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# Weighted Composition Operators from the Besov Spaces into the Bloch Spaces

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**Abstract.** Let $\varphi$ be an analytic self-map of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ and let $u$ be a fixed analytic function on $\mathbb{D}$. The weighted composition operator is defined on the space $H(\mathbb{D})$ of analytic functions on $\mathbb{D}$ by $uC_\varphi f = u\cdot(f\circ\varphi),\ \ f\in H(\mathbb{D})$. In this work, we characterize the bounded and the compact weighted composition operators from the Besov spaces $B_p$ $(1<p<\infty)$ into the Bloch space as well as into the little Bloch space.

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## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the class of all complex-valued functions analytic on $\mathbb{D}$. An analytic self-map $\varphi$ of $\mathbb{D}$ induces the composition operator $C_\varphi$ on $H(\mathbb{D})$, defined by $C_\varphi(f) = f\circ\varphi$ for $f$ analytic on $\mathbb{D}$. Let $u$ be a fixed analytic function on $\mathbb{D}$. The functions $\varphi$ and $u$ induce a linear operator $uC_\varphi$ on the space $H(\mathbb{D})$ as follows:

$$uC_\varphi f = u\cdot(f\circ\varphi),\ \ f\in H(\mathbb{D}),$$

where the dot denotes pointwise multiplication. An operator of the form $uC_\varphi$ is called a *weighted composition operator*. The functions $u$ and $\varphi$ are called the *symbols* of the operator $uC_\varphi$. We may regard this operator as a generalization of a multiplication operator and a composition operator. An interesting problem in operator theory is to provide a function theoretic characterization of the symbols of the bounded or the compact weighted composition operators on various spaces. For an in-depth study of the composition operators, see [5] and [17].

An analytic function $f$ on $\mathbb{D}$ is said to belong to the *Bloch space* $\mathscr{B}$ if

$$B(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

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The expression $B(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\mathscr{B}} = |f(0)| + B(f)$. Under this norm $\mathscr{B}$ is a Banach space. Let $\mathscr{B}_0$ denote the subspace of $\mathscr{B}$, called the *little Bloch space*, consisting of those $f \in \mathscr{B}$ which satisfy the condition

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

For $p \in (1,\infty)$, the *analytic Besov space* $B_p$ is the set of all $f \in H(\mathbb{D})$ for which

$$L_p^p(f) = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty,$$

where $dA$ is the normalized area measure on $\mathbb{D}$. The correspondence

$$f \mapsto \|f\|_{B_p} = |f(0)| + L_p(f)$$

defines a norm which yields a Banach space structure on $B_p$. In particular, $B_2$ is the classical Dirichlet space with an equivalent norm.

The composition operators on the Bloch spaces $\mathscr{B}$ and $\mathscr{B}_0$ were studied by Madigan and Matheson in [13]. In [16], Ohno and Zhao extended their results by characterizing the bounded and the compact weighted composition operators on these spaces. In [14], Ohno characterized the bounded and the compact weighted composition operators between $H^\infty$ and the Bloch space $\mathscr{B}$. The weighted composition operator from Bergman-type spaces and the Zygmund spaces into the Bloch spaces were investigated by the second author and Stević in [10, 12], respectively. The issues of boundedness and compactness of the composition operators and of the weighted composition operators between different analytic function spaces on the unit disc $\mathbb{D}$, the unit ball, as well as on bounded homogeneous domains in $\mathbb{C}^n$ have been studied by several authors, for example, in [1–4, 6–16, 18–20, 24, 25, 27–30] (see also related references therein).

In this paper we study the weighted composition operators from the Besov space $B_p$ into the Bloch space $\mathscr{B}$ and the little Bloch space $\mathscr{B}_0$. Specifically, in Section 2, we characterize the bounded weighted composition operators from the Besov space into the Bloch space and, in Section 3, we give compactness criteria for such operators. In Section 4, we characterize the bounded and the compact weighted composition operators from $B_p$ into the little Bloch space. Finally, in Section 5, we highlight the results for the component operators $C_\varphi$ and the multiplication operator $M_u$ defined as $M_u(f) = u \cdot f$. In particular, we point out that, for $1 < p < \infty$, a composition operator from $B_p$ to $\mathscr{B}$ is compact if and only if it is compact as an operator acting on several other analytic function spaces (and likewise mapping into the Bloch space).

Throughout this paper, we adopt the convention of denoting by $C$ a positive constant which may differ from one occurrence to the next. The notation $a \preceq b$ means that there is a positive constant $C$ such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, we use the notation $a \asymp b$.

## 2. Boundedness into $\mathscr{B}$

In this section we characterize the bounded weighted composition operator $uC_\varphi : B_p \to \mathscr{B}$. In order to prove the main results of this paper, we shall need the following lemmas.

**Lemma 2.1.** [26] *Suppose that* $z \in \mathbb{D}$, $t > -1$. *Then*

$$I_t(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t}} dA(w) \asymp \ln \frac{2}{1-|z|^2}, \ \text{as } |z| \to 1.$$

**Lemma 2.2.** [4] *For an analytic self map* $\varphi$ *on* $\mathbb{D}$ *and a function* $u \in \mathscr{B}$, *the following statements are equivalent:*

(a) *The sequence* $\{\|u\varphi^n\|_{\mathscr{B}}\}_{n\in\mathbb{N}}$ *is bounded.*
(b) $\sup_{z\in\mathbb{D}}((1-|z|^2)|u(z)\varphi'(z)|)/(1-|\varphi(z)|^2) < \infty$.
(c) $\sup_{w\in\mathbb{D}} \|uC_\varphi L_{\varphi(w)}\|_{\mathscr{B}} < \infty$, *where for* $a \in \mathbb{D}$, $L_a(z) = (a-z)/(1-\bar{a}z)$, $z \in \mathbb{D}$.

For $w \in \mathbb{D}$, define the function

$$f_w(z) = \Big(\ln \frac{2}{1-|w|^2}\Big)^{-1/p} \Big(\ln \frac{2}{1-\overline{w}z}\Big), \ z \in \mathbb{D}.$$

We shall now use the family $\{f_{\varphi(\lambda)} : \lambda \in \mathbb{D}\}$ and the sequence $(p_n)_{n\geq 0}$ defined by $p_n(z) = z^n$ to characterize the bounded weighted composition operators from $B_p$ to $\mathscr{B}$. Note that, however, $p_n$ is unbounded in $B_p$ and thus the characterizing condition of boundedness we obtain in terms of $uC_\varphi p_n = u\varphi^n$ does not follow immediately from the boundedness of $uC_\varphi$.

**Theorem 2.1.** *Suppose that* $1 < p < \infty$, $u \in H(\mathbb{D})$ *and* $\varphi$ *is an analytic self-map of* $\mathbb{D}$. *Then the following statements are equivalent.*

(a) $uC_\varphi : B_p \to \mathscr{B}$ *is bounded.*
(b) $\sup_{\lambda\in\mathbb{D}} \|uC_\varphi f_{\varphi(\lambda)}\|_{\mathscr{B}} < \infty$ *and* $L := \sup_{n\geq 0} \|u\varphi^n\|_{\mathscr{B}} < \infty$.
(c)

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|u'(z)|\Big(\ln \frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p} < \infty \tag{2.1}$$

*and*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\varphi'(z)||u(z)|}{1-|\varphi(z)|^2} < \infty. \tag{2.2}$$

*Proof.* (a) $\Rightarrow$ (b) Suppose that $uC_\varphi : B_p \to \mathscr{B}$ is bounded. Then applying this operator to the functions $f(z) = z$ and $f(z) = 1$, we obtain that the quantities

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|u(z)\varphi'(z) + u'(z)\varphi(z)| \quad \text{and} \quad \sup_{z\in\mathbb{D}} (1-|z|^2)|u'(z)|$$

are finite. These facts and the boundedness of the function $\varphi(z)$ indicate that

$$M := \sup_{z\in\mathbb{D}} (1-|z|^2)|u(z)\varphi'(z)| < \infty. \tag{2.3}$$

Further, for $\lambda \in \mathbb{D}$, consider

$$g_\lambda(z) = L_{\varphi(\lambda)}(z) = \frac{\varphi(\lambda) - z}{1-\overline{\varphi(\lambda)}z}, \ z \in \mathbb{D}.$$

By the Möbius invariance of the Besov space seminorm $L_p$, given $f \in B_p$, we have $L_p(f \circ g_\lambda) = L_p(f)$ for all $\lambda \in \mathbb{D}$. In particular, a straightforward calculation shows that for $f(z) = z$,

$$L_p^p(g_\lambda) = L_p^p(f) = \int_{\mathbb{D}} (1-|z|^2)^{p-2} dA(z) = \frac{1}{p-1}. \tag{2.4}$$

Hence

$$\sup_{\lambda\in\mathbb{D}}\|g_\lambda\|_{B_p}=\sup_{\lambda\in\mathbb{D}}(|\varphi(\lambda)|+L_p(g_\lambda))\le 1+\frac{1}{(p-1)^{1/p}}.$$

Moreover $g_\lambda(\varphi(\lambda))=0$ and

$$g'_\lambda(\varphi(\lambda))=\frac{1}{1-|\varphi(\lambda)|^2}.$$

Thus,

$$\Big(1+\frac{1}{(p-1)^{1/p}}\Big)\|uC_\varphi\|\ge\|uC_\varphi g_\lambda\|_{\mathscr{B}}\ge\frac{(1-|\lambda|^2)|u(\lambda)\varphi'(\lambda)|}{1-|\varphi(\lambda)|^2},$$

which yields (2.2). Since $u=uC_\varphi 1\in\mathscr{B}$, we may apply Lemma 2.2 and conclude that $L<\infty$.

Furthermore, it follows easily from Lemma 2.1 that $f_w\in B_p$, and $C:=\sup_{w\in\mathbb{D}}\|f_w\|_{B_p}$ is finite. Thus, by the boundedness of $uC_\varphi:B_p\to\mathscr{B}$, we have

$$\|uC_\varphi f_w\|_{\mathscr{B}}\le C\|uC_\varphi\|.$$

Therefore,

$$\sup_{w\in\mathbb{D}}\|uC_\varphi f_w\|_{\mathscr{B}}<\infty,$$

which in particular yields (b).

(b) $\Rightarrow$ (c) Assume (b) holds. Condition (2.2) follows immediately from Lemma 2.2, which can be applied since $u\in\mathscr{B}$ (indeed, taking $n=0$, we have $\|u\|_{\mathscr{B}}\le L<\infty$).

Next observe that, for $\lambda\in\mathbb{D}$,

$$\begin{aligned}\|uC_\varphi f_{\varphi(\lambda)}\|_{\mathscr{B}}\quad\ge\quad&\Big|\frac{(1-|\lambda|^2)|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|}{1-|\varphi(\lambda)|^2}\Big(\ln\frac{2}{1-|\varphi(\lambda)|^2}\Big)^{-1/p}\\&-(1-|\lambda|^2)|u'(\lambda)|\Big(\ln\frac{2}{1-|\varphi(\lambda)|^2}\Big)^{1-1/p}\Big|.\end{aligned}$$

Thus,

$$\begin{aligned}&(1-|\lambda|^2)|u'(\lambda)|\Big(\ln\frac{2}{1-|\varphi(\lambda)|^2}\Big)^{1-1/p}\\&\le\|uC_\varphi f_{\varphi(\lambda)}\|_{\mathscr{B}}+\frac{(1-|\lambda|^2)|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|}{1-|\varphi(\lambda)|^2}\Big(\ln\frac{2}{1-|\varphi(\lambda)|^2}\Big)^{-1/p}\\&\le\|uC_\varphi f_{\varphi(\lambda)}\|_{\mathscr{B}}+\frac{(1-|\lambda|^2)|u(\lambda)\varphi'(\lambda)|}{1-|\varphi(\lambda)|^2}\Big(\ln\frac{2}{1-|\varphi(\lambda)|^2}\Big)^{-1/p}.\end{aligned}\tag{2.5}$$

Taking the supremum over all $\lambda\in\mathbb{D}$, (2.5) yields (2.1).

(c) $\Rightarrow$ (a) Let $q$ denote the conjugate index of $p$, i.e. $1/p+1/q=1$. For $f\in B_p$, using the Hölder inequality and Lemma 2.1, we obtain

$$\begin{aligned}&|f(w)-f(0)|=\Big|\int_{\mathbb{D}}f'(z)\frac{w}{1-w\bar z}dA(z)\Big|\\&\le\int_{\mathbb{D}}|f'(z)|(1-|z|^2)^{1-2/p}\frac{|w|(1-|z|^2)^{1-2/q}}{|1-w\bar z|}dA(z)\\&\le\Big(\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-2}dA(z)\Big)^{1/p}\Big(\int_{\mathbb{D}}\frac{|w|^q(1-|z|^2)^{q-2}}{|1-w\bar z|^q}dA(z)\Big)^{1/q}\end{aligned}$$

$$\leq \|f\|_{B_p}\Big(\int_{\mathbb{D}}\frac{(1-|z|^2)^{q-2}}{|1-w\bar{z}|^q}dA(z)\Big)^{1/q}$$
$$\leq C\|f\|_{B_p}\Big(\ln\frac{2}{1-|w|^2}\Big)^{1-1/p}.$$

Since $|f(0)| \leq \|f\|_{B_p}$, we deduce that

$$|f(w)| \leq C\|f\|_{B_p}\Big(\ln\frac{2}{1-|w|^2}\Big)^{1-1/p}. \tag{2.6}$$

The inequality

$$|f'(z)| \leq \frac{C}{1-|z|^2}\|f\|_{B_p},\ z\in\mathbb{D}, \tag{2.7}$$

for some positive constant $C$, holds since $B_p$ is continuously embedded into $\mathscr{B}$.

For an arbitrary $z$ in $\mathbb{D}$ and for $f \in B_p$, (2.6) and (2.7) show that

$$\begin{aligned}
&(1-|z|^2)|(uC_\varphi f)'(z)|\\
&\leq (1-|z|^2)|u'(z)||f(\varphi(z))| + (1-|z|^2)|f'(\varphi(z))||u(z)\varphi'(z)|\\
&\leq C(1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p}\|f\|_{B_p}\\
&\quad + C\frac{(1-|z|^2)|u(z)||\varphi'(z)|}{1-|\varphi(z)|^2}\|f\|_{B_p}.
\end{aligned}\tag{2.8}$$

Taking the supremum in (2.8) over all $z \in \mathbb{D}$ and then utilizing conditions (2.1) and (2.2), we conclude that the operator $uC_\varphi : B_p \to \mathscr{B}$ is bounded. ∎

## 3. Compactness into $\mathscr{B}$

We begin this section by stating three preliminary lemmas which will be used to characterize the compact weighted composition operators from $B_p$ into the Bloch space.

**Lemma 3.1.** [4] *For an analytic self map $\varphi$ on $\mathbb{D}$ and a function $u \in \mathscr{B}$ satisfying the condition*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u'(z)| = 0,$$

*the following statements are equivalent:*

(a) $\lim_{n\to\infty}\|u\varphi^n\|_{\mathscr{B}} = 0$.

(b) $\lim_{|\varphi(z)|\to 1}((1-|z|^2)|u(z)\varphi'(z)|)/(1-|\varphi(z)|^2) = 0$.

(c) $\lim_{|\lambda|\to 1}\sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)(C_\varphi L_\lambda)'(z)| = 0$.

The following criterion for compactness follows from Lemma 3.7 of [24].

**Lemma 3.2.** *Suppose that $1 < p < \infty$, $u \in H(\mathbb{D})$ and $\varphi$ is an analytic self-map of $\mathbb{D}$. The operator $uC_\varphi : B_p \to \mathscr{B}$ is compact if and only if for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $B_p$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|uC_\varphi f_n\|_{\mathscr{B}} \to 0$ as $n \to \infty$.*

We are now ready to prove the main result of this section.

**Theorem 3.1.** *Suppose that $1 < p < \infty$, $u \in H(\mathbb{D})$, $\varphi$ is an analytic self-map of $\mathbb{D}$, and $uC_\varphi : B_p \to \mathscr{B}$ is bounded. Then the following statements are equivalent.*

(a) *$uC_\varphi : B_p \to \mathscr{B}$ is compact.*

(b)

$$\lim_{|\varphi(z)|\to 1} \|uC_\varphi f_{\varphi(z)}\|_{\mathscr{B}} = 0 \text{ and } \lim_{n\to\infty} \|u\varphi^n\|_{\mathscr{B}} = 0. \tag{3.1}$$

(c)

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)|u'(z)|\left(\ln \frac{2}{1-|\varphi(z)|^2}\right)^{1-1/p} = 0 \tag{3.2}$$

*and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)||\varphi'(z)|}{1-|\varphi(z)|^2} = 0. \tag{3.3}$$

*Proof.* (a) $\Rightarrow$ (b) Suppose $uC_\varphi : B_p \to \mathscr{B}$ is compact. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such a sequence does not exist conditions (3.2) and (3.3) are automatically satisfied, so suppose such a sequence exists. For $n \in \mathbb{N}$ and $z \in \mathbb{D}$, define

$$f_n(z) = f_{\varphi(z_n)} = \left(\ln \frac{2}{1-|\varphi(z_n)|^2}\right)^{-1/p}\left(\ln \frac{2}{1-\overline{\varphi(z_n)}z}\right).$$

Then, the sequence $(f_n)_{n\in\mathbb{N}}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ and, as was observed in Section 2, $\sup_{n\in\mathbb{N}} \|f_n\|_{B_p}$ is finite. Since $uC_\varphi$ is compact, by Lemma 3.2 we have

$$\|uC_\varphi f_{\varphi(z_n)}\|_{\mathscr{B}} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, the first condition in (3.1) is satisfied. By Lemma 3.1, to prove that $\|u\varphi^n\|_{\mathscr{B}} \to 0$ as $n \to \infty$, it suffices to show that

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)|u'(z)| = 0 \tag{3.4}$$

and that (3.3) holds. Condition (3.4) follows at once from the boundedness of $uC_\varphi$ and condition (2.1).

To prove (3.3), assume $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. For $n \in \mathbb{N}$ and $z \in \mathbb{D}$, define

$$g_n(z) = \frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z}\,\frac{\varphi(z_n)-z}{1-\overline{\varphi(z_n)}z}.$$

Then $(g_n)_{n\in\mathbb{N}}$ converges to 0 uniformly on every compact subset of $\mathbb{D}$, $g_n(\varphi(z_n)) = 0$ and

$$g_n'(\varphi(z_n)) = \frac{1}{1-|\varphi(z_n)|^2}.$$

We now prove that $(g_n)_{n\in\mathbb{N}}$ is a bounded sequence in $B_p$. A straightforward calculation shows that

$$g_n(L_{\varphi(z_n)}(z)) = z - \overline{\varphi(z_n)}z^2, \; z \in \mathbb{D}.$$

Thus, by the conformal invariance of the Besov seminorm, using (2.4), we have

$$L_p^p(g_n) = L_p^p(g_n \circ L_{\varphi(z_n)}) = \int |1-2\overline{\varphi(z_n)}z|^p(1-|z|^2)^{p-2}\,dA(z)$$

$$\leq 3^p \int (1-|z|^2)^{p-2}\, dA(z) = \frac{3^p}{p-1}.$$

Therefore,

$$\|g_n\|_{B_p} = |g_n(0)| + L_p(g_n) \leq (1-|\varphi(z_n)|^2)|\varphi(z_n)| + \frac{3}{(p-1)^{1/p}} \leq 1 + \frac{3}{(p-1)^{1/p}},$$

proving the boundedness of $(g_n)_{n\in\mathbb{N}}$ in $B_p$. Then

$$\frac{(1-|z_n|^2)|u(z_n)\varphi'(z_n)|}{1-|\varphi(z_n)|^2} \leq \sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi g_n)'(z)| \leq \|uC_\varphi g_n\|_{\mathscr{B}} \to 0 \tag{3.5}$$

as $n \to \infty$. From (3.5) it follows that condition (3.3) holds, as desired.

(b) $\Rightarrow$ (c) Assume (b) holds and that $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. As noted in the proof of Theorem 2.1, we have

$$\begin{aligned}\|uC_\varphi f_{\varphi(z_n)}\|_{\mathscr{B}} &\geq \sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi f_{\varphi(z_n)})'(z)| \\ &\geq \Big|\frac{(1-|z_n|^2)|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|}{1-|\varphi(z_n)|^2}\Big(\ln\frac{2}{1-|\varphi(z_n)|^2}\Big)^{-1/p} \\ &\quad - (1-|z_n|^2)|u'(z_n)|\Big(\ln\frac{2}{1-|\varphi(z_n)|^2}\Big)^{1-1/p}\Big|.\end{aligned}$$

Thus, from (3.1), we obtain

$$\begin{aligned}&\lim_{|\varphi(z_n)|\to 1}\frac{(1-|z_n|^2)|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|}{1-|\varphi(z_n)|^2}\Big(\ln\frac{2}{1-|\varphi(z_n)|^2}\Big)^{-1/p} \\ &= \lim_{|\varphi(z_n)|\to 1}(1-|z_n|^2)|u'(z_n)|\Big(\ln\frac{2}{1-|\varphi(z_n)|^2}\Big)^{1-1/p},\end{aligned} \tag{3.6}$$

provided that one of these two limits exists.

As argued in (a) $\Rightarrow$ (b), due to the boundedness of $uC_\varphi$, condition (3.4) holds, so we may apply Lemma 3.1. It follows that (3.3) holds.

From (3.3) we have

$$\lim_{n\to\infty}\frac{(1-|z_n|^2)|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|}{1-|\varphi(z_n)|^2}\Big(\ln\frac{2}{1-|\varphi(z_n)|^2}\Big)^{-1/p} = 0.$$

Therefore by (3.6), we get

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p} = 0,$$

which yields (3.2).

(c) $\Rightarrow$ (a) First assume conditions (3.2) and (3.3) hold. In light of Lemma 3.2, the compactness of $uC_\varphi$ will be proved if it can be shown that $\|uC_\varphi f_n\|_{\mathscr{B}} \to 0$ as $n \to \infty$ for any sequence $(f_n)_{n\in\mathbb{N}}$ bounded in $B_p$ converging to 0 uniformly on compact subsets of $\mathbb{D}$. Let $(f_n)_{n\in\mathbb{N}}$ be such a sequence and set $Q = \sup_{n\in\mathbb{N}}\|f_n\|_{B_p}$.

By (3.2) and (3.3), for any $\varepsilon > 0$, there is a constant $\delta$, $0 < \delta < 1$, such that $\delta < |\varphi(z)| < 1$ implies

$$(1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p} < \varepsilon/Q$$

and

$$|u(z)|\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} < \varepsilon/Q.$$

Let $K=\{w\in\mathbb{D}:|w|\le\delta\}$ and set $E=\{z\in\mathbb{D}:\ \delta<|\varphi(z)|<1\}=\mathbb{D}\backslash\varphi^{-1}(K)$. By the boundedness of $uC_\varphi : B_p\to\mathscr{B}$, which in particular yields $u=uC_\varphi 1\in\mathscr{B}$, we have

$$\begin{aligned}
B(uC_\varphi f_n) &= \sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi f_n)'(z)| \\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)f_n(\varphi(z))| + \sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)f_n'(\varphi(z))\varphi'(z)| \\
&\le \sup_{z\in\varphi^{-1}(K)}(1-|z|^2)|u'(z)f_n(\varphi(z))| + \sup_{z\in E}(1-|z|^2)|u'(z)f_n(\varphi(z))| \\
&\quad + \sup_{z\in\varphi^{-1}(K)}(1-|z|^2)|u(z)\varphi'(z)||f_n'(\varphi(z))| \\
&\quad + \sup_{z\in E}(1-|z|^2)|u(z)\varphi'(z)||f_n'(\varphi(z))| \\
&\le \|u\|_{\mathscr{B}}\sup_{w\in K}|f_n(w)| + M\sup_{w\in K}|f_n'(w)| \\
&\quad + C\sup_{z\in E}(1-|z|^2)|u'(z)|\left(\ln\frac{2}{1-|\varphi(z)|^2}\right)^{1-1/p}\|f_n\|_{B_p} \\
&\quad + C\sup_{z\in E}\frac{(1-|z|^2)|u(z)\varphi'(z)|}{1-|\varphi(z)|^2}\|f_n\|_{B_p} \\
&\le \|u\|_{\mathscr{B}}\sup_{w\in K}|f_n(w)| + M\sup_{w\in K}|f_n'(w)| + C\varepsilon,
\end{aligned}$$

where $M=\sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)\varphi'(z)|$. Therefore,

$$\begin{aligned}
\|uC_\varphi f_n\|_{\mathscr{B}} &= |u(0)f_n(\varphi(0))| + B(uC_\varphi f_n) \\
&\le \|u\|_{\mathscr{B}}\sup_{w\in K}|f_n(w)| + M\sup_{w\in K}|f_n'(w)| + C\varepsilon + |u(0)f_n(\varphi(0))|.
\end{aligned} \tag{3.7}$$

Since $K$ is compact and $f_n\to 0$ pointwise, it follows that,

$$\lim_{n\to\infty}\sup_{w\in K}|f_n(w)|=0$$

and $\lim_{n\to\infty}|u(0)f_n(\varphi(0))|=0$. On the other hand, by Cauchy's estimates, since $f_n$ converges to zero uniformly on compact subsets of $\mathbb{D}$, so does $f_n'$. From (3.7), letting $n\to\infty$, we obtain

$$\limsup_{n\to\infty}\|uC_\varphi f_n\|_{\mathscr{B}}\le C\varepsilon.$$

Since $\varepsilon$ is an arbitrary positive number, it follows that $\lim_{n\to\infty}\|uC_\varphi f_n\|_{\mathscr{B}}=0$. Therefore, $uC_\varphi : B_p\to\mathscr{B}$ is compact. ∎

## 4. Boundedness and compactness into the little Bloch space

We begin the section with two preliminary lemmas.

**Lemma 4.1.** *Suppose that* $1 < p < \infty$*,* $u \in H(\mathbb{D})$ *and* $\varphi$ *is an analytic self-map of* $\mathbb{D}$*. Then,*

$$\lim_{|z|\to 1}(1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p}=0 \tag{4.1}$$

*if and only if* $u \in \mathscr{B}_0$ *and*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p}=0. \tag{4.2}$$

*Proof.* Suppose that (4.1) holds, then

$$(\ln 2)^{1-1/p}(1-|z|^2)|u'(z)| \le (1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p}\to 0$$

as $|z|\to 1$. Thus, $u\in\mathscr{B}_0$. Moreover, if $|\varphi(z)|\to 1$, then $|z|\to 1$, so (4.2) holds.

Conversely, suppose $u\in\mathscr{B}_0$ and (4.2) holds. Then, for every $\varepsilon>0$, there exists $r\in(0,1)$ such that

$$(1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p}<\varepsilon$$

when $r<|\varphi(z)|<1$. Since $u\in\mathscr{B}_0$, there exists a $\sigma\in(0,1)$,

$$(1-|z|^2)|u'(z)|\le\varepsilon.$$

when $\sigma<|z|<1$.

Therefore, when $\sigma<|z|<1$ and $r<|\varphi(z)|<1$, we have

$$(1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p}<\varepsilon. \tag{4.3}$$

On the other hand, if $|\varphi(z)|\le r$ and $\sigma<|z|<1$, then

$$\begin{aligned}&(1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^{1-1/p}\\ &<(1-|z|^2)|u'(z)|\Big(\ln\frac{2}{1-r^2}\Big)^{1-1/p}<\Big(\ln\frac{2}{1-r^2}\Big)^{1-1/p}\varepsilon.\end{aligned} \tag{4.4}$$

Combining (4.3) with (4.4), we obtain the desired result. ∎

The following lemma can be verified by using the method adopted in the above proof.

**Lemma 4.2.** *Suppose* $u\in H(\mathbb{D})$ *and* $\varphi$ *is an analytic self-map of* $\mathbb{D}$*. Then*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|u(z)\varphi'(z)|}{1-|\varphi(z)|^2}=0$$

*if and only if*

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|u(z)\varphi'(z)|}{1-|\varphi(z)|^2}=0$$

*and*

$$\lim_{|z|\to 1}(1-|z|^2)|u(z)\varphi'(z)|=0.$$

**Lemma 4.3.** [13] *A closed set* $K$ *in* $\mathscr{B}_0$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)|f'(z)|=0.$$

Using Lemmas 4.1, 4.2 and 4.3, and arguing as in the proof of Theorems 4.4 and 4.5 of [10], we derive the following two results. We omit the details.

**Theorem 4.1.** *Suppose that $1 < p < \infty$, $u \in H(\mathbb{D})$ and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then $uC_\varphi : B_p \to \mathscr{B}_0$ is bounded if and only if $uC_\varphi : B_p \to \mathscr{B}$ is bounded, $u \in \mathscr{B}_0$ and*

$$\lim_{|z|\to 1}(1-|z|^2)|u(z)\varphi'(z)| = 0.$$

**Theorem 4.2.** *Suppose that $1 < p < \infty$, $u \in H(\mathbb{D})$ and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then $uC_\varphi : B_p \to \mathscr{B}_0$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)|u'(z)|\left(\ln\frac{2}{1-|\varphi(z)|^2}\right)^{1-1/p} = 0$$

*and*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)|u(z)\varphi'(z)|}{1-|\varphi(z)|^2} = 0.$$

## 5. The component operators

We end the paper by discussing the boundedness and the compactness of the component operators, the multiplication operator $M_u$ and the operator $C_\varphi$.

### 5.1. The operator $M_u$

In the special case when $\varphi(z) = z$, for $z \in \mathbb{D}$, condition (2.1) implies that $u \in \mathscr{B}_0$. From Theorems 2.1 and 4.1 we deduce the following result.

**Corollary 5.1.** *Let $u$ be analytic on $\mathbb{D}$ and $1 < p < \infty$. The following statements are equivalent.*

(a) *$M_u : B_p \to \mathscr{B}$ is bounded.*
(b) *$M_u : B_p \to \mathscr{B}_0$ is bounded.*
(c) *$u \in H^\infty$ and $\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)|\left(\ln(2/(1-|z|^2))\right)^{1-1/p} < \infty$.*

From Theorems 3.1 and 4.2, we obtain the following corollary.

**Corollary 5.2.** *Let $u$ be analytic on $\mathbb{D}$ and $1 < p < \infty$. The following statements are equivalent.*

(a) *$M_u : B_p \to \mathscr{B}$ is compact.*
(b) *$M_u : B_p \to \mathscr{B}_0$ is compact.*
(c) *$u$ is identically 0.*

### 5.2. The operator $C_\varphi$

By the Schwartz-Pick Lemma, we see that the operator $C_\varphi : B_p \to \mathscr{B}$ is bounded for any analytic self-map $\varphi$ of $\mathbb{D}$ and $1 < p < \infty$.

From Theorem 3.1, we obtain the following result which was proved in [4] in the special case of the Dirichlet space $\mathscr{D}$.

**Corollary 5.3.** *Suppose that $1 < p < \infty$ and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent:*

(a) $C_\varphi : B_p \to \mathscr{B}$ *is compact.*
(b) $\lim_{n\to\infty} \|\varphi^n\|_{\mathscr{B}} = 0$.
(c) $\lim_{|\varphi(z)|\to 1}((1-|z|^2)|\varphi'(z)|)/(1-|\varphi(z)|^2) = 0$.

Recall that the space BMOA is defined as the Banach space of functions $f$ in the Hardy space $H^2$ such that

$$\|f\|_* = \sup_{\lambda\in\mathbb{D}} \|f\circ L_\lambda - f(\lambda)\|_{H^2} < \infty$$

with norm $\|f\|_{BMOA} = |f(0)| + \|f\|_*$, where

$$\|g\|_{H^2}^2 = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^2\, d\theta.$$

In [23], Theorem 3.1, Tjani characterized the compact composition operators from the Besov spaces, the space BMOA, and the Bloch space into $\mathscr{B}$ in terms of the family of automorphisms $\{L_\lambda : \lambda \in \mathbb{D}\}$.

**Theorem 5.1.** [23] *Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and let $X = B_p$ $(1 < p < \infty)$,* BMOA*, or $\mathscr{B}$. Then $C_\varphi : X \to \mathscr{B}$ is a compact operator if and only if*

$$\lim_{|\lambda|\to 1} \|C_\varphi L_\lambda\|_{\mathscr{B}} = 0.$$

Furthermore, in [4], it was shown that a composition operator $C_\varphi : H^\infty \to \mathscr{B}$ is compact if and only if

$$\lim_{n\to\infty} \|\varphi^n\|_{\mathscr{B}} = 0.$$

From this, Theorem 5.1, and Corollary 5.3, we obtain the following result:

**Corollary 5.4.** *Suppose that $1 < p < \infty$ and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent:*

(a) $C_\varphi : B_p \to \mathscr{B}$ *is compact.*
(b) $C_\varphi : BMOA \to \mathscr{B}$ *is compact.*
(c) $C_\varphi : \mathscr{B} \to \mathscr{B}$ *is compact.*
(d) $C_\varphi : H^\infty \to \mathscr{B}$ *is compact.*

From Theorem 3 and using the remarks in [13] (see Section 1), we obtain the following result.

**Corollary 5.5.** *Suppose that $1 < p < \infty$ and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $C_\varphi : B_p \to \mathscr{B}_0$ *is bounded.*
(b) $C_\varphi : \mathscr{B}_0 \to \mathscr{B}_0$ *is bounded.*
(c) $\varphi \in \mathscr{B}_0$.

Finally, from Theorem 4 and Theorem 1 in [13], we can easily arrive at the following corollary.

**Corollary 5.6.** *Suppose that $1 < p < \infty$ and $\varphi$ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $C_\varphi : B_p \to \mathscr{B}_0$ *is compact.*
(b) $C_\varphi : \mathscr{B}_0 \to \mathscr{B}_0$ *is compact.*

(c) $\lim_{|z|\to 1}((1-|z|^2)|\varphi'(z)|)/(1-|\varphi(z)|^2) = 0$.

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