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The Quarter-Sweep Geometric Mean Method for Solving Second Kind Linear Fredholm Integral Equations

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Abstract. Solving large linear systems is a fundamental problem in large scale scientific and engineering computations. In this paper, the formulation and implementation of the Quarter-Sweep Geometric Mean (QSGM) iterative method for solving large dense nonsymmetric linear systems associated with numerical solution of second kind linear Fredholm integral equations are explained. Furthermore, an analysis of computational complexity and numerical results by solving test problems are also included to verify the performance of the method.

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1. Introduction

Integral equations are used as mathematical models for many and varied physical circumstances, and also occur as reformulations of other mathematical problems. As witnessed by the literature, integral equations of the Fredholm type are one of the most practical and frequently investigated. Therefore, in this paper, numerical solutions for linear Fredholm integral equations of the second kind are considered. The standard form for second kind linear Fredholm integral equations can be represented mathematically as follows:

$$\lambda \varphi(x) - \int_{\Gamma} K(x,t)\varphi(t)dt = f(x), \Gamma = [\alpha, \beta], \lambda \neq 0, \tag{1.1}$$

where the parameter λ, kernel K and function f are given, and φ is the unknown function to be determined. The kernel function $K(x,t)$ is assumed to be absolutely integrable and satisfy properties that are sufficient to imply the Fredholm alternative theorem (refer Theorem 1.1).

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Meanwhile, Eq. (1.1) also can be rewritten in the equivalent operator form as follows:

$$(\lambda - \kappa)\varphi = f \tag{1.2}$$

where the integral operator defines as

$$\kappa\varphi(t) = \int_\Gamma K(x,t)\varphi(t)dt. \tag{1.3}$$

Theorem 1.1. [2] *Let χ be a Banach space and $\kappa : \chi \to \chi$ be compact. Then the equation $(\lambda - \kappa)\varphi = f$, $\lambda \neq 0$ has a unique solution $\varphi \in \chi$ if and only if the homogeneous equation $(\lambda - \kappa)z = 0$ has only the trivial solution $z = 0$. In such a case, the operator $\lambda - \kappa : \chi \underset{onto}{\overset{1-1}{\longrightarrow}} \chi$ has a bounded inverse $(\lambda - \kappa)^{-1}$.*

A numerical approach to the solution of integral equations (1.1) is an essential branch of scientific inquiry. Basically, linear Fredholm integral equations are solved numerically by discretizing the problems to the solution of linear systems. Some valid numerical methods for discretizing problem (1.1) have been developed in recent years; refer [3], [4], [7], [10] and [14]. However, these discretization schemes mostly lead to dense linear systems which can be prohibitively expensive to solve using direct methods as the order of the linear systems increases. Thus, iterative methods are the alternative option for efficient solutions.

Among the existing iterative methods, two-stage iterative methods have been widely accepted to be one of the efficient methods for solving nonsingular linear systems. The Geometric Mean (GM) (also known as Full-Sweep Geometric Mean (FSGM)) iterative method [16] is a particular example of two-stage iterative method. Apart from standard GM method, the variants of GM which are the Half-Sweep Geometric Mean (HSGM) [16] and Quarter-Sweep Geometric Mean (QSGM) [15] methods also have been proposed. Fundamentally, the HSGM and QSGM methods are derived by combining the standard GM method with the complexity reduction approach based on half- [1] and quarter-sweep [13] iteration concepts, respectively.

In a series of papers, the implementation of the GM methods were studied and tested on various types of systems that arise from scientific problems. Recently, FSGM and HSGM methods have been applied for solving second kind linear Fredholm integral equations ([8, 9, 12]). Consequently, in this paper, we extend the applications of another GM iterative method i.e. QSGM method for solving large dense nonsymmetric linear system, which arise from the discretization of linear Fredholm integral equations of the second kind using Nyström method. The performance of QSGM method will be compared with the existing FSGM and HSGM methods.

The remainder of this paper is organized in following way. In Section 2, derivation of the full-, half- and quarter-sweep Nyström approximation equations will be presented. The latter sections of this paper will discuss the implementation and computational complexity of FSGM, HSGM and QSGM methods for solving problem (1.1). Some numerical results will be shown in Section 5 to assess the performance of the methods. Concluding remarks are given in Section 6.

2. Nyström approximation equations

In this paper, Nyström method based on first order composite closed Newton-Cotes quadrature scheme is utilized in order to construct approximation equations for problem (1.1). Let the interval $[\alpha, \beta]$ be divided uniformly into N subintervals and the discrete set of points of x and t, respectively, given by $x_i = \alpha + ih(i = 0,1,2,\cdots,N-2,N-1,N)$ and $t_j = \alpha + jh(j = 0,1,2,\cdots,N-2,N-1,N)$, h is the constant step size defined as:

$$h = \frac{\beta - \alpha}{N}. \tag{2.1}$$

The following notations will be used subsequently for simplicity:

$$\begin{cases} K_{i,j} = K(x_i,t_j) \\ \hat{\varphi}_i = \hat{\varphi}(x_i) \\ \hat{\varphi}_j = \hat{\varphi}(t_j) \\ f_i = f(x_i) \end{cases} \tag{2.2}$$

An implementation of the Nyström method reduces problem (1.1) to

$$\lambda \hat{\varphi}_i - \sum_{j=0}^{N} w_j K_{i,j} \hat{\varphi}_j = f_i, i = 0,1,2,\cdots,N-2,N-1,N \tag{2.3}$$

where solution $\hat{\varphi}$ is an approximation of the exact solution to (1.1) and w_j is the weights of quadrature method. The standard Nyström approximation equations defined in Eq. (2.3) is also referred as full-sweep Nyström approximation equations.

In formulating the half- and quarter-sweep Nyström approximation equations, let consider the following finite grid networks that show the distribution of node points.

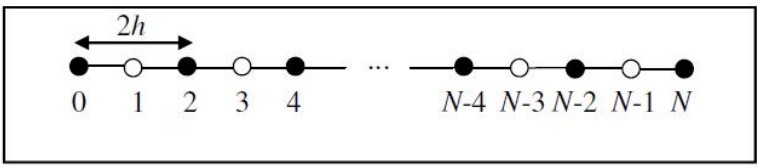

Figure 1. Distribution of uniform node points for the half-sweep case

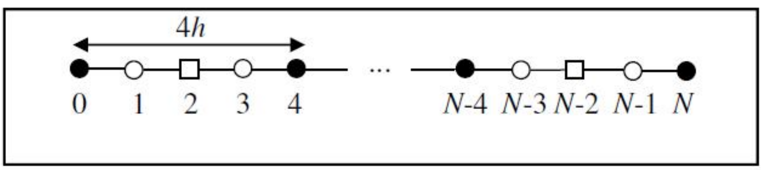

Figure 2. Distribution of uniform node points for the quarter-sweep case

Based on Figures 1 and 2, only node points of type • will be involved during the iteration process. After convergence is achieved, the approximation solution at the other remaining points will be evaluated directly; see [1], [11] and [13] for details.

By applying half- and quarter-sweep iteration concepts, the generalized full-, half- and quarter-sweep Nyström approximation equations is

$$\lambda \hat{\varphi}_i - \sum_{j=0,p,2p}^{N} w_j K_{i,j} \hat{\varphi}_j = f_i \tag{2.4}$$

for $i = 0, p, 2p, \cdots, N-2p, N-p, N$ where the value of p corresponds to one, two and four respective represent to the full-, half- and quarter-sweep Nyström approximation equations. Moreover, approximation equations (2.4) can thus be represented in matrix form as

$$A\hat{\varphi} = f \tag{2.5}$$

where

$$A = \begin{bmatrix} \lambda - w_0K_{0,0} & -w_pK_{0,p} & \cdots & -w_{N-p}K_{0,N-p} & -w_NK_{0,N} \\ -w_0K_{p,0} & \lambda - w_pK_{p,p} & \cdots & -w_{N-p}K_{p,N-p} & -w_NK_{p,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -w_0K_{N-p,0} & -w_pK_{N-p,p} & \cdots & \lambda - w_{N-p}K_{N-p,N-p} & -w_NK_{N-p,N} \\ -w_0K_{N,0} & -w_pK_{N,p} & \cdots & -w_{N-p}K_{N,N-p} & \lambda - w_NK_{N,N} \end{bmatrix}_{(\frac{N}{p}+1)\times(\frac{N}{p}+1)},$$

$$\hat{\varphi} = \begin{bmatrix} \hat{\varphi}_0 & \hat{\varphi}_p & \cdots & \hat{\varphi}_{N-p} & \hat{\varphi}_N \end{bmatrix}^T$$

and

$$f = \begin{bmatrix} f_0 & f_p & \cdots & f_{N-p} & f_N \end{bmatrix}^T.$$

It is noticeable that applications of the half- and quarter-sweep iteration concepts reduce the size of original matrix from $(N+1)\times(N+1)$ to $(\frac{N}{2}+1)\times(\frac{N}{2}+1)$ and $(\frac{N}{4}+1)\times(\frac{N}{4}+1)$, respectively. Based on first order composite closed Newton-Cotes quadrature scheme, the weights quadrature, w_j, will satisfies the following relations

$$w_j = \begin{cases} \frac{1}{2}ph, & j = 0, N \\ ph, & otherwise \end{cases}. \tag{2.6}$$

3. Geometric Mean iterative methods

In this section, the formulation and implementation of FSGM, HSGM and QSGM methods to solve generated nonsymmetric linear systems (2.5) will be discussed. Essentially, each iteration of the FSGM, HSGM and QSGM methods consists of solving two independent systems i.e. $\hat{\varphi^1}$ and $\hat{\varphi^2}$. To develop formulation for all three GM methods, let consider the following splitting

$$A = D - L - U \tag{3.1}$$

where

$$D = \begin{bmatrix} \lambda - w_0K_{0,0} & & & & \\ & \lambda - w_pK_{p,p} & & 0 & \\ & & \ddots & & \\ & 0 & & \lambda - w_{N-p}K_{N-p,N-p} & \\ & & & & \lambda - w_NK_{N,N} \end{bmatrix},$$

$$-L = \begin{bmatrix} & & & & \\ -w_0K_{p,0} & & & & \\ -w_0K_{2p,0} & -w_pK_{2p,p} & & 0 & \\ \vdots & \vdots & \ddots & & \\ -w_0K_{N,0} & -w_pK_{N,p} & \cdots & -w_{N-p}K_{N,N-p} & \end{bmatrix}$$

and

$$-U = \begin{bmatrix} -w_p K_{0,p} & -w_{2p} K_{0,2p} & \cdots & -w_N K_{0,N} \\ & -w_{2p} K_{p,2p} & \cdots & -w_N K_{p,N} \\ 0 & & \ddots & \vdots \\ & & & -w_N K_{N-p,N} \end{bmatrix}.$$

Thus, for nonsingular $(D-\omega L)$ and $(D-\omega U)$ matrices, the general formulation for FSGM, HSGM and QSGM methods is defined as follows

$$\begin{cases} (D-\omega L)\hat{\varphi}^1 &= [(1-\omega)D+\omega U]\hat{\varphi}^{(k)}+\omega f \\ (D-\omega U)\hat{\varphi}^2 &= [(1-\omega)D+\omega L]\hat{\varphi}^{(k)}+\omega f \\ \hat{\varphi}^{(k+1)} &= (\hat{\varphi}^1 \circ \hat{\varphi}^2)^{\frac{1}{2}} \end{cases} \tag{3.2}$$

where ω, $\circ$ and $(.)^{\frac{1}{2}}$ denote the acceleration parameter, Hadamard product and Hadamard power respectively.

Based on the formulation (3.2), iterative forms of the FSGM, HSGM and QSGM methods for solving linear systems (2.5) are of the form

$$\hat{\varphi}^{(k+1)} = T_{FSGM}\hat{\varphi}^{(k)} + c_{FSGM} f, \tag{3.3}$$

$$\hat{\varphi}^{(k+1)} = T_{HSGM}\hat{\varphi}^{(k)} + c_{HSGM} f \tag{3.4}$$

and

$$\hat{\varphi}^{(k+1)} = T_{QSGM}\hat{\varphi}^{(k)} + c_{QSGM} f \tag{3.5}$$

respectively, where

$$T_{FSGM} = T_{HSGM} = T_{QSGM} = [(D-\omega L)^{-1}((1-\omega)D+\omega U)(D-\omega U)^{-1}((1-\omega)D+\omega L)]^{\frac{1}{2}} \tag{3.6}$$

and

$$c_{FSGM} = c_{HSGM} = c_{QSGM} = [\omega^2 (D-\omega L)^{-1}(D-\omega U)^{-1}]^{\frac{1}{2}}. \tag{3.7}$$

The general conditions which guarantee the convergence of FSGM, HSGM and QSGM methods for solving generated linear systems (2.5) are described in the following theorems and definition.

Theorem 3.1. *Let T_{FSGM}, T_{HSGM} and T_{QSGM} be $(N+1)\times(N+1)$, $(\frac{N}{2}+1)\times(\frac{N}{2}+1)$ and $(\frac{N}{4}+1)\times(\frac{N}{4}+1)$ matrices, respectively. Then, the successive approximations* (3.3), (3.4) *and* (3.5) *for $k = 0,1,2,\cdots$ convergence for each $c_{FSGM} \in \mathbf{C}^{N+1}$, $c_{HSGM} \in \mathbf{C}^{\frac{N}{2}+1}$ and $c_{QSGM} \in \mathbf{C}^{\frac{N}{4}+1}$ respectively and, each $\hat{\varphi}^{(0)} \in \mathbf{C}^{N+1}$, $\hat{\varphi}^{(0)} \in \mathbf{C}^{\frac{N}{2}+1}$ and $\hat{\varphi}^{(0)} \in \mathbf{C}^{\frac{N}{4}+1}$ respectively if and only if the spectral radius of the iteration matrices i.e. T_{FSGM}, T_{HSGM} and T_{QSGM} is less than one, that are, $\rho(T_{FSGM}) < 1$, $\rho(T_{HSGM}) < 1$ and $\rho(T_{QSGM}) < 1$.*

Proof. The proof runs parallel to a standard proof given in [5]. ∎

Theorem 3.2. *A necessary condition for the FSGM, HSGM and QSGM methods to be convergent is that $0 < \omega < 2$.*

Proof. Since the eigenvalues, μ_j of T_{FSGM}, T_{HSGM} and T_{QSGM} are the zeroes of the characteristic polynomial, thus determinants of the T_{FSGM}, T_{HSGM} and T_{QSGM} satisfy the following relations

$$\det(T_{FSGM}) = \prod_{j=0}^{N} \mu_j,$$

$$\det(T_{HSGM}) = \prod_{j=0,2,4}^{N} \mu_j$$

and

$$\det(T_{QSGM}) = \prod_{j=0,4,8}^{N} \mu_j$$

respectively (where multiple eigenvalues are repeated according to their algebraic multiplicity). Since $(D-\omega L)^{\frac{1}{2}}$, $(D-\omega U)^{\frac{1}{2}}$, $((1-\omega)D+\omega U)^{\frac{1}{2}}$ and $((1-\omega)D+\omega L)^{\frac{1}{2}}$ are triangular matrices, it follows that

$$\begin{aligned}
&\det(T_{FSGM})\\
&= \det[(D-\omega L)^{-1}((1-\omega)D+\omega U)(D-\omega U)^{-1}((1-\omega)D+\omega L)]^{\frac{1}{2}}\\
&= \det[((D-\omega L)^{\frac{1}{2}})^{-1}((1-\omega)D+\omega U)^{\frac{1}{2}}((D-\omega U)^{\frac{1}{2}})^{-1}((1-\omega)D+\omega L)^{\frac{1}{2}}]\\
&= \det((D-\omega L)^{\frac{1}{2}})^{-1}\det((1-\omega)D+\omega U)^{\frac{1}{2}}\det((D-\omega U)^{\frac{1}{2}})^{-1}\det((1-\omega)D+\omega L)^{\frac{1}{2}}\\
&= ((1-\omega)^{\frac{1}{2}})^{N+1}((1-\omega)^{\frac{1}{2}})^{N+1}\\
&= (1-\omega)^{\frac{N+1}{2}}(1-\omega)^{\frac{N+1}{2}}\\
&= (1-\omega)^{N+1}
\end{aligned}$$

$$\begin{aligned}
&\det(T_{HSGM})\\
&= \det[(D-\omega L)^{-1}((1-\omega)D+\omega U)(D-\omega U)^{-1}((1-\omega)D+\omega L)]^{\frac{1}{2}}\\
&= \det[((D-\omega L)^{\frac{1}{2}})^{-1}((1-\omega)D+\omega U)^{\frac{1}{2}}((D-\omega U)^{\frac{1}{2}})^{-1}((1-\omega)D+\omega L)^{\frac{1}{2}}]\\
&= \det((D-\omega L)^{\frac{1}{2}})^{-1}\det((1-\omega)D+\omega U)^{\frac{1}{2}}\det((D-\omega U)^{\frac{1}{2}})^{-1}\det((1-\omega)D+\omega L)^{\frac{1}{2}}\\
&= ((1-\omega)^{\frac{1}{2}})^{\frac{N}{2}+1}((1-\omega)^{\frac{1}{2}})^{\frac{N}{2}+1}\\
&= (1-\omega)^{\frac{1}{2}(\frac{N}{2}+1)}(1-\omega)^{\frac{1}{2}(\frac{N}{2}+1)}\\
&= (1-\omega)^{\frac{N}{2}+1}
\end{aligned}$$

and

$$\begin{aligned}
&\det(T_{QSGM})\\
&= \det[(D-\omega L)^{-1}((1-\omega)D+\omega U)(D-\omega U)^{-1}((1-\omega)D+\omega L)]^{\frac{1}{2}}\\
&= \det[((D-\omega L)^{\frac{1}{2}})^{-1}((1-\omega)D+\omega U)^{\frac{1}{2}}((D-\omega U)^{\frac{1}{2}})^{-1}((1-\omega)D+\omega L)^{\frac{1}{2}}]\\
&= \det((D-\omega L)^{\frac{1}{2}})^{-1}\det((1-\omega)D+\omega U)^{\frac{1}{2}}\det((D-\omega U)^{\frac{1}{2}})^{-1}\det((1-\omega)D+\omega L)^{\frac{1}{2}}\\
&= ((1-\omega)^{\frac{1}{2}})^{\frac{N}{4}+1}((1-\omega)^{\frac{1}{2}})^{\frac{N}{4}+1}
\end{aligned}$$

$$= (1-\omega)^{\frac{1}{2}(\frac{N}{4}+1)}(1-\omega)^{\frac{1}{2}(\frac{N}{4}+1)}$$
$$= (1-\omega)^{\frac{N}{4}+1}.$$

This now implies,

$$\rho(T_{FSGM}) \geq (\mid 1-\omega \mid^{(N+1)})^{\frac{1}{N+1}} \\ = \mid 1-\omega \mid,$$

$$\rho(T_{HSGM}) \geq (\mid 1-\omega \mid^{(\frac{N}{2}+1)})^{\frac{1}{\frac{N}{2}+1}} \\ = \mid 1-\omega \mid$$

and

$$\rho(T_{QSGM}) \geq (\mid 1-\omega \mid^{(\frac{N}{4}+1)})^{\frac{1}{\frac{N}{4}+1}} \\ = \mid 1-\omega \mid.$$

Therefore, from Theorem 3.1, FSGM, HSGM and QSGM methods must satisfy the condition of $0 < \omega < 2$ for convergence. ∎

Definition 3.1. *The elements of vectors $\hat{\varphi}^1$ and $\hat{\varphi}^2$ must be a nonnegative value for calculating $\hat{\varphi}^{(k+1)}$. The following relations i.e.*

i) $-(\hat{\varphi}^1{}_i\hat{\varphi}^2{}_i)^{\frac{1}{2}}$ *if* $\hat{\varphi}^1{}_i < 0$ *and* $\hat{\varphi}^2{}_i < 0$
ii) $\hat{\varphi}^1{}_i - |\hat{\varphi}^1{}_i\hat{\varphi}^2{}_i|^{\frac{1}{2}}$ *if* $\hat{\varphi}^1{}_i > 0$ *and* $\hat{\varphi}^2{}_i < 0$
iii) $\hat{\varphi}^2{}_i - |\hat{\varphi}^1{}_i\hat{\varphi}^2{}_i|^{\frac{1}{2}}$ *if* $\hat{\varphi}^1{}_i < 0$ *and* $\hat{\varphi}^2{}_i > 0$

are hold in calculating $\hat{\varphi}_i^{(k+1)}$ for $i = 0, p, 2p, \cdots, N-2p, N-p, N$ when negative elements involve.

The algorithm for FSGM, HSGM and QSGM methods associated with full-, half and quarter-sweep Nyström approximation equations, respectively, to solve problem (1.1) is described in Algorithm 1. By referring Algorithm 1, the GM algorithms are explicitly performed by using all equations at Levels 1 and 2 alternately until the solution satisfied a specified convergence criterion i.e. the maximum norm $\| \hat{\varphi}^{(k+1)} - \hat{\varphi}^{(k)} \| \leq \varepsilon$ where ε is the convergence criterion.

Algorithm 1. FSGM, HSGM and QSGM schemes
i. Set $\hat{\varphi}^{(0)} = \hat{\varphi}^1 = \hat{\varphi}^2$ and initialize all the parameters
ii. Iteration cycle
a. Stage 1
1. Level 1
for $i = 0, p, 2p, \cdots, N-2p, N-p, N$
Compute
$\hat{\varphi}^1{}_i \leftarrow (1-\omega)\hat{\varphi}_i^{(k)} + \frac{\omega}{\lambda - w_i K_{i,i}}(f_i - \sum_{j=0,p,2p}^{i-p} w_j K_{i,j}\hat{\varphi}^1{}_j - \sum_{j=i+p,i+2p,i+3p}^{N} w_j K_{i,j}\hat{\varphi}_j^{(k)})$

b. Stage 2
1) Level 2
for $i = N, N-p, N-2p, \cdots, 2p, p, 0$
Compute

$\hat{\varphi^2}_i \leftarrow (1-\omega)\hat{\varphi}_i^{(k)} + \frac{\omega}{\lambda - w_i K_{i,i}}(f_i - \sum_{j=0,p,2p}^{i-p} w_j K_{i,j}\hat{\varphi}_j^{(k)} - \sum_{j=i+p,i+2p,i+3p}^{N} w_j K_{i,j}\hat{\varphi^2}_j)$

2) **for** $i = 0, p, 2p, \cdots, N-2p, N-p, N$
Compute

$$\hat{\varphi}_i^{(k+1)} \leftarrow \begin{cases} (\hat{\varphi^1}_i\hat{\varphi^2}_i)^{\frac{1}{2}} & if \quad \hat{\varphi^1}_i \geqslant 0 \wedge \hat{\varphi^2}_i \geqslant 0 \quad (Case\ 1) \\ -(\hat{\varphi^1}_i\hat{\varphi^2}_i)^{\frac{1}{2}} & if \quad \hat{\varphi^1}_i < 0 \wedge \hat{\varphi^2}_i < 0 \quad (Case\ 2) \\ \hat{\varphi^1}_i - (|\,\hat{\varphi^1}_i\hat{\varphi^2}_i\,|)^{\frac{1}{2}} & if \quad \hat{\varphi^1}_i > 0 \wedge \hat{\varphi^2}_i < 0 \quad (Case\ 3) \\ \hat{\varphi^2}_i - (|\,\hat{\varphi^1}_i\hat{\varphi^2}_i\,|)^{\frac{1}{2}} & if \quad \hat{\varphi^1}_i < 0 \wedge \hat{\varphi^2}_i > 0 \quad (Case\ 4) \end{cases}$$

iii. Check the convergence. If the converge criterion is satisfied, go to Step (iv), otherwise, repeat the iteration cycle (i.e., go to Step (ii))
iv. Stop

After the iteration process, additional calculation is required for HSGM and QSGM methods to calculate the remaining points. In this paper, second order Lagrange interpolation technique [11] will be applied to compute the remaining points. The formulations to calculate remaining points using second order Lagrange interpolation technique for HSGM and QSGM are defined as

$$\hat{\varphi}_i = \begin{cases} \frac{3}{8}\hat{\varphi}_{i-1} + \frac{3}{4}\hat{\varphi}_{i+1} - \frac{1}{8}\hat{\varphi}_{i+3}, & i = 1,3,5,\cdots,N-3 \\ \frac{3}{4}\hat{\varphi}_{i-1} + \frac{3}{8}\hat{\varphi}_{i+1} - \frac{1}{8}\hat{\varphi}_{i-3}, & i = N-1 \end{cases} \tag{3.8}$$

and

$$\hat{\varphi}_i = \begin{cases} \frac{3}{8}\hat{\varphi}_{i-2} + \frac{3}{4}\hat{\varphi}_{i+2} - \frac{1}{8}\hat{\varphi}_{i+6}, & i = 2,6,10,\cdots,N-6 \\ \frac{3}{4}\hat{\varphi}_{i-2} + \frac{3}{8}\hat{\varphi}_{i+2} - \frac{1}{8}\hat{\varphi}_{i-6}, & i = N-2 \\ \frac{3}{8}\hat{\varphi}_{i-1} + \frac{3}{4}\hat{\varphi}_{i+1} - \frac{1}{8}\hat{\varphi}_{i+3}, & i = 1,3,5,\cdots,N-3 \\ \frac{3}{4}\hat{\varphi}_{i-1} + \frac{3}{8}\hat{\varphi}_{i+1} - \frac{1}{8}\hat{\varphi}_{i-3}, & i = N-1 \end{cases} \tag{3.9}$$

respectively.

4. Computational complexity analysis

An estimation amount of the computational work has been conducted to measure the computational complexity of FSGM, HSGM and QSGM methods with corresponding Nyström approximation equations for solving problem (1.1). The computational work is estimated by considering the arithmetic operations performed per iteration. Based on Algorithm 1, it can be observed that there are four different cases for all three GM methods. In estimating the computational work for GM iterative methods, the value for kernel K, function f and weights w_j are stored beforehand.

Based on Algorithm 1, it can be observed that $6N+16$, $3N+16$ and $\frac{3N}{2}+16$ arithmetic operations are involved for Case 1 of FSGM, HSGM and QSGM methods respectively in computing a value for each node point in the solution domain. Whereas, for Case 2, 3 and 4, $6N+17$, $3N+17$ and $\frac{3N}{2}+17$ operations are required for FSGM, HSGM and QSGM methods respectively. However, for HSGM and QSGM methods, the iteration processes are carried out only on $\frac{N}{2}+1$ and $\frac{N}{4}+1$ mesh points respectively. Thus, additional eight arithmetic operations are involved to calculate a remaining node point after convergence by using second order Lagrange interpolation technique. Hence, the total arithmetic operations involved for FSGM, HSGM and QSGM methods for solving problem (1.1) are summarized in Table 1.

Table 1. Total computing operations involved for the FSGM, HSGM and QSGM methods

Case	Methods	Total Arithmetic Operations	
		Per Iteration	After Convergence
Case 1	FSGM	$6N^2+22N+16$	-
	HSGM	$\frac{3}{2}N^2+11N+16$	$4N$
	QSGM	$\frac{3}{8}N^2+\frac{11}{2}N+16$	$6N$
Case 2	FSGM	$6N^2+23N+17$	-
	HSGM	$\frac{3}{2}N^2+\frac{23}{2}N+17$	$4N$
	QSGM	$\frac{3}{8}N^2+\frac{23}{4}N+17$	$6N$
Case 3	FSGM	$6N^2+23N+17$	-
	HSGM	$\frac{3}{2}N^2+\frac{23}{2}N+17$	$4N$
	QSGM	$\frac{3}{8}N^2+\frac{23}{4}N+17$	$6N$
Case 4	FSGM	$6N^2+23N+17$	-
	HSGM	$\frac{3}{2}N^2+\frac{23}{2}N+17$	$4N$
	QSGM	$\frac{3}{8}N^2+\frac{23}{4}N+17$	$6N$

5. Numerical experiments

In order to compare the performances of the methods, several tests were carried out on the following two well-posed second kind linear Fredholm integral equations which will generate dense nonsymmetric matrix A.

Test Problem 1 [17]
Consider the Fredholm integral equation of the second kind

$$\varphi(x)-\int_0^1(4xt-x^2)\varphi(t)dt=x, x\in[0,1] \tag{5.1}$$

and the exact solution is given by

$$\varphi(x)=24x-9x^2.$$

Test Problem 2 [10]
Consider the Fredholm integral equation of the second kind

$$\varphi(x)-\int_0^1(x^2+t^2)\varphi(t)dt=x^6-5x^3+x+10, x\in[0,1] \tag{5.2}$$

with the exact solution

$$\varphi(x)=x^6-5x^3+\frac{1045}{28}x^2+x+\frac{2141}{84}.$$

The value of initial iteration is set to be zero for both test problems. Meanwhile, the experimental values of ω for FSGM, HSGM and QSGM methods are chosen within ± 0.01 to be an optimal value by a trial and error process. All the simulations were implemented by a computer with processor Intel(R) Core(TM) 2 CPU 1.66GHz and programming codes were written in C language. Throughout the simulations, the convergence test considered $\varepsilon=10^{-10}$ and carried out on several different values of N. In addition, numerical results of the standard Gauss-Seidel (GS) method with standard Nyström approximation equations for

solving both test problems are also included. For the numerical experiments, the following parameters are defined to make a comparative analysis
k - Number of iterations
rk - Ratio of iteration number of an iterative method to the iteration number of the GS method
CPU - CPU time (in seconds) when the converged solution is obtained
$\| e_N \|$ - Error defines as $\| e_N \| \simeq (\frac{1}{N}\sum_{i=0}^{N} e_N^2(\varphi_i))^{\frac{1}{2}}$ where $e(x_i) = \varphi(x_i) - \hat{\varphi}(x_i)$ for $i = 0, 1, 2, \cdots, N-2, N-1, N$ [6].

The numerical results of the tested methods for test problems 1 and 2 are tabulated in Tables 2 and 3 respectively. Furthermore, reduction percentages in terms of number of iterations and CPU time for the FSGM, HSGM and QSGM methods compared with GS method are summarized in Table 4. Meanwhile, Tables 5 to 16 show the approximation solutions of φ at some discrete points for both test problems.

Table 2. Numerical results of the iterative methods for test problem 1

N	Methods	k	rk	CPU	$\| e_N \|$
240	GS	193	1.000000	1.58	1.423744×10^{-03}
	FSGM	83	0.430052	0.95	1.423745×10^{-03}
	HSGM	83	0.430052	0.60	5.697426×10^{-03}
	QSGM	83	0.430052	0.37	2.282894×10^{-02}
480	GS	194	1.000000	5.87	3.554797×10^{-04}
	FSGM	83	0.427835	3.81	3.554801×10^{-04}
	HSGM	83	0.427835	1.00	1.422074×10^{-03}
	QSGM	83	0.427835	0.66	5.690739×10^{-03}
960	GS	194	1.000000	23.49	8.881489×10^{-05}
	FSGM	83	0.427835	14.92	8.881525×10^{-05}
	HSGM	83	0.427835	4.24	3.552711×10^{-04}
	QSGM	83	0.427835	1.38	1.421238×10^{-03}
1920	GS	195	1.000000	92.41	2.219668×10^{-05}
	FSGM	83	0.425641	65.31	2.219698×10^{-05}
	HSGM	83	0.425641	16.64	8.878912×10^{-05}
	QSGM	83	0.425641	5.08	3.551666×10^{-04}
3840	GS	195	1.000000	328.71	5.547964×10^{-06}
	FSGM	83	0.425641	242.23	5.548273×10^{-06}
	HSGM	83	0.425641	75.19	2.219372×10^{-05}
	QSGM	83	0.425641	23.11	8.877606×10^{-05}
7680	GS	195	1.000000	1042.47	1.386506×10^{-06}
	FSGM	83	0.425641	828.14	1.386818×10^{-06}
	HSGM	83	0.425641	288.38	5.547865×10^{-06}
	QSGM	83	0.425641	104.17	2.219209×10^{-05}

Table 3. Numerical results of the iterative methods for test problem 2

N	Methods	k	rk	CPU	$\| e_N \|$
	GS	56	1.000000	0.77	1.334164×10^{-03}
240	FSGM	32	0.571429	0.75	1.334164×10^{-03}
	HSGM	32	0.571429	0.35	5.337101×10^{-03}
	QSGM	32	0.571429	0.17	2.135750×10^{-02}
	GS	56	1.000000	2.37	3.330043×10^{-04}
480	FSGM	32	0.571429	2.32	3.330043×10^{-04}
	HSGM	32	0.571429	0.84	1.332034×10^{-03}
	QSGM	32	0.571429	0.43	5.328645×10^{-03}
	GS	56	1.000000	14.83	8.318448×10^{-05}
960	FSGM	32	0.571429	14.39	8.318453×10^{-05}
	HSGM	32	0.571429	3.18	3.327379×10^{-04}
	QSGM	32	0.571429	1.18	1.330977×10^{-03}
	GS	56	1.000000	58.27	2.078778×10^{-05}
1920	FSGM	32	0.571429	57.21	2.078783×10^{-05}
	HSGM	32	0.571429	17.41	8.315122×10^{-05}
	QSGM	32	0.571429	4.20	3.326057×10^{-04}
	GS	56	1.000000	135.75	5.195839×10^{-06}
3840	FSGM	32	0.571429	131.32	5.195890×10^{-06}
	HSGM	32	0.571429	71.67	2.078366×10^{-05}
	QSGM	32	0.571429	23.34	8.313469×10^{-05}
	GS	56	1.000000	279.34	1.298760×10^{-06}
7680	FSGM	32	0.571429	270.98	1.298810×10^{-06}
	HSGM	32	0.571429	164.37	5.195369×10^{-06}
	QSGM	32	0.571429	90.72	2.078160×10^{-05}

Table 4. Reduction percentages of the FSGM, HSGM and QSGM methods compared with GS method

Methods	Test problem 1		Test problem 2	
	k (%)	CPU (%)	k (%)	CPU(%)
FSGM	56.99 - 57.44	20.55 - 39.88	42.85 - 42.86	1.81 - 3.27
HSGM	56.99 - 57.44	62.02 - 82.97	42.85 - 42.86	41.15 - 78.56
QSGM	56.99 - 57.44	74.68 - 94.51	42.85 - 42.86	67.52 - 92.80

Table 5. Numerical solutions for case $N = 240$ of test problem 1

x	$N = 240$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.10	2.3100000000	2.3103260882	2.3103260883	2.3113049139	2.3152286431
0.20	4.4400000000	4.4406271729	4.4406271731	4.4425097704	4.4500563675
0.30	6.3900000000	6.3909032539	6.3909032542	6.3936145697	6.4044831731
0.40	8.1600000000	8.1611543314	8.1611543318	8.1646193116	8.1785090599
0.50	9.7500000000	9.7513804054	9.7513804058	9.7555239962	9.7721340279
0.60	11.1600000000	11.1615814758	11.1615814762	11.1663286235	11.1853580772
0.70	12.3900000000	12.3917575426	12.3917575431	12.3970331934	12.4181812078
0.80	13.4400000000	13.4419086058	13.4419086063	13.4476377060	13.4706034195
0.90	14.3100000000	14.3120346655	14.3120346660	14.3181421613	14.3426247125
1.00	15.0000000000	15.0021357216	15.0021357221	15.0085465593	15.0342450867

Table 6. Numerical solutions for case $N = 480$ of test problem 1

x	$N = 480$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.10	2.3100000000	2.3100815132	2.3100815133	2.3103260883	2.3113049139
0.20	4.4400000000	4.4401567762	4.4401567764	4.4406271731	4.4425097704
0.30	6.3900000000	6.3902257890	6.3902257892	6.3909032542	6.3936145697
0.40	8.1600000000	8.1602885515	8.1602885518	8.1611543318	8.1646193116
0.50	9.7500000000	9.7503450639	9.7503450642	9.7513804058	9.7555239962
0.60	11.1600000000	11.1603953260	11.1603953264	11.1615814762	11.1663286235
0.70	12.3900000000	12.3904393380	12.3904393384	12.3917575431	12.3970331934
0.80	13.4400000000	13.4404770997	13.4404771001	13.4419086063	13.4476377060
0.90	14.3100000000	14.3105086112	14.3105086117	14.3120346660	14.3181421613
1.00	15.0000000000	15.0005338725	15.0005338730	15.0021357221	15.0085465593

Table 7. Numerical solutions for case $N = 960$ of test problem 1

x	$N = 960$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.10	2.3100000000	2.3100203776	2.3100203777	2.3100815133	2.3103260883
0.20	4.4400000000	4.4400391928	4.4400391930	4.4401567764	4.4406271731
0.30	6.3900000000	6.3900564454	6.3900564457	6.3902257892	6.3909032542
0.40	8.1600000000	8.1600721356	8.1600721359	8.1602885518	8.1611543318
0.50	9.7500000000	9.7500862632	9.7500862636	9.7503450642	9.7513804058
0.60	11.1600000000	11.1600988284	11.1600988288	11.1603953264	11.1615814762
0.70	12.3900000000	12.3901098310	12.3901098315	12.3904393384	12.3917575431
0.80	13.4400000000	13.4401192712	13.4401192716	13.4404771001	13.4419086063
0.90	14.3100000000	14.3101271488	14.3101271493	14.3105086117	14.3120346660
1.00	15.0000000000	15.0001334640	15.0001334645	15.0005338730	15.0021357221

Table 8. Numerical solutions for case $N = 1920$ of test problem 1

x	$N = 1920$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.10	2.3100000000	2.3100050943	2.3100050944	2.3100203777	2.3100815133
0.20	4.4400000000	4.4400097980	4.4400097981	4.4400391930	4.4401567764
0.30	6.3900000000	6.3900141110	6.3900141112	6.3900564457	6.3902257892
0.40	8.1600000000	8.1600180335	8.1600180337	8.1600721359	8.1602885518
0.50	9.7500000000	9.7500215653	9.7500215656	9.7500862636	9.7503450642
0.60	11.1600000000	11.1600247065	11.1600247069	11.1600988288	11.1603953264
0.70	12.3900000000	12.3900274571	12.3900274575	12.3901098315	12.3904393384
0.80	13.4400000000	13.4400298171	13.4400298175	13.4401192716	13.4404771001
0.90	14.3100000000	14.3100317865	14.3100317869	14.3101271493	14.3105086117
1.00	15.0000000000	15.0000333653	15.0000333657	15.0001334645	15.0005338730

Table 9. Numerical solutions for case $N = 3840$ of test problem 1

x	$N = 3840$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.10	2.3100000000	2.3100012735	2.3100012736	2.3100050944	2.3100203777
0.20	4.4400000000	4.4400024493	4.4400024495	4.4400097981	4.4400391930
0.30	6.3900000000	6.3900035275	6.3900035277	6.3900141112	6.3900564457
0.40	8.1600000000	8.1600045080	8.1600045083	8.1600180337	8.1600721359
0.50	9.7500000000	9.7500053909	9.7500053913	9.7500215656	9.7500862636
0.60	11.1600000000	11.1600061762	11.1600061765	11.1600247069	11.1600988288
0.70	12.3900000000	12.3900068638	12.3900068642	12.3900274575	12.3901098315
0.80	13.4400000000	13.4400074538	13.4400074542	13.4400298175	13.4401192716
0.90	14.3100000000	14.3100079461	14.3100079465	14.3100317869	14.3101271493
1.00	15.0000000000	15.0000083408	15.0000083412	15.0000333657	15.0001334645

Table 10. Numerical solutions for case $N = 7680$ of test problem 1

x	$N = 7680$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.10	2.3100000000	2.3100003183	2.3100003184	2.3100012736	2.3100050944
0.20	4.4400000000	4.4400006121	4.4400006123	4.4400024495	4.4400097981
0.30	6.3900000000	6.3900008816	6.3900008818	6.3900035277	6.3900141112
0.40	8.1600000000	8.1600011267	8.1600011270	8.1600045083	8.1600180337
0.50	9.7500000000	9.7500013473	9.7500013477	9.7500053913	9.7500215656
0.60	11.1600000000	11.1600015436	11.1600015440	11.1600061765	11.1600247069
0.70	12.3900000000	12.3900017155	12.3900017159	12.3900068642	12.3900274575
0.80	13.4400000000	13.4400018629	13.4400018633	13.4400074542	13.4400298175
0.90	14.3100000000	14.3100019860	14.3100019864	14.3100079465	14.3100317869
1.00	15.0000000000	15.0000020847	15.0000020851	15.0000083412	15.0000333657

Table 11. Numerical solutions for case $N = 240$ of test problem 2

x	$N = 240$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	25.4880952381	25.4889079053	25.4889079053	25.4913462671	25.5011051219
0.10	25.9563105239	25.9571368056	25.9571368056	25.9596160174	25.9695383635
0.20	27.1410163810	27.1418835066	27.1418835066	27.1444852681	27.1548980882
0.30	29.0127528096	29.0136880082	29.0136880082	29.0164940193	29.0277242961
0.40	31.5436198096	31.5446503104	31.5446503104	31.5477422711	31.5601169871
0.50	34.7090773810	34.7102304133	34.7102304133	34.7136900233	34.7275361612
0.60	38.4904655239	38.4917683168	38.4917683169	38.4956772760	38.5113218185
0.70	42.8782442381	42.8797240210	42.8797240210	42.8841640292	42.9019339589
0.80	47.8759535239	47.8776375258	47.8776375258	47.8826902829	47.9029125825
0.90	53.5048933810	53.5068088313	53.5068088313	53.5125560371	53.5355576892
1.00	59.8095238096	59.8116979374	59.8116979374	59.8182212918	59.8443292791

Table 12. Numerical solutions for case $N = 480$ of test problem 2

x	$N = 480$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	25.4880952381	25.4882983993	25.4882983993	25.4889079053	25.4913462671
0.10	25.9563105239	25.9565170885	25.9565170885	25.9571368056	25.9596160174
0.20	27.1410163810	27.1412331563	27.1412331564	27.1418835066	27.1444852681
0.30	29.0127528096	29.0129866027	29.0129866027	29.0136880082	29.0164940193
0.40	31.5436198096	31.5438774276	31.5438774276	31.5446503104	31.5477422711
0.50	34.7090773810	34.7093656310	34.7093656311	34.7102304133	34.7136900233
0.60	38.4904655239	38.4907912130	38.4907912130	38.4917683169	38.4956772760
0.70	42.8782442381	42.8786141735	42.8786141735	42.8797240210	42.8841640292
0.80	47.8759535239	47.8763745125	47.8763745126	47.8776375258	47.8826902829
0.90	53.5048933810	53.5053722301	53.5053722302	53.5068088313	53.5125560371
1.00	59.8095238096	59.8100673263	59.8100673263	59.8116979374	59.8182212918

Table 13. Numerical solutions for case $N = 960$ of test problem 2

x	$N = 960$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	25.4880952381	25.4881460280	25.4881460280	25.4882983993	25.4889079053
0.10	25.9563105239	25.9563621646	25.9563621646	25.9565170885	25.9571368056
0.20	27.1410163810	27.1410705744	27.1410705744	27.1412331564	27.1418835066
0.30	29.0127528096	29.0128112574	29.0128112574	29.0129866027	29.0136880082
0.40	31.5436198096	31.5436842136	31.5436842136	31.5438774276	31.5446503104
0.50	34.7090773810	34.7091494429	34.7091494430	34.7093656311	34.7102304133
0.60	38.4904655239	38.4905469455	38.4905469456	38.4907912130	38.4917683169
0.70	42.8782442381	42.8783367213	42.8783367213	42.8786141735	42.8797240210
0.80	47.8759535239	47.8760587702	47.8760587703	47.8763745126	47.8776375258
0.90	53.5048933810	53.5050130924	53.5050130924	53.5053722302	53.5068088313
1.00	59.8095238096	59.8096596877	59.8096596877	59.8100673263	59.8116979374

Table 14. Numerical solutions for case $N = 1920$ of test problem 2

x	$N = 1920$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	25.4880952381	25.4881079355	25.4881079356	25.4881460280	25.4882983993
0.10	25.9563105239	25.9563234340	25.9563234340	25.9563621646	25.9565170885
0.20	27.1410163810	27.1410299293	27.1410299293	27.1410705744	27.1412331564
0.30	29.0127528096	29.0127674214	29.0127674215	29.0128112574	29.0129866027
0.40	31.5436198096	31.5436359105	31.5436359105	31.5436842136	31.5438774276
0.50	34.7090773810	34.7090953964	34.7090953964	34.7091494430	34.7093656311
0.60	38.4904655239	38.4904858792	38.4904858792	38.4905469456	38.4907912130
0.70	42.8782442381	42.8782673588	42.8782673589	42.8783367213	42.8786141735
0.80	47.8759535239	47.8759798353	47.8759798354	47.8760587703	47.8763745126
0.90	53.5048933810	53.5049233087	53.5049233088	53.5050130924	53.5053722302
1.00	59.8095238096	59.8095577789	59.8095577790	59.8096596877	59.8100673263

Table 15. Numerical solutions for case $N = 3840$ of test problem 2

x	$N = 3840$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	25.4880952381	25.4880984124	25.4880984125	25.4881079356	25.4881460280
0.10	25.9563105239	25.9563137513	25.9563137514	25.9563234340	25.9563621646
0.20	27.1410163810	27.1410197680	27.1410197681	27.1410299293	27.1410705744
0.30	29.0127528096	29.0127564625	29.0127564625	29.0127674215	29.0128112574
0.40	31.5436198096	31.5436238347	31.5436238348	31.5436359105	31.5436842136
0.50	34.7090773810	34.7090818848	34.7090818848	34.7090953964	34.7091494430
0.60	38.4904655239	38.4904706126	38.4904706127	38.4904858792	38.4905469456
0.70	42.8782442381	42.8782500182	42.8782500183	42.8782673589	42.8783367213
0.80	47.8759535239	47.8759601016	47.8759601017	47.8759798354	47.8760587703
0.90	53.5048933810	53.5049008628	53.5049008629	53.5049233088	53.5050130924
1.00	59.8095238096	59.8095323018	59.8095323019	59.8095577790	59.8096596877

Table 16. Numerical solutions for case $N = 7680$ of test problem 2

x	$N = 7680$				
	Exact Solution	GS	FSGM	HSGM	QSGM
0.00	25.4880952381	25.4880960317	25.4880960317	25.4880984125	25.4881079356
0.10	25.9563105239	25.9563113307	25.9563113307	25.9563137514	25.9563234340
0.20	27.1410163810	27.1410172277	27.1410172277	27.1410197681	27.1410299293
0.30	29.0127528096	29.0127537227	29.0127537228	29.0127564625	29.0127674215
0.40	31.5436198096	31.5436208158	31.5436208159	31.5436238348	31.5436359105
0.50	34.7090773810	34.7090785069	34.7090785069	34.7090818848	34.7090953964
0.60	38.4904655239	38.4904667960	38.4904667960	38.4904706127	38.4904858792
0.70	42.8782442381	42.8782456831	42.8782456831	42.8782500183	42.8782673589
0.80	47.8759535239	47.8759551682	47.8759551683	47.8759601017	47.8759798354
0.90	53.5048933810	53.5048952514	53.5048952514	53.5049008629	53.5049233088
1.00	59.8095238096	59.8095259325	59.8095259326	59.8095323019	59.8095577790

6. Conclusions

In this paper, an application of the QSGM iterative method for solving large dense nonsymmetric matrices that arise from the linear Fredholm integral equations of the second kind is examined. Through numerical results presented in Tables 2 to 4, it clearly shows that applications of the GM methods reduce number of iterations and computational time compared to GS method. Meanwhile, among the GM methods, QSGM method compute with fastest time compared to FSGM and HSGM methods for all considered N. In terms of accuracy, numerical solution obtained by using QSGM method is comparable with the solutions generated via GS, FSGM and HSGM methods. Furthermore, through the observation in Tables 5 to 16, increment of N improved the accuracy of numerical solutions and maximum error of the numerical solutions occurred at point $x = 1.00$ for both test problems.

Finally, it can be concluded that the QSGM method is superior to GS, FSGM and HSGM methods, particularly in the sense of computational time. This mainly because of the reduction in terms of computational complexity; since the QSGM method will only consider approximately quarter of all interior node points in a solution domain during iteration process. Based on Table 1, computational complexity of the QSGM method is at least 93.75% and 75% less than FSGM and HSGM methods respectively.

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