)|^2 dt \le \frac{C_3}{c_\varepsilon \eta^2(u_k)} \to 0, \quad k \to +\infty,$$

which implies that

$$\int_{\Psi_k(\alpha_\varepsilon,\beta_\varepsilon)} \frac{|V'(Au_k(t))|}{|Au_k(t)|} |Av_k(t)|^2 dt \le d_\varepsilon \int_{\Psi_k(\alpha_\varepsilon,\beta_\varepsilon)} |Av_k(t)|^2 dt \to 0, \quad k \to +\infty.$$

Thus there is k_0 such that

$$\int_{\Psi_k(\alpha_\varepsilon,\beta_\varepsilon)} \frac{|V'(Au_k(t))|}{|Au_k(t)|} |Av_k(t)|^2 dt \le \varepsilon, \quad k \ge k_0. \tag{3.18}$$

Now the combination of (3.15), (3.17), and (3.18) implies that for $k \ge k_0$

$$\int_{\mathbb{R}} \frac{(V'(Au_k(t)), Av_k(t))}{|Au_k(t)|} |Av_k(t)| dt \le \int_{\mathbb{R}} \frac{|V'(Au_k(t))|}{|Au_k(t)|} |Av_k(t)|^2 dt \le 3\varepsilon,$$

which contradicts (3.14). The proof is completed. ∎

Now we are in the position to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Obviously, $I(0) = 0$. Similar to Step 4 in the proof of Theorem 1.1, we know that $u_k \rightharpoonup u$ in E as $k \to +\infty$ for some $u \in E$ and $I'(u) = 0$.

We claim that u is a nontrivial solution. To complete the proof of this conclusion, we at first show that $J_1'(u) := \int_{\mathbb{R}} V(Au(t))dt$ is weakly continuous, that is, if $u_n \rightharpoonup u_*$ weakly in E, then $J_1'(u_n) \to J_1'(u_*)$ in E^*. To this aim, any choose $\varphi \in E$, for fixed $\iota > 0$ and let $R_\iota > 0$ be so large that

$$\int_{|t|>R_\iota} \varphi^2(t) dt < \iota^2.$$

Since $u_n \rightharpoonup u_*$ weakly in E, u_n is bounded in E, i.e., there exists $R_1 > 0$ such that $\|u_n\|_{L^2(\mathbb{R},\mathbb{R})} \le R_1$ and $\|u_*\|_{L^2(\mathbb{R},\mathbb{R})} \le R_1$. Then by $(\mathscr{H}_1')$ and the Hölder inequality, we have

$$|(J_1'(u_n) - J_1'(u_*))\varphi|$$

$$
\begin{aligned}
&= |\int_{\mathbb{R}} (V'(Au_n(t)) - V'(Au_*(t)), A\varphi(t))dt| \\
&\leq |\int_{-R_\iota-1}^{R_\iota+1} (V'(Au_n(t)) - V'(Au_*(t)), A\varphi(t))dt| \\
&\quad + |\int_{|t|\geq R_\iota+1} (V'(Au_n(t)) - V'(Au_*(t)), A\varphi(t))dt| \\
&\leq \int_{-R_\iota-1}^{R_\iota+1} |V'(Au_n(t)) - V'(Au_*(t))||A\varphi(t)|dt + C\int_{|t|\geq R_\iota+1} (|Au_n(t)| + |Au_*(t)|)|A\varphi(t)|dt \\
&\leq \int_{-R_\iota-1}^{R_\iota+1} |V'(Au_n(t)) - V'(Au_*(t))||A\varphi(t))|dt + 2\iota C(\int_{|t|\geq R_\iota+1} (|u_n(t)| + |Au_*(t)|)^2 dt)^{\frac{1}{2}}.
\end{aligned}
$$

The first integral tends to zero as $n \to +\infty$, because $u_n \to u_*$ strongly in $L^\infty_{loc}(\mathbb{R})$; the second integral is bounded independently of n. The fact that ι is arbitrary concludes J_1' is weakly continuous.

In what follows, we show that u is nontrivial. Obviously, $I'(u_k) \to 0$ as $u_k \rightharpoonup u$ and $I'(u) = 0$. By the definition of I and I', we obtain that

$$
\begin{aligned}
(I'(u_k) - I'(u), u_k - u) &= \int_{\mathbb{R}} [c^2|u_k(t) - u(t)|^2 + d_1|u_k(t) - u(t)|^2]dt \\
&\quad - \int_{\mathbb{R}} (V'(Au_k(t)) - V'(Au(t)), Au_k(t) - Au(t))dt \\
&\geq \min\{c^2, d_1\}\|u_k - u\|^2 - (J_1'(u_k) - J_1'(u), u_k - u).
\end{aligned}
$$

Combining this with the compactness of J_1', we deduce that $\|u_k - u\| \to 0$ as $k \to \infty$, that is $u_k \to u$ in E, So $I(u) \geq \alpha > 0$. Therefore, u is nontrivial.

Combining the above with Lemmas 3.1, 3.2, 3.3, and Theorem 2.1, we have that I possesses at least one nontrivial critical point. Therefore, (1.3) possesses at least one nontrivial homoclinic solution. The proof is completed. ∎

Proof of Theorem 1.3. Similar to the proof of Theorem 1.1, the proof of Theorem 1.3 will be carried out in four steps.

Step 1. We now show that there exist constants $\rho, \alpha > 0$ such that I satisfies the assumptions of Theorem 2.1 with these constants. In view of $(\mathscr{H}_5)$, for any given $\sigma > 0$, there exists $r_\sigma > 0$ such that

$$
|g(u)| \leq \sigma|u|^2, \quad |u| \leq r_\sigma. \tag{3.19}
$$

Since $W'(u) = o(u)$ as $u \to 0$, for the above σ, there exists $s_\sigma > 0$ such that

$$
|W(u)| \leq \sigma|u|^2, \quad |u| \leq s_\sigma. \tag{3.20}
$$

Combining (3.19) with (3.20), we get that

$$
|W(u)| \leq \sigma|u|^2, |g(u)| \leq \sigma|u|^2, \quad |u| \leq \min\{r_\sigma, s_\sigma\}. \tag{3.21}
$$

Choose $\|u\| = \min\{r_\sigma, s_\sigma\}/(1 + D_\infty)$. Then, by Lemma 2.1 and (2.1), we have $\|u\|_\infty \leq \min\{r_\sigma, s_\sigma\}, \|Au\|_\infty \leq \min\{r_\sigma, s_\sigma\}$. It follows from Lemma 2.1, (2.12) and (3.21) that

$$
I(u) = \int_{\mathbb{R}} \left[\frac{c^2}{2}|\dot{u}(t)|^2 + \frac{d_1}{2}u^2\right] dt - \int_{\mathbb{R}} V(Au(t))dt - \int_{\mathbb{R}} g(u(t))dt
$$

$$\geq \frac{1}{2}\min\{c^2,d_1\}\|u\|^2 - \int_{\mathbb{R}}\left[\frac{c_0^2}{2}|Au(t)|^2 + W(A(u(t)))\right]dt - \sigma\int_{\mathbb{R}}|u(t)|^2dt$$
$$\geq \frac{1}{2}(\min\{c^2,d_1\}-c_0^2)\|u\|^2 - \sigma\|Au\|_2^2 - \sigma\|u\|_2^2$$
$$\geq \frac{1}{2}(\min\{c^2,d_1\}-c_0^2)\|u\|^2 - (1+D_2^2)\sigma\|u\|^2$$
$$= \left[\frac{1}{2}(\min\{c^2,d_1\}-c_0^2) - (1+D_2^2)\sigma\right]\|u\|^2.$$

Hence, by fixing $\sigma \in (0, (\min\{c^2,d_1\}-c_0^2)/2(1+D_2^2))$ and letting $\|u\| = \rho := \min\{r_\sigma, s_\sigma\}/(1+D_\infty) > 0$ small enough, it is easy to see that there is $\alpha > 0$ such that

$$\inf_{\|u\|=\rho} I(u) \geq \alpha.$$

Step 2. We show that there exists $\mathrm{e} \in E$ such that $\|\mathrm{e}\| > \rho$ and $I(\mathrm{e}) \leq 0$. Obviously, $I(0) = 0$. Choosing $v \in E\setminus\{0\}$, we have $\|Av\|_\infty > 0$ and $\mathrm{meas}\{t \in \mathbb{R} : |Av(t)| \geq b_3\} \geq b_4$, which b_3 and b_4 are positive constants. By Lemma 2.6 and (2.12), we can get the following inequality

$$I(\lambda v) \leq \frac{\lambda^2}{2}\max\{c^2,d_1\}\|v\|^2 - \int_{\mathbb{R}} W(A\lambda v(t))dt - \int_{\mathbb{R}}\frac{c_0^2}{2}(A\lambda v(t))^2dt - \int_{\mathbb{R}} g(\lambda v(t))dt$$
$$\leq \frac{\lambda^2}{2}\max\{c^2,d_1\}\|v\|^2 - \lambda^\mu\int_{\{t\in\mathbb{R}\,|Av(t)|\geq b_3\}} W\left(\frac{A\lambda v(t)}{|A\lambda v(t)|}\right)|Av(t)|^\mu dt + \frac{c_0^2\lambda^2}{2}\|v\|^2$$
$$\leq \frac{\lambda^2}{2}\max\{c^2,d_1\}\|v\|^2 - \lambda^\mu M' b_4 b_3^\mu + \frac{c_0^2\lambda^2}{2}\|v\|^2$$

for $\lambda > 1/b_3, \lambda \in \mathbb{R}$. Since $\mu > 2$, there exists $\mathrm{e} := \lambda_1 v \in E, \lambda_1 > 1/b_3$ such that $\|\mathrm{e}\| \geq \rho$ and $I(\mathrm{e}) \leq 0$.

Step 3. Based on Steps 1 and 2, Theorem 2.1 implies that there is a sequence $\{u_k\} \subset E$ such that $I(u_k) \to a \geq \alpha > 0$ and

$$(1+\|u_k\|)\|I'(u_k)\|_{E^*} \to 0, \quad k \to +\infty. \tag{3.22}$$

We now prove the sequence $\{u_k\}$ is bounded. It follows from (3.22) that there exist positive constants C_4 and C_5 such that

$$C_4 \geq 2I(u_k) - (I'(u_k), u_k)$$
$$= \int_{\mathbb{R}}[(W'(Au_k(t)), Au_k(t)) - 2W(Au_k(t))]dt + \int_{\mathbb{R}}[(g'(u_k(t)), u_k(t)) - 2g(u_k(t))]dt \tag{3.23}$$

and

$$I(u_k(t)) \leq C_5. \tag{3.24}$$

It follows from the conditions of Theorem 1.3 that, for any given $\sigma' > 0$, (3.21) holds. By $(\mathscr{H}_5)$

$$(g'(u), u) \geq 2g(u) \geq 0, \tag{3.25}$$

and

$$g(u) \leq (\kappa + \gamma|u|^{\gamma_1})[(g'(u), u) - 2g(u)], \quad |u| \geq \min\{r_{\sigma'}, s_{\sigma'}\}. \tag{3.26}$$

Then it follows from (2.12), (3.23), (3.24), (3.25), (3.26), (3.21) and Lemma 2.1 that

(3.27)
$$\begin{aligned}
&\frac{1}{2}(\min\{c^2,d_1\}-c_0^2)\|u_k\|^2\\
&\le \frac{1}{2}\eta^2(u_k)-\frac{1}{2}c_0^2\|u_k\|^2 \le I(u_k)+\int_{\mathbb{R}} W(Au_k(t))dt+\int_{\mathbb{R}} g(u_k(t))dt\\
&\le C_5+\int_{\{t\in\mathbb{R}:|Au_k(t)|\le\min\{r_{\sigma'},s_{\sigma'}\}\}} W(Au_k(t))dt+\int_{\{t\in\mathbb{R}:|Au_k(t)|\ge\min\{r_{\sigma'},s_{\sigma'}\}\}} W(Au_k(t))dt\\
&\quad+\int_{\{t\in\mathbb{R}:|u_k(t)|\ge\min\{r_{\sigma'},s_{\sigma'}\}\}} g(u_k(t))dt+\int_{\{t\in\mathbb{R}:|u_k(t)|\le\min\{r_{\sigma'},s_{\sigma'}\}\}} g(u_k(t))dt\\
&\le C_5+(1+D_2^2)\sigma'\|u_k\|^2+\frac{1}{\mu-2}\int_{\mathbb{R}}[(W'(Au_k(t)),Au_k(t))-2W(Au_k(t))]dt\\
&\quad+\int_{\mathbb{R}}(\kappa+\gamma|u_k(t)|^{\gamma_1})[(g'(u_k(t)),u(t))-2g(u_k(t))]dt\\
&\le C_5+\max\left\{\frac{1}{\mu-2},(\kappa+\gamma\|u_k\|_\infty^{\gamma_1})\right\}\left\{\int_{\mathbb{R}}[(W'(Au_k(t)),Au_k(t))-2W(Au_k(t))]dt\right.\\
&\quad\left.+\int_{\mathbb{R}}[(g'(u_k(t)),u_k(t))-2g(u_k(t))]dt\right\}+(1+D_2^2)\sigma'\|u_k\|^2\\
&\le C_5+(1+D_2^2)\sigma'\|u_k\|^2+\max\left\{\frac{1}{\mu-2},(\kappa+\gamma D_\infty^{\gamma_1}\|u_k\|^{\gamma_1})\right\}C_4.
\end{aligned}$$

Hence, by fixing $\sigma'\in(0,(\min\{c^2,d_1\}-c_0^2)/2(1+D_2^2))$, it is easy to see that $\{\|u_k\|\}$ is bounded. So we may assume that, up to a subsequence, $u_k\rightharpoonup u$ weakly in E as $k\to+\infty$ for some $u\in E$.

Step 4. Similarly to Step 4 in the proof of Theorem 1.1, we know that $u_k\rightharpoonup u$ in E as $k\to+\infty$ for some $u\in E$ and $I'(u)=0$.

We claim that u is a nontrivial solutions. To complete the proof of this conclusion, we show that for R' sufficiently large, $u_k\nrightarrow 0$ in $L^2([-R',R'],\mathbb{R})$. We proceed arguing by contradiction. Assuming that $u_k\to 0$ in $L^2([-R',R'],\mathbb{R})$ for all $R'>0$. Then there exists constant $m_1>0$, such that

$$\limsup_{k\to\infty}\|u_k\|_{L^2(\mathbb{R},\mathbb{R})}^2\le\frac{m_1}{d_1}. \tag{3.28}$$

In fact, since $\{u_k\}$ is bounded, (3.27) implies that $\{\eta(u_k)\}$ is bounded independently of d_1. Set $m_1=\sup_k\eta^2(u_k)$ and $\varpi_1=\sup_k\|u_k\|_\infty$. Therefore,

$$\|u_k\|_{L^2(\mathbb{R},\mathbb{R})}^2=\int_{-R'}^{R'}|u_k(t)|^2dt+\int_{\mathbb{R}\setminus[-R',R']}|u_k(t)|^2dt\le\int_{-R'}^{R'}|u_k(t)|^2dt+\frac{m_1}{d_1}.$$

By letting $k\to\infty$, (3.28) holds. Since $W'(u)=o(u)$ and $g'(u)=o(u)$ as $|u|\to 0$, similarly to Step 4 in the proof of Theorem 1.1, one has that

$$|g'(u)u-2g(u)|\le M_6|u|^2,\quad |u|\le\varpi_1, \tag{3.29}$$

$$|W'(u)u-2W(u)|\le M_7|u|^2,\quad |u|\le 2\varpi_1, \tag{3.30}$$

where M_6 and M_7 are two positive constants independently of d_1. By using (3.23), (3.28), (3.29) and (3.30), one has

$$\begin{aligned}2a &= \lim_{k\to\infty} 2I(u_k) - (I'(u_k), u_k) \\ &= \lim_{k\to\infty} \int_{\mathbb{R}} [(W'(Au_k(t)), Au_k(t)) - 2W(Au_k(t))]dt + \int_{\mathbb{R}} [(g'(u_k(t)), u_k(t)) - 2g(u_k(t))]dt \\ &\leq (M_6 + 4M_7) \limsup_{k\to\infty} \|u_k\|^2_{L^2(\mathbb{R},\mathbb{R})} \leq \frac{m_1(M_6+4M_7)}{d_1}.\end{aligned}$$

This contradicts d_1 is large enough. Hence, there exists a $R' > 0$ such that $u_k \nrightarrow 0$ in $L^2([-R', R'], \mathbb{R})$, that is, $u \neq 0$.

Combining the above with Steps 1–3 and Theorem 2.1, we have that I possesses at least one nontrivial critical point. Therefore, (1.3) possesses at least one nontrivial homoclinic solution. The proof is completed. ∎

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References

[1] G. Arioli, F. Gazzola and S. Terracini, Multibump periodic motions of an infinite lattice of particles, *Math. Z.* **223** (1996), no. 4, 627–642.

[2] G. Arioli and F. Gazzola, Periodic motions of an infinite lattice of particles with nearest neighbor interaction, *Nonlinear Anal.* **26** (1996), no. 6, 1103–1114.

[3] S. Aubry and P. Y. Le Daeron, The discrete Frenkel-Kontorova model and its extensions. I. Exact results for the ground-states, *Phys. D* **8** (1983), no. 3, 381–422.

[4] I. Ekeland, *Convexity Methods in Hamiltonian Mechanics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 19, Springer, Berlin, 1990.

[5] N. Forcadel, C. Imbert and R. Monneau, Homogenization of fully overdamped Frenkel-Kontorova models, *J. Differential Equations* **246** (2009), no. 3, 1057–1097.

[6] S. Guo, J. S. W. Lamb and B. W. Rink, Branching patterns of wave trains in the FPU lattice, *Nonlinearity* **22** (2009), no. 2, 283–299.

[7] B. Hu, W.-X. Qin and Z. Zheng, Rotation number of the overdamped Frenkel-Kontorova model with ac-driving, *Phys. D* **208** (2005), no. 3–4, 172–190.

[8] M. Izydorek and J. Janczewska, Homoclinic solutions for a class of the second order Hamiltonian systems, *J. Differential Equations* **219** (2005), no. 2, 375–389.

[9] C.-F. Kreiner and J. Zimmer, Travelling wave solutions for the discrete sine-Gordon equation with nonlinear pair interaction, *Nonlinear Anal.* **70** (2009), no. 9, 3146–3158.

[10] P. D. Makita, Periodic and homoclinic travelling waves in infinite lattices, *Nonlinear Anal.* **74** (2011), no. 6, 2071–2086.

[11] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, 74, Springer, New York, 1989.

[12] P. H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, *Proc. Roy. Soc. Edinburgh Sect. A* **114** (1990), no. 1–2, 33–38.

[13] B. Ruf and P. N. Srikanth, On periodic motions of lattices of Toda type via critical point theory, *Arch. Rational Mech. Anal.* **126** (1994), no. 4, 369–385.

[14] D. Smets and M. Willem, Solitary waves with prescribed speed on infinite lattices, *J. Funct. Anal.* **149** (1997), no. 1, 266–275.

[15] D. Smets, Travelling waves for an infinite lattice: Multibump type solutions, *Topol. Methods Nonlinear Anal.* **12** (1998), no. 1, 79–90.

[16] J. Sun, H. Chen and J. J. Nieto, Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems, *J. Math. Anal. Appl.* **373** (2011), no. 1, 20–29.

[17] X. H. Tang and L. Xiao, Homoclinic solutions for nonautonomous second-order Hamiltonian systems with a coercive potential, *J. Math. Anal. Appl.* **351** (2009), no. 2, 586–594.
[18] X. H. Tang and X. Lin, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Proc. Roy. Soc. Edinburgh Sect. A* **141** (2011), no. 5, 1103–1119.
[19] L. Wan and C. Tang, Homoclinic orbits for a class of the second order Hamiltonian systems, *Acta Math. Sci. Ser. B Engl. Ed.* **30** (2010), no. 1, 312–318.
[20] L. Xiao, Existence of periodic solutions for second order Hamiltonian system, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 3, 785–801.
[21] X. Zhang and P. Zhou, An existence result on periodic solutions of an ordinary p-Laplace system, *Bull. Malays. Math. Sci. Soc. (2)* **34** (2011), no. 1, 127–135.
[22] L. Zhang and Y. Chen, Existence of periodic solutions of $p(t)$-Laplacian systems, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 1, 25–38.
[23] Z. Zhang and R. Yuan, Homoclinic solutions for a class of asymptotically quadratic Hamiltonian systems, *Nonlinear Anal. Real World Appl.* **11** (2010), no. 5, 4185–4193.
[24] Z. Zhang and R. Yuan, Homoclinic solutions of some second order non-autonomous systems, *Nonlinear Anal.* **71** (2009), no. 11, 5790–5798.