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Finite Groups with Some $\mathscr{M}$-Permutable Primary Subgroups

[1]HONGWEI BAO AND [2]LONG MIAO
[1]Department of Mathematics and Physics, Bengbu College, Bengbu 233000, People's Republic of China
[2]School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, People's Republic of China
[1]big_bao2003@163.com, [2]lmiao@yzu.edu.cn

Abstract. Let d be the smallest generator number of a finite p-group P and $\mathscr{M}_d(P) = \{P_1, P_2, \cdots P_d\}$ be the set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$. Then P is called $\mathscr{M}$-permutable in a finite group G, if there exists a subgroup B of G such that $G = PB$ and $P_iB < G$ for every P_i of $\mathscr{M}_d(P)$. In this paper, we investigate the structure of finite groups by some $\mathscr{M}$-permutable subgroups of the Sylow p-subgroup. Some new results about p-supersolvable groups and p-nilpotent groups are obtained.

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1. Introduction

All groups considered in this paper will be finite. We shall adhere to the notation employed in [4] and [17].

It seems that getting some critical information about some subgroups of Sylow subgroups of a group G often helps us to understand the structure of finite groups. Many authors have investigated the structure of a finite group when some subgroups of Sylow subgroups are well situated in the group. For instance, Wang [20] introduced the concept of c-normal subgroups and proved that if every maximal subgroup of Sylow subgroup of G is c-normal in G, then G is supersolvable. Afterwards this result was generalized [5–11,13,14]. In 2007, as an interesting application of these generalizations, Skiba [18] fixed in every noncyclic Sylow subgroup P of G a group D satisfying $1 < |D| < |P|$, and then investigated the structure of G under the assumption that all subgroups H with $|H| = |D|$ are weakly s-permutable in G. Recently, Miao and Lempken [15] considered $\mathscr{M}$-supplemented subgroups and obtained some new characterization of saturated formations containing all supersolvable groups. More recently, Miao and Lempken [16] generalized the $\mathscr{M}$-supplemented subgroups with $\mathscr{M}$-permutable subgroups and obtained some new results of supersolvable groups.

In this article, we continue to consider the p-nilpotency and p-supersolvability of finite group by using some $\mathscr{M}$-permutable primary subgroups.

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Definition 1.1. *Let G be a group and p be a prime divisor of $|G|$. A p-subgroup $P \neq 1$ of G is called $\mathscr{M}$-permutable in G if there exists a set $\mathscr{M}_d(P) = \{P_1, \cdots, P_d\}$ of maximal subgroup P_i of P and a subgroup B of G such that*

1) $\bigcap_{i=1}^{d} P_i = \Phi(P)$ *and* $|P:\Phi(P)| = p^d$ *(so d is the smallest generator number of P);*
2) $G = PB$ *and* $P_iB = BP_i < G$ *for every P_i of $\mathscr{M}_d(P)$.*

Remark 1.1. Suppose that the prime p divides the order of a finite group G. Let P be a Sylow p-subgroup of G. If G is a p-nilpotent group, then P is an $\mathscr{M}$-permutable subgroup of G. In fact, this is true if G is a p-supersolvable group.(see Lemma 2.11)

However, if, in addition, we assume that P is a cyclic group of order p^2, then P has exactly one maximal subgroup P_1. Then P_1 is not $\mathscr{M}$-permutable in G.

Remark 1.2. In a finite group G, no p-subgroup of $\Phi(G)$ can be $\mathscr{M}$-permutable in G.

Remark 1.3. Let G be a finite group. Consider an abelian minimal normal subgroup N of G. Then we have the following.

1) No proper subgroup of N can be $\mathscr{M}$-permutable in G.
2) If N is $\mathscr{M}$-permutable in G, then N is cyclic and $N \cap \Phi(G) = 1$.

Recall that, a subgroup H is called $\mathscr{M}$-supplemented in a finite group G, if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for every maximal subgroup H_1 of H([15]). Obviously, if a p-subgroup H is $\mathscr{M}$-supplemented in G, then H is also $\mathscr{M}$-permutable in G. The following example shows that the converse is not true.

Example 1.1. $G = <s,a> \times <t,b>$ where $|a| = |b| = 3, |s| = |t| = 2$ and $<s,a> \cong <t,b> \cong S_3$. Clearly, $P = <a,b> \in Syl_3(G)$, $d = 2$ and $\mathscr{M}_2(P) = \{<a>, <b>\}$. Choose $B = <s,t>$. $<a>B = B<a>, <b>B = B<b>$, but $<ab>B \neq B<ab>$. Therefore we conclude that Sylow 3-subgroup of G is $\mathscr{M}$-permutable in G, but is not $\mathscr{M}$-supplemented in G.

2. Preliminaries

Firstly, we list here some known results which will be useful in the sequel.

Lemma 2.1. [16, Lemma 2.1] *Let G be a group and $P \neq 1$ be a p-subgroup of G for some $p \in \pi(G)$. Assume that P is $\mathscr{M}$-permutable in G with respect to $\mathscr{M}_d(P)$ and that L is a normal subgroup of G contained in P. Then the following statements hold:*

1) *There exists a subgroup B of G such that $G = PB$ and $|G:P_iB| = p$ for every $P_i \in \mathscr{M}_d(P)$; moreover, $P \cap B = P_i \cap B = \Phi(P) \cap B$.*
2) *If $P \leq H \leq G$, then P is $\mathscr{M}$-permutable in H.*
3) *$L \leq \Phi(G)$ if and only if $L \leq \Phi(P)$.*
4) *If $L \leq \Phi(P)$, then P/L is $\mathscr{M}$-permutable in G/L.*
5) *If L is minimal normal in G and $L \nleq \Phi(P)$, then $|L| = p$.*
6) *If K is a normal p'-subgroup, then PK/K is $\mathscr{M}$-permutable in G/K.*

Lemma 2.2. [3, Main Theorem] *Suppose a group G has a Hall π-subgroup where π is a set of primes not containing 2. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.3. [4, Theorem 1.8.17] *Let N be a nontrivial solvable normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which is contained in N.*

Lemma 2.4. [12, Theorem I.6.6] *If H is a subgroup of G with $|G:H|=p$, where p is the smallest prime divisor of $|G|$, then $H \trianglelefteq G$.*

Lemma 2.5. [4, Lemma 3.6.10] *Let K be a normal subgroup of G and P be a p-subgroup of G where p is a prime divisor of $|G|$. Then $N_{G/K}(PK/K)=N_G(P_1)K/K$, here P_1 is a Sylow p-subgroup of PK.*

Lemma 2.6. [2, Theorem 3.2, Chapter 6] *If G is a p-solvable group where p is a prime divisor of $|G|$, then $C_G(F_p(G)) \leq F_p(G)$.*

Lemma 2.7. [8, Lemma 2.6] *Let G be a p-solvable group. Suppose that G has a chief series:*

$$1 \leq \cdots \leq \Phi(G)=K_0 \leq K_1 \cdots \leq K_s = F_p(G) \leq \cdots \leq G$$

such that K_i/K_{i-1} are cyclic groups of order p or p'-groups for all $1 \leq i \leq s$. Then G is p-supersolvable.

Lemma 2.8. [16, Lemma 2.11] *Let G be a group and $P \in Syl_p(G)$, where p is the smallest prime divisor of $|G|$. Then G is p-nilpotent if and only if every maximal subgroup of P is $\mathscr{M}$-permutable in G or has a p-nilpotent supplement in G .*

Lemma 2.9. [19] *If P is a Sylow p-subgroup of a group G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.10. *Suppose that G is a group, $\pi(G)=\{p_1,p_2=p,p_3,\cdots,p_n, p_1<p_2<\cdots<p_n\}$ and P is a Sylow p-subgroup of G. If P is $\mathscr{M}$-permutable in G, then G is p-solvable.*

Proof. Induction on the order of G. By the hypotheses, P is $\mathscr{M}$-permutable in G. There exists a suitable set $\mathscr{M}_d(P)$ and a subgroup B of G such that $G=PB$ and $P_iB<G$ for every $P_i \in \mathscr{M}_d(P) (i=1,2,\cdots,d)$. By Lemma 2.1(1), $|G:P_iB|=p$ and $P\cap B=P_i\cap B=\Phi(P)\cap B$. Clearly, $G/(P_iB)_G$ is isomorphic to a subgroup of the symmetric group S_p and $|G/(P_iB)_G| = p_1^{\alpha_1}p$. Therefore $G/\cap_{i=1}^d (P_iB)_G$ is p-solvable. If $\cap_{i=1}^d (P_iB)_G=1$, then G is solvable by Burnside Theorem. So we may assume $\cap_{i=1}^d (P_iB)_G \neq 1$. Since $P\cap(\cap_{i=1}^d (P_iB)_G) \leq P\cap(\cap_{i=1}^d P_iB)=\cap_{i=1}^d (P\cap P_iB)=\cap_{i=1}^d\{P_i(P\cap B)\}=\Phi(P)$, $\cap_{i=1}^d (P_iB)_G$ is p-nilpotent by Lemma 2.9. Thereby G is p-solvable. ∎

Lemma 2.11. *If G is a p-supersolvable groups where $p \in \pi(G)$, then Sylow p-subgroup P of G is $\mathscr{M}$-permutable in G.*

Proof. Induction on the order of G. Let L be a minimal normal subgroup of G. Since G is p-supersolvable, L is a p'-subgroup or $|L|=p$. If L is a p'-subgroup, then G/L satisfies the condition and hence PL/L is $\mathscr{M}$-permutable in G/L. It is easy to see that P is $\mathscr{M}$-permutable in G, a contradiction. On the other hand, if $\Phi(G)\neq 1$, then we may get the same contradiction by Lemma 2.1(3)(4). So we have $\Phi(G)=1$ and $F_p(G)=O_p(G)=L_1\times L_2\cdots L_t$ by Lemma 2.3 where $|L_i|=p$. By Lemma 2.6 and the hypotheses, $C_G(O_p(G))\leq O_p(G)$. Moreover, since $|L|=p$, $L\leq Z(P)$ and $O_p(G)\leq Z(P)$. It follows that $P=C_G(O_p(G))=O_p(G)$ and hence P is $\mathscr{M}$-permutable in G. The proof is over. ∎

3. Main results

Theorem 3.1. *Let G be a group and $P \in Syl_p(G)$ where p is the smallest prime divisor of $|G|$. Then G is p-nilpotent if and only if either P is cyclic or P has a subgroup D with*

$1 < D < P$, and every subgroup E of P with order $|D|$ not having a p-nilpotent supplement is $\mathscr{M}$-permutable in G.

Proof. Since the necessity part is obvious, we only need to consider the sufficiency part. Assume the theorem is false and we may choose G to be a counterexample of the minimal order.

If P is cyclic, then G is p-nilpotent by Burnside p-nilpotence Theorem, a contradiction. By hypotheses, we may assume that P is not cyclic, then there exists a subgroup D with $1 < D < P$ and every subgroup E of P with order $|D|$ not having a p-nilpotent supplement is $\mathscr{M}$-permutable in G. Fix a subgroup E of P with order $|D|$. Let P_1 be a maximal subgroup of P with $E \leq P_1$.

If E has a p-nilpotent supplement K in G, then $G = EK = P_1K$ and $K_{p'}$ is a Hall p'-subgroup of K. Hence $G = P_1N_G(K_{p'}) = PN_G(K_{p'})$. Since G is not p-nilpotent, obviously, $N_G(K_{p'})$ is not normal in G; in particular, $p^2 \mid |G : N_G(K_{p'})|$. So we may assume that $P \cap N_G(K_{p'}) \leq L_2 < L_1$ where L_1 is a maximal subgroup of P and L_2 is maximal in L_1. By the hypotheses, L_1 contains a subgroup T with order $|D|$. If T has a p-nilpotent supplement in G, then L_1 also has a p-nilpotent supplement in G and hence there exists a p-nilpotent subgroup H such that $G = TH = L_1H$. With the similar discussion as above we obtain $G = L_1N_G(H_{p'})$ where $H_{p'}$ is a Hall p'-subgroup of H and of course of G. By Lemma 2.2, there exists an element x of P such that $N_G(K_{p'}) = (N_G(H_{p'}))^x$. Therefore $G = L_1N_G(H_{p'}) = (L_1N_G(H_{p'}))^x = L_1N_G(K_{p'})$. Furthermore, $P = P \cap L_1N_G(K_{p'}) = L_1(P \cap N_G(K_{p'})) = L_1$, a contradiction.

So we may assume T is $\mathscr{M}$-permutable in G. There exists a suitable set $\mathscr{M}_d(T)$ and a subgroup B of G such that $G = TB = L_1B$ and $T_iB < G$ for every T_i of $\mathscr{M}_d(T)$. If $|D| = |L_1|$, then G is p-nilpotent by Lemma 2.8, a contradiction. If $|D| < |L_1|$, then $|G : T_iB| = p$ by Lemma 2.1(1) and hence $T_iB \trianglelefteq G$ by Lemma 2.4. So we have $G = TB = PB = PT_iB$ and $P \cap T_iB = T_i(P \cap B)$ is a Sylow p-subgroup of T_iB. Clearly, $T_i(P \cap B)$ is maximal in P. If any subgroup N of $T_i(P \cap B)$ with order $|D|$ has no p-nilpotent supplement in T_iB, then N also has no p-nilpotent supplement in G and hence is $\mathscr{M}$-permutable in G, furthermore, N is $\mathscr{M}$-permutable in T_iB by Lemma 2.1(2). Therefore T_iB satisfies the hypotheses of the theorem and hence T_iB is p-nilpotent by the minimal choice of G. On the other hand, since T_iB is normal in G and $|G : T_iB| = p$, we get G is p-nilpotent, a contradiction.

Final contradiction completes our proof. ∎

Theorem 3.2. *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and P has a nontrivial subgroup D such that every subgroup E of P with $|E| = |D|$ is $\mathscr{M}$-permutable in G.*

Proof. Based on the definition of $\mathscr{M}$-permutable subgroups, the necessity part is obvious. Next we only need to consider the sufficiency part. Assume that the assertion is false and choose G to be a counterexample of the minimal order. Then we consider the following steps.

1) $O_{p'}(G) = 1$. In fact, if $O_{p'}(G) \neq 1$, then we consider the quotient group $G/O_{p'}(G)$. By Lemma 2.1(6) and 2.5, $G/O_{p'}(G)$ satisfies the condition of the theorem, the minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent and hence G is p-nilpotent, a contradiction.

2) If S is a proper subgroup of G containing P, then S is p-nilpotent. Clearly, $N_S(P) \leq N_G(P)$ and hence $N_S(P)$ is p-nilpotent. Applying Lemma 2.1(2) and 2.5, S satisfies the hypotheses of our theorem. Then the minimal choice of G implies that S is p-nilpotent.
3) $G = PQ$, where Q is the Sylow q-subgroup of G with $q \neq p$. Since G is not p-nilpotent, by Glauberman-Thompson theorem [2, Theorem 3.1, Chapter 8], there exists a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P with $H < K \leq P$. Since $N_G(P) \leq N_G(H)$ and $N_G(H)$ is not p-nilpotent, we have $N_G(P) < N_G(H)$. Then by 2), we have $N_G(H) = G$. This leads to $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P such that $O_p(G) < K \leq P$. Now by Glauberman-Thompson theorem [2, Theorem 3.1, Chapter 8], again, we see that $G/O_p(G)$ is p-nilpotent and therefore, G is p-solvable. Since G is p-solvable, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that $PQ = QP$ is a subgroup of G by Gorenstein [2, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by 2). This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$ by Lemma 2.6. Since $O_{p'}(G) = 1$, a contradiction. Thus we have proven that $G = PQ$.
4) $\Phi(G) = 1$. Assume that $\Phi(G) \neq 1$. Notice that, by definition of $\mathscr{M}$-permutable subgroup and Remark 1.2, no subgroup of $\Phi(G)$ can be $\mathscr{M}$-permutable in G. Hence $|D| > |\Phi(G)|$ and let E be a subgroup of P with $|D| = |E|$ and $\Phi(G) < E$. By Lemma 2.1(3) we have that $\Phi(G) \leq \Phi(E)$, since E is $\mathscr{M}$-permutable in G. By Lemma 2.1(4) we have that $E/\Phi(G)$ is $\mathscr{M}$-permutable in $G/\Phi(G)$. By minimality of G, the group $G/\Phi(G)$ is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, this implies that G is p-nilpotent. This contradicts our assumption. Hence, $\Phi(G) = 1$.
5) Every minimal normal subgroup of G is cyclic. Let N be a minimal normal subgroup of G. By Remark 1.3, we have that $|N| \leq |D|$. Let E be an $\mathscr{M}$-permutable subgroup of G such that $N \leq E$. Then there exists a subgroup B such that $G = BE$ and a set $\mathscr{M}$ of maximal subgroups of E such that BE_i is a proper subgroup of G for every $E_i \in \mathscr{M}$. By Lemma 2.1(3) and step 4), $E = NE_i$ for some $E_i \in \mathscr{M}$. Then $G = (BE_i)N$. This implies that BE_i is a maximal subgroup of G and $BE_i \cap N = 1$. Hence $|N| = |G : BE_i| = p$. Therefore N is cyclic.
6) Final contradiction. Now apply [1, Theorem 2.3.24] and G is supersolvable. Since $O_q(G) = 1$, this implies that P is normal in G. But then $G = N_G(P)$ is p-nilpotent. This is a contradiction.

The final contradiction completes our proof. ∎

Corollary 3.1. *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is $\mathscr{M}$-permutable in G, then G is p-nilpotent.*

Corollary 3.2. *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and P is $\mathscr{M}$-permutable in G.*

Corollary 3.3. *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every minimal subgroup of P is $\mathscr{M}$-permutable in G, then G is p-nilpotent.*

Theorem 3.3. *Let G be a p-solvable group and P be a Sylow p-subgroup of G. Suppose that P has a subgroup D such that $1 < D \leq P$ and every subgroup E of P with $|E| = |D|$ is $\mathscr{M}$-permutable in G, then G is p-supersolvable.*

Proof. Assume that the assertion is false and choose G to be a counterexample of the minimal order. Furthermore, we have

1) $O_{p'}(G) = 1$. If $L = O_{p'}(G) \neq 1$, then PL/L is a Sylow p-subgroup of G/L. By hypotheses, P has a subgroup D such that $1 < D < P$ and every subgroup E of P with $|E| = |D|$ is $\mathscr{M}$-permutable in G. Clearly, EL/L is also $\mathscr{M}$-permutable in G/L by Lemma 2.1(6). Therefore G/L satisfies the condition of the theorem. The minimal choice of G implies that G/L is p-supersolvable, and hence G is p-supersolvable, a contradiction.
2) $O_p(G) \neq 1$. Since G is p-solvable and $O_{p'}(G) = 1$, we obtain that the minimal normal subgroup of G is an abelian p-group and hence $O_p(G) \neq 1$.
3) $\Phi(G) = 1$. Assume that $\Phi(G) \neq 1$. Notice that, by definition of $\mathscr{M}$-permutable subgroup and Remark 1.2, no subgroup of $\Phi(G)$ can be $\mathscr{M}$-permutable in G. Hence $|D| > |\Phi(G)|$ and let E be a subgroup of P with $|D| = |E|$ and $\Phi(G) < E$. By Lemma 2.1(3) we have that $\Phi(G) \leq \Phi(E)$, since E is $\mathscr{M}$-permutable in G. By Lemma 2.1(4) we have that $E/\Phi(G)$ is $\mathscr{M}$-permutable in $G/\Phi(G)$. By minimality of G, the group $G/\Phi(G)$ is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, this implies that G is p-nilpotent. This contradicts our assumption. Hence, $\Phi(G) = 1$.

By Lemma 2.3, $O_p(G) = R_1 \times \ldots \times R_t$ with minimal normal subgroups $R_1, \ldots, R_t$ of G. Let L be any minimal normal subgroup of G contained in $O_p(G)$. Assume that $|D| < |L|$ for some $L \in \{R_1, \ldots, R_t\}$ and let $E < L$ with $|E| = |D|$. By hypotheses, E is $\mathscr{M}$-permutable in G, i.e. there exists a set $\mathscr{M}_d(E)$ and a subgroup B of G such that $G = EB$ and $E_iB < G$ for every $E_i \in \mathscr{M}_d(E)$. Since Lemma 2.1(1) and $\Phi(E) \leq \Phi(O_p(G)) = 1$, we also have $E \cap B = \Phi(E) \cap B = 1$. Now $G = EB = LB$ and thus $1 \neq L \cap B \trianglelefteq G$. Since L is minimal normal in G, we get $L \leq B$ and hence $G = LB = B$, a contradiction.

Now let $L \leq E \leq P$ with $|E| = |D|$. Assume that E is $\mathscr{M}$-permutable in G; i.e. there exists $B \leq G$ and a set $\mathscr{M}_d(E)$ such that $G = EB$ and $E_lB < G$ with $|G : E_lB| = p$ by Lemma 2.1(1). Since $O_p(G) \cap \Phi(G) = 1$, there exists $E_i \in \mathscr{M}_d(E)$ with $L \nleq E_i$ and hence $E = LE_i$ as well as $G = EB = LE_iB$ and $L \cap E_iB \trianglelefteq G$. As L is minimal normal in G, we get $L \nleq E_iB$ and thus $|L| = |G : E_iB| = p$, otherwise, if $L \leq E_iB$, then $E_iB = LE_iB = EB = G$, a contradiction.

Thus $O_p(G)$ is the direct product of some minimal normal subgroup of order p of G by Lemma 2.3. Since G is p-solvable, $C_G(O_p(G)) = O_p(G)$. It follows from $G/C_G(O_p(G)) = G/O_p(G)$ is p-supersolvable and Lemma 2.7 that G is p-supersolvable, a final contradiction.

The final contradiction completes our proof. ∎

Corollary 3.4. *Let G be a group, $\pi(G) = \{p_1, p_2 = p, \cdots, p_n, p_1 < p_2 < \cdots < p_n\}$ and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < D \leq P$ and every subgroup T of H with $|T| = |D|$ is $\mathscr{M}$-permutable in G, then G is p-supersolvable.*

Proof. According to Theorem 3.3, we only need to prove G is p-solvable. Assume that the claim is false and choose G to be a counterexample of the minimal order. Then we have

1) $O_{p'}(G) = 1$. Assume that $L = O_{p'}(G) \neq 1$, then we consider the quotient group G/L. Obviously, PL/L is a Sylow p-subgroup of G/L and DL/L is a subgroup of

PL/L with $1 < DL/L \leq PL/L$. For every subgroup T/L of PL/L with $|TL/L| = |DL/L|$, we have $T = T_1L$ with $|T_1| = |D|$. Since T_1 is $\mathscr{M}$-permutable in G, we get that T/L is $\mathscr{M}$-permutable in G/L by Lemma 2.1(6). Therefore G/L satisfies the condition of the theorem. The minimal choice of G implies that G/L is p-solvable and hence G is p-solvable, a contradiction.

2) G is p-solvable. If $D = P$, then G is p-solvable by Lemma 2.10, a contradiction. So we may assume that $D < P$. Let T be a subgroup of P with $|T| = |D|$. By hypotheses, there exists a suitable set $\mathscr{M}_d(T)$ and a subgroup B of G such that $G = TB$ and $T_iB < G$ for every T_i of $\mathscr{M}_d(T)$. Applying Lemma 2.1(1), we have $|G : T_iB| = p$. Then $G/(T_iB)_G$ is isomorphic to a subgroup of the symmetric group S_p and $|G/(T_iB)_G| = p_1^{\alpha_1}p$, therefore $G/(T_iB)_G$ is p-solvable. Clearly, for every T_i of $\mathscr{M}_d(T)$, we have $(T_iB)_G \neq 1$ and hence $T_iB \neq 1$. Let $L = (T_iB)_p$ be a Sylow p-subgroup of T_iB. Then L is a maximal subgroup of P. If $|D| = |L|$, then L is $\mathscr{M}$-permutable in T_iB by Lemma 2.1(2) and hence T_iB is p-solvable by Lemma 2.10. This implies that $(T_iB)_G$ is p-solvable.

So we may assert that $|D| < |L|$. It follows from Lemma 2.1(2) that every subgroup A of L with $|A| = |D|$ is $\mathscr{M}$-permutable in T_iB. The minimal choice of G implies that T_iB is p-solvable and hence $(T_iB)_G$ is p-solvable. Therefore G is p-solvable and G is p-supersolvable by Theorem 3.3. ∎

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