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# Convergence Theorems for Generalized Mixed Equilibrium and Variational Inclusion Problems of Strict-Pseudocontractive Mappings

[1]NAWITCHA ONJAI-UEA AND [2]POOM KUMAM
[1]Department of Mathematics, Faculty of Science and Technology, Nakhon Pathom Rajabhat University (NPRU), Muang, Nakhon Pathom 73000, Thailand
[2]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Bangkok 10140, Thailand
[1]nawitcha@hotmail.com, [2]poom.kum@kmutt.ac.th

**Abstract.** In this paper, we introduce and analyze a new iterative algorithm for finding a common element of the set of fixed points of strictly-pseudocontractive, the set of common solutions of generalized mixed equilibrium problems and the set of common solutions of the variational inclusion with inverse-strongly accretive mappings in Banach spaces. Using our new iterative scheme, we prove strong convergence theorems for approximation of common element of the three above mentioned sets. The results obtained in this paper extend the corresponding results announced by many authors and the previously known results in this area.

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## 1. Introduction

Let $E$ be a real Banach space with norm $\|\cdot\|$ and $C$ be a nonempty, closed and convex subset of $E$. Let $E^*$ be the dual space of $E$ and $\langle\cdot,\cdot\rangle$ denote the pairing between $E$ and $E^*$. For $q>1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\},$$

for all $x \in E$. In particular, if $q = 2$, the mapping $J_2$ is called the *normalized duality mapping* and usually write $J_2 = J$.

Let $\varphi : E \to \mathbb{R}\cup\{+\infty\}$ be a proper extended real-valued function and let $F$ be a bifunction of $E\times E$ into $\mathbb{R}$ such that $E\cap \operatorname{dom}\varphi \neq \emptyset$, where $\mathbb{R}$ is the set of real numbers and $\operatorname{dom}\varphi = \{x \in E : \varphi(x) < +\infty\}$.
The *generalized mixed equilibrium problem* for finding $x \in E$ such that

$$F(x,y) + \langle Bx, y-x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in E. \tag{1.1}$$

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The set of solutions of (1.1) is denoted by $GMEP(F,\varphi,B)$, that is,

$$GMEP(F,\varphi,B)=\{x\in E : F(x,y)+\langle Bx,y-x\rangle+\varphi(y)-\varphi(x)\geq 0,\quad \forall y\in E\}. \tag{1.2}$$

We see that $x$ is a solution of a problem (1.1) implies that $x\in \mathrm{dom}\varphi=\{x\in E : \varphi(x)<+\infty\}$.

**Special Examples**.

(1) If $B=0$, the problem (1.1) is reduced into the *mixed equilibrium problem* for finding $x\in E$ such that

$$F(x,y)+\varphi(y)-\varphi(x)\geq 0,\quad \forall y\in E. \tag{1.3}$$

Problem (1.3) was studied by Ceng and Yao [11]. The set of solutions of (1.3) is denoted by $MEP(F,\varphi)$.

(2) If $\varphi=0$, the problem (1.1) is reduced into the *generalized equilibrium problem* for finding $x\in E$ such that

$$F(x,y)+\langle Bx,y-x\rangle\geq 0,\quad \forall y\in E. \tag{1.4}$$

Problem (1.4) was studied by Takahashi and Toyoda [41]. The set of solutions of (1.4) is denoted by $GEP(F,B)$.

(3) If $B=0$ and $\varphi=0$, the problem (1.1) is reduced into the *equilibrium problem* for finding $x\in E$ such that

$$F(x,y)\geq 0,\quad \forall y\in E. \tag{1.5}$$

Problem (1.5) was studied by Blum and Oettli [3]. The set of solutions of (1.5) is denoted by $EP(F)$.

(4) If $F=0$, the problem (1.1) is reduced into the *mixed variational inequality of Browder type* for finding $x\in E$ such that

$$\langle Bx,y-x\rangle+\varphi(y)-\varphi(x)\geq 0,\quad \forall y\in E. \tag{1.6}$$

Problem (1.6) was studied by Browder [4]. The set of solutions of (1.6) is denoted by $VI(E,B,\varphi)$.

The generalized mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems and the equilibrium problem as special cases. Numerous problems in physics, optimization and economics reduce to find a solution of (1.1). Many authors have proposed some useful methods for solving the $GMEP(F,\varphi,B)$, $MEP(F,\varphi)$ and $EP(F)$; see, for instance [8, 10–19, 28, 29, 35, 36, 40, 43, 46].

In this paper, we use $F(S)$ to denote the set of fixed points of the mapping $S$, that is, $F(S)=\{x\in E : Sx=x\}$. Recall that the mapping $S$ is said to be *nonexpansive*, if

$$\|Sx-Sy\|\leq\|x-y\|,\quad \forall x,y\in E.$$

A mapping $T$ is said to be $\lambda$*-strictly pseudo-contractive*, if there exists a constant $\lambda\in[0,1)$ such that

$$\langle Tx-Ty,J(x-y)\rangle\leq\|x-y\|^2-\lambda\|(I-T)x-(I-T)y\|^2,\ \forall x,y\in C.$$

A mapping $T$ is said to be *pseudo-contraction*, if there exists a constant $\lambda\in(0,1)$ such that

$$\langle Tx-Ty,J(x-y)\rangle\leq\|x-y\|^2,\ \forall x,y\in C.$$

Recall that an operator $A$ of $E$ into itself is said to be *accretive* if

$$\langle Ax - Ay, J(x-y)\rangle \geq 0, \ \forall x, y \in E.$$

For $\alpha > 0$, recall that an operator $A$ of $E$ into itself is said to be *$\alpha$-inverse strongly accretive* if

$$\langle Ax - Ay, J(x-y)\rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in E.$$

The class of strictly pseudo-contractive falls into the one between classes of nonexpansive mappings and pseudo-contraction. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of fixed points for strictly pseudo-contractive. In 1967, Browder and Petryshyn [6] introduced a convex combination method to study strictly pseudo-contractive in Hilbert spaces. On the other hand, Marino and Xu [27] and Zhou [48] developed some iterative scheme for finding a fixed point of a strictly pseudo-contractive mapping. More precisely, take $k \in [0,1)$ and define a mapping $S_k$ by

$$S_k x = kx + (1-k)Sx, \quad \forall x \in E,$$

where $S$ is a strictly pseudo-contractive. Under appropriate restrictions on $k$, it is proved the mapping $S_k$ is nonexpansive. Therefore, the techniques of studying nonexpansive mappings can be applied to study more general strictly pseudo-contractive. Variational inclusions are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in the optimization and control, economics and transportation equilibrium, engineering science. For these reasons, many existence result and iterative algorithms for various variational inclusion have been studied extensively many authors. For detail, see [1, 21–26] and references therein. In this paper, we consider the problem so-called *quasi-variational inclusions problems*, which is to find $u \in E$ such that

$$0 \in A(u) + M(u), \tag{1.7}$$

where $A : E \to E$ and $M : E \to 2^E$ are nonlinear mappings. The problem (1.7) denoted by $VI(E, A, M)$.

**Definition 1.1.** [47] *Let $M : E \to 2^E$ be a multi-valued maximal accretive mapping. The single-valued mapping $J_{(M,\rho)} : E \to E$ defined by*

$$J_{(M,\rho)}(u) = (I + \rho M)^{-1}(u), \ \forall u \in E$$

*is called the* resolvent *operator associated with $M$, where $\rho$ is any positive number and $I$ is the identity mapping.*

Let $D$ be a subset of $C$ and $Q$ be a mapping of $C$ into $D$. Then $Q$ is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q$ of $C$ into itself is called a *retraction* if $Q^2 = Q$. If a mapping $Q$ of $C$ into itself is a retraction, then $Qz = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of $P$. A subset $D$ of $C$ is called a *sunny nonexpansive retract* of $C$ if there exists a sunny nonexpansive retraction from $C$ onto $D$.

In 2006, Aoyama, Iiduka and Takahashi [2] considered the following problem for finding $u \in C$ such that

$$\langle Au, J(v-u)\rangle \geq 0, \ \forall v \in C. \tag{1.8}$$

The variational inequality (1.8) is equivalent to a fixed point problem. The element $u \in C$ is a solution of the variational inequality (1.8) if and only if $u \in C$ satisfies the equation

$$u = Q_C(u - \lambda Au), \tag{1.9}$$

where $\lambda > 0$ is constant and $Q_C$ is sunny nonexpansive retraction from $E$ onto $C$, see the definition below.
In order to find a solution of the problem (1.8), Aoyama *et al.* [2] introduced the following iterative algorithm in Banach spaces:

$$\begin{cases} x_1 \in E \text{ chosen arbitrary}, \\ y_n = Q_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n, \ \forall n \geq 0, \end{cases} \tag{1.10}$$

where $Q_C$ is a sunny nonexpansive retraction from $E$ onto $C$.
Very recently, Ceng *et al.* [9] introduced iterative scheme for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a $k$-strictly pseudo-contractive mapping defined in the setting of real Hilbert space $H$: $x_0 \in H$, let $C$ be a nonempty closed and convex subset of $H$ and then by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)Su_n, \end{cases} \tag{1.11}$$

where $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (k,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and $S$ is $k$-strictly pseudo-contractive mapping. Further, they proved $\{x_n\}$ and $\{u_n\}$ converge weakly to $q \in F(S) \cap EP(F)$, where $q = P_{F(S)\cap EP(F)}x_0$.
In 2008, Zhang, Lee and Chan [47] considered the problem (1.7) in Hilbert spaces. To be more precise, they introduced the new following iterative process:

$$\begin{cases} x_0 \in H, \ chosen\ arbitrary, \\ y_n = J_{(M,\rho)}(x_n - \rho Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Sy_n \end{cases} \tag{1.12}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\lambda \in (0, 2\alpha]$ satisfying some mild condition. They proved that if $F(S) \cap VI(H,A,M) \neq \emptyset$, where $F(S)$ is denoted for the set of fixed point of a nonexpansive mapping, then $\{x_n\}$ generated by (1.14) converges strongly to $x_0$, where $x_0 = P_{F(S)\cap VI(H,A,M)}f(x_0)$.

In 2010, Qin, Cho and Kang [33] proved the following theorem.

**Theorem 1.1** (Theorem QCK)**.** [33] *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $F$ be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-A(4) (in section 2) and $B : C \to H$ a $\lambda$-inverse-strongly monotone mapping. Let $S : C \to C$ be a $k$-strict pseudo-contraction, $A_1 : C \to H$ an $\alpha$-inverse-strongly monotone mapping, $A_2 : C \to H$ an $\beta$-inverse-strongly monotone mapping. Assume that $\mathscr{F} := EP(F,B) \cap VI(C,A_1) \cap VI(C,A_2) \cap F(S)$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequences in (0,1). Let $\{t_n\}$ be a* *sequences in* $(0,2\alpha)$, $\{s_n\}$ *a sequences in* $(0,2\beta)$ *and* $\{r_n\}$ *a sequences in* $(0,2\lambda)$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$(1.13)\quad \begin{cases} x_1 \in C, \text{ chosen arbitrary}, \\ u_n \in C \text{ such that } F(u_n,u) + \langle Bx_n, u-u_n\rangle + \frac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \geq 0,\ \forall u \in C, \\ z_n = P_C(u_n - s_nA_2u_n), \\ y_n = P_C(z_n - t_nA_1z_n), \\ x_{n+1} = \alpha_nx_n + (1-\alpha_n)(\beta_ny_n + (1-\beta_n)Sy_n),\ \forall n \geq 1. \end{cases}$$

*Assume that the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{t_n\}$, $\{s_n\}$ *and* $\{r_n\}$ *satisfy the following restrictions:*

(a) $0 < a \leq \alpha_n \leq a' < 1$;
(b) $0 < k \leq \beta_n \leq b < 1$;
(c) $0 < c \leq r_n \leq d < 2\lambda$, $0 < c' \leq s_n \leq d' < 2\beta$ *and* $0 < c'' \leq t_n \leq d'' < 2\alpha$.

*Then the sequence* $\{x_n\}$ *generated in (1.14) converges weakly to some point* $\bar{x} \in \mathscr{F}$, *where* $\bar{x} = \lim_{n\to\infty} P_{\mathscr{F}}x_n$ *and* $P_{\mathscr{F}}$ *is the projection of* $H$ *onto set* $\mathscr{F}$.

Next, Petrot *et al.* [31] introduced the new following iterative process for finding the set of solution of quasi-variational inclusion problem and the set of fixed point of a nonexpansive mapping. The sequence generate by

$$(1.14)\quad \begin{cases} x_0 \in H, \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_nf(x_n) + \beta_nx_n + \gamma_nSz_n \\ z_n = J_{M,\lambda}(y_n - \lambda Ay_n), \\ y_n = J_{M,\rho}(x_n - \rho Ax_n), \end{cases}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $S$ is nonexpansive self mapping on $C$, $A : H \to H$ an $\alpha$-inverse-strongly monotone mapping $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ and $\lambda \in (0,2\alpha]$. They proved that $\{x_n\}$ generated by (1.14) converges strongly to $z_0$ which is the unique solution in $F(S) \cap VI(H,A,M)$.

In 2010, Kumam *et al.* [20] introduced the shrinking projection method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of variational inclusion problems in Hilbert spaces. Starting with an arbitrary $C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$ define sequence $\{x_n\}$, $\{z_n\}$, $\{v_n\}$ and $\{y_n\}$ as follows

$$(1.15)\quad \begin{cases} F(u_n,y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0,\ \forall y \in C, \\ y_n = J_{M,\delta_n}(u_n - \delta_nAu_n), \\ v_n = J_{M,\lambda_n}(y_n - \lambda_nAy_n), \\ z_n = \alpha_nx_n + (1-\alpha_n)K_nv_n,\ n \geq 1, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\},\ n \geq 1, \\ x_{n+1} = P_{C_{n+1}}x_0,\ n \geq 1, \end{cases}$$

where $K_n$ be the $K$-mapping, $A$, $B$ be $\beta$, $\xi$-inverse-strongly monotone mapping of $C$ into $H$. They proved that if the sequences $\{\alpha_n\}$, $\{r_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ is generated by (1.15) converges strongly to $P_{\cap_{i=1}^{\infty}F(S_i)\cap GMEP(F,\varphi,B)\cap VI(E,A,M)}x_0$.

Very recently, Qin *et al.* [34] considered the problem of finding the solutions of a general system of variational inclusion with $\alpha$-inverse strongly accretive mappings. To be more precise, they obtained the following result:

**Theorem 1.2** (Theorem QCCK)**.** [34] *Let $E$ be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant $K$ (see Lemma 2.5 in section 2). Let $M_i : E \to 2^E$ be a maximal monotone mapping, and $A_i : E \to E$ a $\gamma_i$-inverse-strongly accretive mapping, respectively for each $i = 1,2$. Let $T : E \to E$ be a $\lambda$-strict pseudo-contraction with fixed point. Define a mapping $S$ by $Sx = (1-\frac{\lambda}{K^2})x + \frac{\lambda}{K^2}Tx, \forall x \in E$. Assume that $\Theta = F(T) \cap VI(E,A,M) \neq \emptyset$. Let $x_1 = u \in E$ and $\{x_n\}$ a sequence generated by*

$$
(1.16) \qquad \begin{cases} z_n = J_{(M_2,\rho_2)}(x_n - \rho_2 A_2 x_n), \\ y_n = J_{(M_1,\rho_1)}(z_n - \rho_1 A_1 z_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1-\beta_n-\alpha_n)[\mu S x_n + (1-\mu) y_n], \ \forall n \geq 1, \end{cases}
$$

*where $\mu \in (0,1)$, $\rho_1 \in (0,\gamma_1/K^2]$, $\rho_2 \in (0,\gamma_2/K^2]$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1). If the control consequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following restrictions*

*(C1)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and (C2)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

*then $\{x_n\}$ converges strongly to $x^* = P_\Theta u$, where $P_\Theta$ is the sunny nonexpansive retraction from $E$ onto $\Theta$ and $(x^*,y^*)$, where $y^* = J_{(M_2,\rho_2)}(x^* - \rho_2 A_2 x^*)$.*

Could we extend the iterative algorithm (1.11), (1.14) and (1.15) to solve the problem (1.2) and (1.7) from Hilbert spaces to general Banach spaces?

The purpose of this paper is to give affirmative answer to this questions mentioned above. Motivated and inspired by Zhang, Lee and Chan [47], Ceng *et al.* [9] and Qin *et al.* [34], we introduce a new iterative scheme which for finding a common element of the set of fixed points of strict pseudo-contractions, the set of common solutions of a generalized mixed equilibrium problem and the set of common solutions of the quasi-variational inclusion in Banach spaces. Strong convergence theorems are established in uniformly convex and 2-uniformly smooth Banach spaces. The results in this paper extend and improve the corresponding recent results.

## 2. Preliminaries

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space $E$ is said to be *uniformly convex* if, for any $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x,y \in U$, $\|x-y\| \geq \varepsilon$ implies $\|\frac{x+y}{2}\| \leq 1-\delta$.

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space $E$ is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x,y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x,y \in U$. The *modulus of smoothness* of $E$ is defined by

$$
\rho(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x,y \in E, \|x\| = 1, \|y\| = \tau\},
$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that $E$ is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$. Let $q$ be a fixed real number with $1 < q \leq 2$. A Banach space $E$ is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

We note that $E$ is a uniformly smooth Banach space if and only if $J_q$ is single-valued and uniformly continuous on any bounded subset of $E$. Typical examples of both uniformly convex and uniformly smooth Banach spaces are $L^p$, where $p > 1$. More precisely, $L^p$ is $\min\{p,2\}$-uniformly smooth for every $p > 1$.

**Lemma 2.1.** (Zhou [49]) *Let $E$ be a real 2-uniformly smooth Banach space $E$ and $S : E \to E$ be a $\lambda$-strict pseudo-contraction. Then $S_k := (1-\lambda/K^2)I + \lambda/K^2 S$ is nonexpansive and $F(S_k) = F(S)$.*

**Lemma 2.2.** (Bruck [7] and see on Qin *et al.* [32]) *Let $C$ be a nonempty, closed and convex subset of a strictly convex Banach space $E$. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on $E$. Suppose $\cap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\delta_n$ be a sequence of positive number with $\sum_{n=1}^{\infty} \delta_n = 1$. Then a mapping $S$ on $E$ define by*

$$Sx = \sum_{n=1}^{\infty} \delta_n T_n x$$

*for $x \in E$ is well defined, nonexpansive and $F(S) = \cap_{n=1}^{\infty} F(T_n)$ holds.*

For solving the generalized mixed equilibrium problem, let us give the following assumptions for the bifunction $F : C \times C \to \mathbb{R}$, $\varphi : C \to \mathbb{R}$ is convex and lower semi-continuous, the nonlinear mapping $B : C \to E^*$ is continuous and monotone satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) $F$ is monotone, i.e., $F(x,y) + F(y,x) \le 0 \ \forall x,y \in C$;
(A3) for each $x,y,z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \le F(x,y)$;
(A4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semi-continuous;
(B1) for each $x \in E$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, Jz - Jx \rangle < 0; \tag{2.1}$$

(B2) $C$ is a bounded set.

**Lemma 2.3.** (Takahashi and Zembayashi [42]) *Let $C$ be a closed and convex subset of smooth, strictly convex and reflexive Banach space $E$, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfies (A1)-(A4) and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \ \forall y \in C.$$

Motivated by Combettes and Hirstoaga [12] in a Hilbert space and Takahashi and Zembayashi [42] in a Banach space, Zhang [48] and also [30] obtain the following lemma.

**Lemma 2.4.** (Zhang [48]) *Let $C$ be nonempty, closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space $E$. Let $B : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $F : C \times C \to \mathbb{R}$ be a bifunction satisfies (A1)-(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$F(u,y) + \langle Bu, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle, \ \forall y \in C.$$

*Define a mapping $K_r : C \to C$ as follows:*

$$K_r(x) = \left\{ u \in C : F(u,y) + \langle Bu, y - u \rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \ge 0, \ \ \forall y \in C \right\}$$

*for all $x \in C$. Then, the following conclusions hold:*

(i) $K_r$ *is single-valued;*
(ii) $K_r$ *is firmly nonexpansive, i.e., for any* $x,y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \le \langle K_r x - K_r y, Jx - Jy\rangle$;
(iii) $F(K_r) = GMEP(F,\varphi,B)$;
(iv) $GMEP(F,\varphi,B)$ *is closed and convex.*

**Lemma 2.5.** (Xu [44]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2,\ \forall x,y \in E.$$

**Lemma 2.6.** (Suzuki [38]) *Let* $\{x_n\}$ *and* $\{l_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ *for all integers* $n \ge 1$ *and* $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then,* $\lim_{n\to\infty}\|l_n - x_n\| = 0$.

**Proposition 2.1.** (Reich [37]) *Let E be a uniformly smooth Banach space and* $T : C \to C$ *a nonexpansive mapping such that* $F(T) \ne \emptyset$. *For each fixed* $u \in C$ *and every* $t \in (0,1)$, *the unique fixed point* $x_t \in C$ *of the contraction* $C \ni x \mapsto tu + (1-t)Tx$ *converges strongly as* $t \to 0$ *to a fixed point of T. Define* $Q : C \to D$ *by* $Qu = s - \lim_{t\to 0} x_t$. *Then Q is the unique sunny nonexpansive retract from C onto D, that is, Q satisfies the property:*

$$\langle u - Qu, J(y - Qu)\rangle \le 0,\ \forall u \in C,\ y \in D.$$

*Note that we use* $Qu = s - \lim_{t\to 0} x_t$ *to denote strong convergence to* $Q_u$ *of the net* $\{x_t\}$ *as* $t \to 0$.

A Banach space $X$ is said to be satisfying Opial's condition if for any sequence $x_n \rightharpoonup x$ for all $x \in X$ implies

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|,\ \forall y \in X,\ \text{with } x \ne y.$$

**Lemma 2.7.** (Browder [5] (Demi Closed Principle)) *Let C be a nonempty closed convex subset of a reflexive Banach space X which Opial's condition and suppose* $T : C \to X$ *is nonexpansive. Then the mapping* $I - T$ *is demiclosed at zero, i.e.,* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$ *implies* $x = Tx$.

**Lemma 2.8.** (Xu [45]) *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\rho_n)a_n + \sigma_n,\ n \ge 1,$$

*where* $\{\rho_n\}$ *is a sequence in* $(0,1)$ *and* $\{\sigma_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty}\rho_n = \infty$,
(2) $\limsup_{n\to\infty}\frac{\sigma_n}{\rho_n} \le 0$ *or* $\sum_{n=1}^{\infty}|\sigma_n| < \infty$.

*Then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.9.** *Let E be a Banach space. Then for all* $x,y \in E$,
$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle$.

## 3. Main results

In this section, we will use the new viscosity approximation iterative method to prove a strong convergence theorem for finding a common element of the set of fixed points of strictly pseudo-contractive mapping, the set of common solutions of common of a generalized mixed equilibrium problem and the set of a common solutions of the variational inclusion for inverse-strongly accretive mappings in a Banach spaces.

**Lemma 3.1.** *Let $C$ be a nonempty, closed and convex subset of uniformly convex and a real 2-uniformly smooth Banach space $E$ with the smooth constant $K$. Let be $A_1 : C \to C$ an $\gamma_1$-inverse-strongly accretive mapping. If $\rho_1 \in (0, \gamma_1/K^2)$, then $I - \rho_1 A_1$ is nonexpansive.*

*Proof.* For any $x, y \in C$, Lemma 2.5, one has

$$\begin{aligned}\|(I-\rho_1 A_1)x-(I-\rho_1 A_1)y\|^2 &= \|(x-y)-\rho_1(A_1x-A_1y)\|^2\\ &\le \|x-y\|^2 - 2\rho\langle A_1x-A_1y, J(x-y)\rangle + 2K^2\rho_1^2\|Ax-Ay\|^2\\ &\le \|x-y\|^2 - 2\rho_1\gamma_1\|A_1x-A_1y\|^2 + 2K^2\rho_1^2\|A_1x-A_1y\|^2\\ &= \|x-y\|^2 - 2\rho_1(\gamma_1-K^2\rho_1)\|Ax-Ay\|^2\\ &\le \|x-y\|^2,\end{aligned}$$

which implies the mapping $I - \rho_1 A_1$ is nonexpansive. ∎

**Theorem 3.1.** *Let $C$ be a subset of uniformly convex and 2-uniformly smooth Banach space of $E$ with the smooth constant $K$. Let $M_1, M_2 : E \to 2^E$ be maximal accretive mappings and $A_1, A_2 : E \to E$ $\gamma_1, \gamma_2$-inverse-strongly accretive mappings, respectively. Let $F_1$ and $F_2$ be two bifunctions from $E \times E$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let $B_1 : C \to E^*$ be an $\rho$-inverse-strongly accretive mapping, $B_2 : C \to E^*$ be an $\omega$-inverse-strongly accretive mapping. Let $f : E \to E$ be an $\alpha$-contraction with coefficient $\alpha$ $(0 \le \alpha < 1)$. Let $S : E \to E$ be an $\lambda$-strictly pseudo-contractive mapping with a fixed point. Define a mapping $S_k$ by $S_k x = kx + (1-k)Sx$, $\forall x \in E$. Suppose that*

$$\Theta := F(S) \cap VI(E, A_1, M_1) \cap VI(E, A_2, M_2) \cap GMEP(F_1, \varphi, B_1) \cap GMEP(F_2, \varphi, B_2) \neq \emptyset.$$

*Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

(3.1)
$$\begin{cases} F_1(u_n^{(1)}, y_1) + \langle B_1 x_n, y_1 - u_n^{(1)}\rangle + \varphi(y) - \varphi(u_n^{(1)}) + \frac{1}{r}\langle y_1 - u_n^{(1)}, Ju_n^{(1)} - Jx_n\rangle \ge 0, \ \forall y_1 \in C,\\ F_2(u_n^{(2)}, y_2) + \langle B_2 x_n, y_2 - u_n^{(2)}\rangle + \varphi(y) - \varphi(u_n^{(2)}) + \frac{1}{s}\langle y_2 - u_n^{(2)}, Ju_n^{(2)} - Jx_n\rangle \ge 0, \ \forall y_2 \in C,\\ y_n = J_{M_2,\rho_2}(x_n - \rho_2 A_2 x_n),\\ v_n = J_{M_1,\rho_1}(x_n - \rho_1 A_1 x_n),\\ t_n = \mu_n^{(1)} S_k x_n + \mu_n^{(2)} v_n + \mu_n^{(3)} y_n + \mu_n^{(4)} u_n^{(1)} + \mu_n^{(5)} u_n^{(2)},\\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n t_n,\end{cases}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4,5$, $\alpha_n + \beta_n + \gamma_n = 1$, $\rho_1 \in (0, \gamma_1/K^2]$, $\rho_2 \in (0, \gamma_2/K^2]$, $r \in (0, 2\rho)$ and $s \in (0, 2\omega)$. Assume that the control sequences satisfy the following restrictions:*

(C1) $\sum_{i=1}^{5} \mu_n^{(i)} = 1$,
(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(C4) $\lim_{n\to\infty} \mu_n^{(i)} = \mu^{(i)} \in (0,1)$, *where* $i = 1,2,3,4,5$.

*Then* $\{x_n\}$ *converges strongly to* $x^* = Q_\Theta f(x^*)$, *where* $Q_\Theta$ *is the sunny nonexpansive retraction from* $E$ *onto* $\Theta$.

*Proof.* First, we define four functions $H_1, H_2 : C \times C \to \mathbb{R}$ and $\widetilde{K}_r, \widetilde{K}_s : C \to C$ by

$$H_1(u^{(1)}, y_1) = F_1(u^{(1)}, y_1) + \langle B_1 x, y_1 - u^{(1)}\rangle + \varphi(y) - \varphi(u^{(1)}),\ y_1 \in C;$$

$$\widetilde{K}_r(x) = \{u \in C : H_1(u^{(1)}, y_1) + \frac{1}{r}\langle y_1 - u^{(1)}, Ju^{(1)} - Jx\rangle \ge 0,\ \forall y_1 \in C, \}, x \in C, \tag{3.2}$$

and

$$H_2(u^{(2)}, y_2) = F_2(u^2, y_2) + \langle B_2 x, y_2 - u^{(2)}\rangle + \varphi(y) - \varphi(u^{(2)}),\ y_2 \in C;$$

$$\widetilde{K}_s(x) = \{u \in C : H_2(u^{(2)}, y_2) + \frac{1}{s}\langle y_2 - u^{(2)}, Ju^{(2)} - Jx\rangle \ge 0,\ \forall y_2 \in C, \}, x \in C. \tag{3.3}$$

By Lemma 2.4, we know that the functions $H_1, H_2$ satisfy the conditions (A1)-(A4) and $\widetilde{K}_r, \widetilde{K}_s$ satisfy properties (i)-(iv).
We will divide the proof into five steps.

**Step 1**. We claim that $\{x_n\}$ is bounded.

Indeed, let $p \in \Theta$ and Lemma 2.4, we obtain

$$p = J_{M_1,\rho_1}(p - \rho_1 A_1 p) = J_{M_2,\rho_2}(p - \rho_2 A_2 p) = \widetilde{K}_r p = \widetilde{K}_s p.$$

We note that $u_n^{(1)} = \widetilde{K}_r x_n \in \text{dom } \varphi$ and $u_n^{(2)} = \widetilde{K}_s x_n \in \text{dom } \varphi$, and since $\widetilde{K}_r$ and $\widetilde{K}_s$ are nonexpansive mappings, we have

$$\|u_n^{(1)} - p\| = \|\widetilde{K}_r x_n - \widetilde{K}_r p\| \le \|x_n - p\| \tag{3.4}$$

and

$$\|u_n^{(2)} - p\| = \|\widetilde{K}_s x_n - \widetilde{K}_s p\| \le \|x_n - p\|. \tag{3.5}$$

Putting $v_n = J_{M_1,\rho_1}(x_n - \rho_1 A_1 x_n)$ and $y_n = J_{M_2,\rho_2}(x_n - \rho_2 A_2 x_n)$, we get $I - \rho_1 A_1$ and $I - \rho_2 A_2$ are nonexpansive. Thus, we have

$$\begin{aligned}
\|v_n - p\| &= \|J_{M_1,\rho_1}(x_n - \rho_1 A_1 x_n) - J_{M_1,\rho_1}(p - \rho_1 A_1 p)\| \\
&\le \|(x_n - \rho_1 A_1 x_n) - (p - \rho_1 A_1 p)\| \\
&= \|(I - \rho_1 A_1)x_n - (I - \rho_1 A_1)p\| \\
&\le \|x_n - p\|
\end{aligned}$$

and similarly, we also have

$$\|y_n - p\| \le \|x_n - p\|.$$

From Lemma 2.1, we have that $S_k$ is nonexpansive with $F(S_k) = F(S)$. It follows that

$$\begin{aligned}
\|t_n - p\| &= \|\mu_n^{(1)}(S_k x_n - p) + \mu_n^{(2)}(v_n - p) + \mu_n^{(3)}(y_n - p) + \mu_n^{(4)}(u_n^{(1)} - p) + \mu_n^{(5)}(u_n^{(2)} - p)\| \\
&\le \mu_n^{(1)}\|S_k x_n - p\| + \mu_n^{(2)}\|v_n - p\| + \mu_n^{(3)}\|y_n - p\| + \mu_n^{(4)}\|u_n^{(1)} - p\| + \mu_n^{(5)}\|u_n^{(2)} - p\| \\
&\le \mu_n^{(1)}\|x_n - p\| + \mu_n^{(2)}\|x_n - p\| + \mu_n^{(3)}\|x_n - p\| + \mu_n^{(4)}\|x_n - p\| + \mu_n^{(5)}\|x_n - p\| \\
&= \|x_n - p\|,
\end{aligned}$$

which yields that

$$\begin{aligned}\|x_{n+1}-p\| =&\|\alpha_n(f(x_n)-p)+\beta_n(x_n-p)+\gamma_n(t_n-p)\|\\ \leq&\alpha_n\|f(x_n)-p\|+\beta_n\|x_n-p\|+\gamma_n\|t_n-p\|\\ \leq&\alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+\beta_n\|x_n-p\|+\gamma_n\|t_n-p\|\\ \leq&\alpha_n\alpha\|x_n-p\|+\beta_n\|x_n-p\|+\gamma_n\|x_n-p\|+\alpha_n\|f(p)-p\|\\ =&\alpha_n\alpha\|x_n-p\|+(1-\alpha_n)\|x_n-p\|+\alpha_n\|f(p)-p\|\\ =&(1-(1-\alpha)\alpha_n)\|x_n-p\|+(1-\alpha)\alpha_n\frac{\|f(p)-p\|}{1-\alpha}\\ \leq&\max\left\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\alpha}\right\}\\ \leq&\vdots\\ \leq&\max\left\{\|x_1-p\|,\frac{\|f(p)-p\|}{1-\alpha}\right\},\ \forall n\in\mathbb{N}.\end{aligned}\tag{3.6}$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$, $\{v_n\}$, $\{y_n\}$, $\{t_n\}$, $\{f(x_n)\}$, $\{A_1x_n\}$ and $\{A_2x_n\}$.

**Step 2**. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|t_n-x_n\|=0$.

Observing that $u_n^{(1)}=\widetilde{K}_rx_n\in\operatorname{dom}\varphi$ and $u_{n+1}^{(1)}=\widetilde{K}_rx_{n+1}\in\operatorname{dom}\varphi$, by the nonexpansiveness of $\widetilde{K}_r$, we get

$$\|u_{n+1}^{(1)}-u_n^{(1)}\|=\|\widetilde{K}_rx_{n+1}-\widetilde{K}_rx_n\|\leq\|x_{n+1}-x_n\|.\tag{3.7}$$

Similarly, let $u_n^{(2)}=\widetilde{K}_sx_n\in\operatorname{dom}\varphi$ and $u_{n+1}^{(2)}=\widetilde{K}_sx_{n+1}\in\operatorname{dom}\varphi$, we have

$$\|u_{n+1}^{(2)}-u_n^{(2)}\|=\|\widetilde{K}_sx_{n+1}-\widetilde{K}_sx_n\|\leq\|x_{n+1}-x_n\|.\tag{3.8}$$

From $v_n=J_{M_1,\rho_1}(x_n-\rho_1A_1x_n)$ and $y_n=J_{M_2,\rho_2}(x_n-\rho_2A_2x_n)$, we compute

$$\begin{aligned}\|v_{n+1}-v_n\| =&\|J_{M_1,\rho_1}(x_{n+1}-\rho_1A_1x_{n+1})-J_{M_1,\rho_1}(x_n-\rho_1A_1x_n)\|\\ \leq&\|(x_{n+1}-\rho_1A_1x_{n+1})-(x_n-\rho_1A_1x_n)\|\\ =&\|(I-\rho_1A_1)x_{n+1}-(I-\rho_1A_1)x_n\|\\ \leq&\|x_{n+1}-x_n\|.\end{aligned}\tag{3.9}$$

Similarly, we also have

$$\begin{aligned}\|y_{n+1}-y_n\| =&\|J_{M_2,\rho_2}(x_{n+1}-\rho_2A_2x_{n+1})-J_{M_2,\rho_2}(x_n-\rho_2A_2x_n)\|\\ \leq&\|x_{n+1}-x_n\|.\end{aligned}\tag{3.10}$$

Observing that

$$\begin{cases} t_n=\mu_n^{(1)}S_kx_n+\mu_n^{(2)}v_n+\mu_n^{(3)}y_n+\mu_n^{(4)}u_n^{(1)}+\mu_n^{(5)}u_n^{(2)}\\ t_{n+1}=\mu_{n+1}^{(1)}S_kx_{n+1}+\mu_{n+1}^{(2)}v_{n+1}+\mu_{n+1}^{(3)}y_{n+1}+\mu_{n+1}^{(4)}u_{n+1}^1+\mu_{n+1}^{(5)}u_{n+1}^2,\end{cases}$$

we compute

$$\begin{aligned}&\|t_{n+1}-t_n\|\\ \leq&\mu_{n+1}^{(1)}\|S_kx_{n+1}-S_kx_n\|+|\mu_{n+1}^{(1)}-\mu_n^{(1)}|\|S_kx_n\|+\mu_{n+1}^{(2)}\|v_{n+1}-v_n\|\end{aligned}$$

$$
\begin{aligned}
&+|\mu_{n+1}^{(2)}-\mu_n^{(2)}|\|v_n\|+\mu_{n+1}^{(3)}\|y_{n+1}-y_n\|+|\mu_{n+1}^{(3)}-\mu_n^{(3)}|\|y_n\| \\
&+\mu_{n+1}^{(4)}\|u_{n+1}^{1}-u_n^{(1)}\|+|\mu_{n+1}^{(4)}-\mu_n^{(4)}|\|u_n^{(1)}\| \\
&+\mu_{n+1}^{(5)}\|u_{n+1}^{2}-u_n^{(2)}\|+|\mu_{n+1}^{(5)}-\mu_n^{(5)}|\|u_n^{(2)}\| \\
\leq&\mu_{n+1}^{(1)}\|x_{n+1}-x_n\|+|\mu_{n+1}^{(1)}-\mu_n^{(1)}|\|S_kx_n\|+\mu_{n+1}^{(2)}\|v_{n+1}-v_n\| \\
&+|\mu_{n+1}^{(2)}-\mu_n^{(2)}|\|v_n\|+\mu_{n+1}^{(3)}\|y_{n+1}-y_n\|+|\mu_{n+1}^{(3)}-\mu_n^{(3)}|\|y_n\| \\
&+\mu_{n+1}^{(4)}\|u_{n+1}^{1}-u_n^{(1)}\|+|\mu_{n+1}^{(4)}-\mu_n^{(4)}|\|u_n^{(1)}\| \\
&+\mu_{n+1}^{(5)}\|u_{n+1}^{2}-u_n^{(2)}\|+|\mu_{n+1}^{(5)}-\mu_n^{(5)}|\|u_n^{(2)}\|. \qquad (3.11)
\end{aligned}
$$

Substitution of (3.7), (3.8), (3.9) and (3.10) into (3.11), yields that

$$
\begin{aligned}
\|t_{n+1}-t_n\| \leq\ & \mu_{n+1}^{(1)}\|x_{n+1}-x_n\|+|\mu_{n+1}^{(1)}-\mu_n^{(1)}|\|S_kx_n\|+\mu_{n+1}^{(2)}\|x_{n+1}-x_n\|+|\mu_{n+1}^{(2)}-\mu_n^{(2)}|\|v_n\| \\
&+\mu_{n+1}^{(3)}\|x_{n+1}-x_n\|+|\mu_{n+1}^{(3)}-\mu_n^{(3)}|\|y_n\|+\mu_{n+1}^{(4)}\|x_{n+1}-x_n\|+|\mu_{n+1}^{(4)}-\mu_n^{(4)}|\|u_n^{(1)}\| \\
&+\mu_{n+1}^{(5)}\|x_{n+1}-x_n\|+|\mu_{n+1}^{(5)}-\mu_n^{(5)}|\|u_n^{(2)}\| \\
\leq\ & \|x_{n+1}-x_n\|+M_1\Big(|\mu_{n+1}^{(1)}-\mu_n^{(1)}|+|\mu_{n+1}^{(2)}-\mu_n^{(2)}|+|\mu_{n+1}^{(3)}-\mu_n^{(3)}| \\
&+|\mu_{n+1}^{(4)}-\mu_n^{(4)}|+|\mu_{n+1}^{(5)}-\mu_n^{(5)}|\Big), \qquad (3.12)
\end{aligned}
$$

where $M_1$ is an appropriate constant such that

$$M_1=\max\Big\{\sup_{n\geq 1}\|S_kx_n\|,\|v_n\|,\|y_n\|,\|u_n^{(1)}\|,\|u_n^{(2)}\|\Big\}.$$

Putting $x_{n+1}=(1-\beta_n)l_n+\beta_nx_n, \quad \forall n\geq 1$, we have

$$l_n=\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}=\frac{\alpha_nf(x_n)+\gamma_nt_n}{1-\beta_n}.$$

Then, we compute

$$
\begin{aligned}
l_{n+1}-l_n=&\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}t_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nf(x_n)+\gamma_nt_n}{1-\beta_n} \\
=&\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}f(x_n)+t_{n+1}-t_n \\
&+\frac{\alpha_n}{1-\beta_n}t_n-\frac{\alpha_{n+1}}{1-\beta_{n+1}}t_{n+1} \\
=&\frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-t_{n+1})+\frac{\alpha_n}{1-\beta_n}(t_n-f(x_n))+t_{n+1}-t_n. \qquad (3.13)
\end{aligned}
$$

It follows from (3.12) and (3.13), that

$$
\begin{aligned}
\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|\leq&\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\|+\|t_{n+1}\|)+\frac{\alpha_n}{1-\beta_n}(\|t_n\|+\|f(x_n)\|) \\
&+\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\| \\
\leq&\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\|+\|t_{n+1}\|)+\frac{\alpha_n}{1-\beta_n}(\|t_n\|+\|f(x_n)\|)
\end{aligned}
$$

$$+M_1\Big(|\mu_{n+1}^{(1)}-\mu_n^{(1)}|+|\mu_{n+1}^{(2)}-\mu_n^{(2)}|+|\mu_{n+1}^{(3)}-\mu_n^{(3)}| + |\mu_{n+1}^{(4)}-\mu_n^{(4)}|+|\mu_{n+1}^{(5)}-\mu_n^{(5)}|\Big).$$

This together with (C2)-(C4), imply that

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence, by Lemma 2.6, we obtain $\|l_n-x_n\|\to 0$ as $n\to\infty$. It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|l_n-x_n\|=0. \tag{3.14}$$

From (3.7), (3.8) and (3.14), we have

$$\lim_{n\to\infty}\|u_{n+1}^1-u_n^{(1)}\|=\lim_{n\to\infty}\|u_{n+1}^2-u_n^{(2)}\|=0.$$

From (3.9), (3.10) and (3.14), we have

$$\lim_{n\to\infty}\|v_{n+1}-v_n\|=\lim_{n\to\infty}\|y_{n+1}-y_n\|=0.$$

Moreover, from condition (C4), (3.12) and (3.14), we also get

$$\lim_{n\to\infty}\|t_{n+1}-t_n\|=0.$$

Observe that

$$x_{n+1}-x_n= \qquad \alpha_n(f(x_n)-x_n)+\gamma_n(t_n-x_n).$$

By conditions (C2), (C3) and (3.14), we have

$$\lim_{n\to\infty}\|t_n-x_n\|=0. \tag{3.15}$$

**Step 3**. We show that $\limsup_{n\to\infty}\langle (f-I)x^*,J(x^*-x_{n+1})\rangle$

First, we will prove that $z\in F(S)\cap I(E,A_1,M_1)\cap I(E,A_2,M_2)\cap GMEP(F_1,\varphi,B_1)\cap GMEP(F_2,\varphi,B_2)$.

Define a mapping $G:E\to E$ by

$$Gx=\mu^{(1)}S_kx+\mu^{(2)}J_{(M_1,\rho_1)}(I-\rho_1A_1)x+\mu^{(3)}J_{(M_2,\rho_2)}(I-\rho_2A_2)x+\mu^{(4)}\widetilde{K}_rx+\mu^{(5)}\widetilde{K}_sx,\quad \forall x\in E,$$

where $\lim_{n\to\infty}\mu_n^{(i)}=\mu^{(i)}\in(0,1)$, where $i=1,2,3,4,5$. Since $\sum_{i=1}^5\mu_n^{(i)}=1$ and by Lemma 2.2, we have $G$ is nonexpansive and

$$\begin{aligned}F(G)=&F(S_k)\cap F(J_{(M_1,\rho_1)}(I-\rho_1A_1))\cap F(J_{(M_2,\rho_2)}(I-\rho_2A_2))\cap F(\widetilde{K}_r)\cap F(\widetilde{K}_s)\\ =&F(S)\cap VI(E,A_1,M_1)\cap VI(E,A_2,M_2)\cap GMEP(F_1,\varphi,B_1)\cap GMEP(F_2,\varphi,B_2).\end{aligned} \tag{3.16}$$

Notice that

$$\begin{aligned}&\|Gx_n-x_n\|\\ \le&\|Gx_n-t_n\|+\|t_n-x_n\|\\ =&\Big\|\Big[\mu^{(1)}S_kx_n+\mu^{(2)}J_{(M_1,\rho_1)}(I-\rho_1A_1)x_n+\mu^{(3)}J_{(M_2,\rho_2)}(I-\rho_2A_2)x_n\\ &+\mu^{(4)}\widetilde{K}_rx_n+\mu^{(5)}T_s^{F_2}(I-rB_2)x_n\Big]-\Big[\mu_n^{(1)}S_kx_n\\ &+\mu_n^{(2)}J_{(M_1,\rho_1)}(I-\rho_1A_1)x_n+\mu_n^{(3)}J_{(M_2,\rho_2)}(I-\rho_2A_2)x_n\end{aligned}$$

$$
\begin{aligned}
&+\mu_n^{(4)}\widetilde{K}_r x_n+\mu_n^{(5)}\widetilde{K}_s x_n\Big]\Bigg\|+\|t_n-x_n\|\\
\le&|\mu^{(1)}-\mu_n^{(1)}|\|S_k x_n\|+|\mu^{(2)}-\mu_n^{(2)}|\|J_{(M_1,\rho_1)}(I-\rho_1 A_1)x_n\|\\
&+|\mu^{(3)}-\mu_n^{(3)}|\|J_{(M_2,\rho_2)}(I-\rho_2 A_2)x_n\|+|\mu^{(4)}-\mu_n^{(4)}|\|\widetilde{K}_r x_n\|\\
&+|\mu^{(5)}-\mu_n^{(5)}|\|\widetilde{K}_s x_n\|+\|t_n-x_n\|\\
\le&K_1\left(\sum_{i=1}^{5}|\mu^{(i)}-\mu_n^{(i)}|\right)+\|t_n-x_n\|,
\end{aligned}
$$

where $K_1$ is an appropriate constant such that

$$
\begin{aligned}
K_1=\max\Big\{&\sup_{n\ge1}\|\widetilde{K}_r x_n\|,\ \sup_{n\ge1}\|\widetilde{K}_s x_n\|,\ \sup_{n\ge1}\|J_{(M_1,\rho_1)}(I-\rho_1 A_1)x_n\|,\\
&\sup_{n\ge1}\|J_{(M_2,\rho_2)}(I-\rho_2 A_2)x_n\|,\ \sup_{n\ge1}\|S_k x_n\|\Big\}.
\end{aligned}
$$

From (C4) and (3.15), we obtain

$$
\lim_{n\to\infty}\|x_n-Gx_n\|=0. \tag{3.17}
$$

Since $Q_\Theta f(x^*)$ is a contraction with the coefficient $\alpha\in[0,1)$, we have that there exists a unique fixed point. We use $x^*$ to denote the unique fixed point to the mapping $Q_\Theta f(x^*)$. That is $x^*=Q_\Theta f(x^*)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $z$. Without loss of generality, we may assume that $\{x_{n_i}\}\rightharpoonup z$. It follows from (3.17), that

$$
\lim_{n\to\infty}\|x_{n_i}-Gx_{n_i}\|=0.
$$

Since $G$ is nonexpansive, it follows from Lemma 2.7 that $z=Gz$, we obtain that $z\in F(G)$. By (3.16), hence we have $z\in\Theta$.

Let be $x^*=\lim_{t\to0}x_t$ with $x_t$ being the fixed point of the contraction

$$
x\mapsto tf(x)+(1-t)Gx.
$$

From Lemma 2.9 that

$$
\begin{aligned}
\|x_t-x_n\|^2=&\|(1-t)(Gx_t-x_n)+t(f(x_t)-x_n)\|^2\\
\le&(1-t)^2\|Gx_t-x_n\|^2+2t\langle f(x_t)-x_n,J(x_t-x_n)\rangle\\
\le&(1-t)^2(\|Gx_t-Gx_n\|+\|Gx_n-x_n\|)^2+2t\langle f(x_t)-x_n,J(x_t-x_n)\rangle\\
\le&(1-t)^2(\|x_t-x_n\|+\|Gx_n-x_n\|)^2+2t\langle f(x_t)-x_n,J(x_t-x_n)\rangle\\
=&(1-t)^2\left[\|x_t-x_n\|^2+2\|x_t-x_n\|\|Gx_n-x_n\|+\|Gx_n-x_n\|^2\right]\\
&+2t\langle f(x_t)-x_t,J(x_t-x_n)\rangle+2t\langle x_t-x_n,J(x_t-x_n)\rangle\\
=&(1-2t+t^2)\|x_t-x_n\|^2+f_n(t)+2t\langle f(x_t)-x_t,J(x_t-x_n)\rangle+2t\|x_t-x_n\|^2,
\end{aligned} \tag{3.18}
$$

where

$$
f_n(t)=(1-t)^2(2\|x_t-x_n\|+\|x_n-Gx_n\|)\|x_n-Gx_n\|\to0,\ \text{as } n\to\infty. \tag{3.19}
$$

It follows from (3.18) that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t} f_n(t). \tag{3.20}$$

Let $n \to \infty$ in (3.20) and note that (3.19) yields

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}M, \tag{3.21}$$

where $M > 0$ is a constant such that $\|x_t - x_n\|^2$ for all $t \in (0,1)$ and $n \ge 1$. Taking $t \to 0$ from (3.21), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0. \tag{3.22}$$

On the other hand, we have

$$\begin{aligned}\langle f(x^*) - x^*, J(x_n - x^*)\rangle =& \langle f(x^*) - x^*, J(x_n - x^*)\rangle - \langle f(x^*) - x^*, J(x_n - x_t)\rangle\\ &+ \langle f(x^*) - x^*, J(x_n - x_t)\rangle - \langle f(x^*) - x_t, J(x_n - x_t)\rangle\\ &+ \langle f(x^*) - x_t, J(x_n - x_t)\rangle - \langle f(x_t) - x_t, J(x_n - x_t)\rangle\\ &+ \langle f(x_t) - x_t, J(x_n - x_t)\rangle\\ =& \langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle + \langle x_t - x^*, J(x_n - x_t)\rangle\\ &+ \langle f(x^*) - f(x_t), J(x_n - x_t)\rangle + \langle f(x_t) - x_t, J(x_n - x_t)\rangle.\end{aligned}$$

It follow that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \le& \limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*) - J(x_n - x_t)\rangle\\ &+ |x_t - x^*\| \limsup_{n\to\infty}\|x_n - x_t\| + \alpha\|x^* - x_t\| \limsup_{n\to\infty}\|x_n - x_t\|\\ &+ \limsup_{n\to\infty}\langle f(x_t) - x_t, J(x_n - x_t)\rangle.\end{aligned}$$

Noticing that $J$ is norm-to-norm uniformly continuous on bounded subset of $C$, it follows from (3.22), we have

$$\begin{aligned}\limsup_{n\to\infty}\big\langle f(x^*) - x^*, J(x^* - x_n)\big\rangle =& \limsup_{t\to 0}\limsup_{n\to\infty}\big\langle f(x^*) - x^*, J(x^* - x_n)\big\rangle\\ \le& 0.\end{aligned} \tag{3.23}$$

Observe that On the other hand, we have

$$\begin{aligned}\langle (f(x^*) - x^*, J(x^* - x_{n+1})\rangle =& \langle f(x^*) - x^*, J(x_n - x_{n+1})\rangle + \langle f(x^*) - x^*, J(x^* - x_n)\rangle\\ \le& \|f(x^*) - x^*\|\|x_n - x_{n+1}\| + \langle f(x^*) - x^*, J(x^* - x_n)\rangle.\end{aligned}$$

From (3.14) and (3.23), we obtain that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x^* - x_{n+1})\rangle \le 0. \tag{3.24}$$

**Step 5**. We claim that $\lim_{n\to\infty}\|x_n - x^*\| = 0$.
Indeed, by (3.1) and using Lemma 2.9, we observe that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 =& \big\|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + (1 - \alpha_n - \beta_n)(t_n - x^*)\big\|^2\\ \le& \Big\|(1 - \alpha_n - \beta_n)(t_n - x^*) + \beta_n(x_n - x^*)\Big\|^2\end{aligned}$$

$$
\begin{aligned}
&+2\alpha_n\Big\langle f(x_n)-x^*,J(x_{n+1}-x^*)\Big\rangle\\
\le&\Big(\beta_n\|x_n-x^*\|+(1-\alpha_n-\beta_n)\|t_n-x^*\|\Big)^2\\
&+2\alpha_n\Big\langle f(x_n)-f(x^*),J(x_{n+1}-x^*)\Big\rangle+2\alpha_n\Big\langle f(x^*)-x^*,J(x_{n+1}-x^*)\Big\rangle\\
\le&\Big(\beta_n\|x_n-x^*\|+(1-\alpha_n-\beta_n)\|x_n-x^*\|\Big)^2\\
&+2\alpha_n\alpha\|x_n-x^*\|\|x_{n+1}-x^*\|+2\alpha_n\Big\langle f(x^*)-x^*,J(x_{n+1}-x^*)\Big\rangle\\
\le&(1-\alpha_n)^2\|x_n-x^*\|^2\\
&+\alpha_n\alpha\Big(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\Big)+2\alpha_n\Big\langle f(x^*)-x^*,J(x_{n+1}-x^*)\Big\rangle\\
=&\Big(1-2\alpha_n+\alpha_n^2+\alpha\alpha_n+2\alpha\alpha_n-2\alpha\alpha_n\Big)\|x_n-x^*\|^2\\
&+\alpha_n\alpha\|x_{n+1}-x^*\|^2+2\alpha_n\Big\langle f(x^*)-x^*,J(x_{n+1}-x^*)\Big\rangle\\
=&\Big(1-\alpha\alpha_n-2\alpha_n(1-\alpha)+\alpha_n^2\Big)\|x_n-x^*\|^2\\
&+\alpha_n\alpha\|x_{n+1}-x^*\|^2+2\alpha_n\Big\langle f(x^*)-x^*,J(x_{n+1}-x^*)\Big\rangle, \qquad (3.25)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2\le&\frac{1-\alpha\alpha_n-2\alpha_n(1-\alpha)+\alpha_n^2}{1-\alpha_n\alpha}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
=&\left[1-\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right]\|x_n-x^*\|^2\\
&+\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\left(\frac{1}{(1-\alpha)}\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle+\frac{\alpha_n}{2(1-\alpha)}M_2\right), \qquad (3.26)
\end{aligned}
$$

where $M_2$ is an appropriate constant such that $M_2\ge\sup_{n\ge1}\{\|x_n-x^*\|^2\}$.
Set $\rho_n=\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}$ and $\sigma_n=\frac{1}{(1-\alpha)}\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle+\frac{\alpha_n}{2(1-\alpha)}M_2$, we can rewrite (3.26) as

$$
\|x_{n+1}-x^*\|^2\le(1-\rho_n)\|x_n-x^*\|^2+\sigma_n, \qquad (3.27)
$$

we can see that $\sum_{n=1}^{\infty}\rho_n=\infty$ and $\limsup_{n\to\infty}\frac{\sigma_n}{\rho_n}\le0$. Applying Lemma 2.8 to (3.27), we conclude that $\{x_n\}$ converges strongly to $x^*$ in norm. This completes the proof. ∎

If the mapping $S$ is nonexpansive, then $S_k=S_0=S$. We can obtain the following result from Theorem 3.1 immediately.

**Corollary 3.1.** *Let $C$ be a subset of uniformly convex and 2-uniformly smooth Banach space of $E$ with the smooth constant $K$. Let $M_1,M_2:E\to2^E$ be maximal accretive mappings and $A_1,A_2:E\to E$ $\gamma_1,\gamma_2$-inverse-strongly accretive mappings, respectively. Let $F_1$ and $F_2$ be two bifunctions from $E\times E$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi:C\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function with either (B1) or (B2). Let $B_1:C\to E^*$ be an $\rho$-inverse-strongly accretive mapping, $B_2:C\to E^*$ be an $\omega$-inverse-strongly accretive mapping. Let $f:E\to E$ be an $\alpha$-contraction with coefficient $\alpha$ $(0\le\alpha<1)$. Let $S:E\to E$ be a nonexpansive mapping. Suppose that*

$$
\Theta:=F(S)\cap VI(E,A_1,M_1)\cap VI(E,A_2,M_2)\cap GMEP(F_1,\varphi,B_1)\cap GMEP(F_2,\varphi,B_2)\ne\emptyset.
$$

*Let $\{x_n\}$ be a sequence generated by the following iterative algorithm (3.1), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4,5$, $\alpha_n+\beta_n+\gamma_n = 1$, $\rho_1 \in (0,\gamma_1/K^2]$, $\rho_2 \in (0,\gamma_2/K^2]$, $r \in (0,2\rho)$ and $s \in (0,2\omega)$. Assume that the control sequences satisfy (C1)-(C4). Then $\{x_n\}$ converges strongly to $x^* = Q_\Theta f(x^*)$, where $Q_\Theta$ is the sunny nonexpansive retraction from $E$ onto $\Theta$.*

**Corollary 3.2.** *Let $C$ be a subset of uniformly convex and 2-uniformly smooth Banach space of $E$ with the smooth constant $K$. Let $M_1, M_2 : E \to 2^E$ be maximal accretive mappings and $A_1, A_2 : E \to E$ $\gamma_1,\gamma_2$-inverse-strongly accretive mappings, respectively. Let $f : E \to E$ be an $\alpha$-contraction with coefficient $\alpha$ $(0 \le \alpha < 1)$. Let $S : E \to E$ be an $\lambda$-strictly pseudo-contractive mapping with a fixed point. Define a mapping $S_k$ by $S_k x = kx+(1-k)Sx$, $\forall x \in E$. Suppose that*

$$\Omega' := F(S)\cap VI(E,A_1,M_1)\cap VI(E,A_2,M_2) \neq \emptyset.$$

*Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} y_n = J_{M_2,\rho_2}(x_n-\rho_2 A_2 x_n), \\ v_n = J_{M_1,\rho_1}(x_n-\rho_1 A_1 x_n), \\ t_n = \upsilon_n^{(1)} S_k x_n + \upsilon_n^{(2)} v_n + \upsilon_n^{(3)} y_n + \upsilon_n^{(4)} x_n, \\ x_{n+1} = \alpha_n f(x_n)+\beta_n x_n+\gamma_n t_n, \end{cases} \tag{3.28}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\upsilon_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4$, $\alpha_n+\beta_n+\gamma_n = 1$, $\rho_1 \in (0,\gamma_1/K^2]$ and $\rho_2 \in (0,\gamma_2/K^2]$. Assume that the control sequences satisfy the following restrictions:*

(C1) $\sum_{i=1}^4 \upsilon_n^{(i)} = 1$,
(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,
(C3) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,
(C4) $\lim_{n\to\infty}\upsilon_n^{(i)} = \upsilon^{(i)} \in (0,1)$, *where* $i = 1,2,3,4$.

*Then $\{x_n\}$ converges strongly to $x^* = Q_{\Omega'} f(x^*)$, where $Q_{\Omega'}$ is the sunny nonexpansive retraction from $E$ onto $\Omega'$.*

*Proof.* Put $F_1(x,y) = F_2(x,y) = 0$ and $B_1 = B_2 = 0$ for all $x,y \in C$ and $s = r = 1$ in Theorem 3.1. Then, we have $u_n^{(1)} = u_n^{(2)} = x_n$. So, by Theorem 3.1, we can conclude the desired result easily. ∎

**Corollary 3.3.** *Let $C$ be a subset of uniformly convex and 2-uniformly smooth Banach space of $E$ with the smooth constant $K$. Let $f : E \to E$ be an $\alpha$-contraction with coefficient $\alpha$ $(0 \le \alpha < 1)$. Let $S : E \to E$ be an $\lambda$-strictly pseudo-contractive mapping with a fixed point. Define a mapping $S_k$ by $S_k x = kx+(1-k)Sx$, $\forall x \in E$. Suppose that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} t_n = \mu_n S_k x_n + (1-\mu_n)x_n, \\ x_{n+1} = \alpha_n f(x_n)+\beta_n x_n+\gamma_n t_n, \end{cases} \tag{3.29}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ are sequences in $(0,1)$. Assume that the control sequences satisfy the following restrictions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,
(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

*Then* $\{x_n\}$ *converges strongly to* $x^* = Q_{F(S)}f(x^*)$*, where* $Q_{F(S)}$ *is the sunny nonexpansive retraction from E onto* $F(S)$*.*

*Proof.* Put $J_{M_1,\rho_1}(I-\rho_1 A_1) = J_{M_2,\rho_2}(I-\rho_2 A_2) = I$ in Corollary 3.2. Then, we have $y_n = v_n = x_n$ So, by Theorem 3.1, we can conclude the desired result easily. ∎

## 4. Applications

If we set $\varphi(x) = 0$ in (1.1) then the problem (1.1) is reduced to the $GEP(F,B)$ (1.4), this problem first introduced by Takahashi and Takahashi [39].
In case $F(x,y) = 0$, problem (1.4) reduced to variational inequality problem $VI(C,B)$, i.e.,

$$\langle Bx, y-x\rangle \geq 0,\ x,y \in C.$$

If $B = 0$, problem (1.4) is reduced to $EP(F)$ (1.5).
We give a mapping $D : C \to E^*$ and $F(x,y) = \langle Dx, y-x\rangle$, then,

$$x \in EP(F) \Leftrightarrow \langle Dx, y-x\rangle \geq 0,\ y \in C. \tag{4.1}$$

If we defined $F(x,y) = \langle Dx, y-x\rangle$, then we can solve a common solution of variational inequalities problems for two operators that is $VI(C,D) \cap VI(C,B)$ (see in Corollary 4.3).

Using our main theorem 3.1, we obtained the following Corollaries.

**Corollary 4.1.** *Let C be a subset of uniformly convex and 2-uniformly smooth Banach space of E with the smooth constant K. Let* $M_1, M_2 : E \to 2^E$ *be maximal accretive mappings and* $A_1, A_2 : E \to E$ $\gamma_1, \gamma_2$*-inverse-strongly accretive mappings, respectively. Let* $F_1$ *and* $F_2$ *be two bifunctions from* $E \times E$ *to* $\mathbb{R}$ *satisfying (A1)-(A4) and let* $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* $f : E \to E$ *be an* $\alpha$*-contraction with coefficient* $\alpha$ $(0 \leq \alpha < 1)$*. Let* $S : E \to E$ *be an* $\lambda$*-strict pseudo-contraction mapping with a fixed point. Define a mapping* $S_k$ *by* $S_k x = kx + (1-k)Sx$, $\forall x \in E$*. Suppose that*

$$\Omega := F(S) \cap VI(E, A_1, M_1) \cap VI(E, A_2, M_2) \cap EP(F_1) \cap EP(F_2) \neq \emptyset.$$

*Let* $\{x_n\}$ *be a sequence generated by the following iterative algorithm:*

$$\begin{cases} F_1(u_n^{(1)}, y_1) + \frac{1}{r}\langle y_1 - u_n^{(1)}, Ju_n^{(1)} - Jx_n\rangle \geq 0,\ \forall y_1 \in C, \\ F_2(u_n^{(2)}, y_2) + \frac{1}{s}\langle y_2 - u_n^{(2)}, Ju_n^{(2)} - Jx_n\rangle \geq 0,\ \forall y_2 \in C, \\ y_n = J_{M_2,\rho_2}(x_n - \rho_2 A_2 x_n), \\ v_n = J_{M_1,\rho_1}(x_n - \rho_1 A_1 x_n), \\ t_n = \mu_n^{(1)} S_k x_n + \mu_n^{(2)} v_n + \mu_n^{(3)} y_n + \mu_n^{(4)} u_n^{(1)} + \mu_n^{(5)} u_n^{(2)}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n t_n, \end{cases}$$

*where* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*,* $\{\mu_n^{(i)}\}$ *are sequences in* $(0,1)$*, where* $i = 1,2,3,4,5$*,* $\alpha_n + \beta_n + \gamma_n = 1$*,* $\rho_1 \in (0, \gamma_1/K^2]$*,* $\rho_2 \in (0, \gamma_2/K^2]$*,* $r \in (0, 2\rho)$ *and* $s \in (0, 2\omega)$*. Assume that the control sequences satisfy condition (C1)-(C4). Then* $\{x_n\}$ *converges strongly to* $x^* = Q_\Omega f(x^*)$*, where* $Q_\Omega$ *is the sunny nonexpansive retraction from E onto* $\Omega$*.*

*Proof.* Put $B_1 = B_2 = 0$ for all $x, y \in C$ in Theorem 3.1. Then, we can conclude the desired result easily. ∎

**Corollary 4.2.** *Let C be a subset of uniformly convex and 2-uniformly smooth Banach space of E with the smooth constant K. Let* $M_1, M_2 : E \to 2^E$ *be maximal accretive mappings and* $A_1, A_2 : E \to E$ $\gamma_1, \gamma_2$*-inverse-strongly accretive mappings, respectively. Let* $B_1 : C \to E^*$ *be* *an $\rho$-inverse-strongly accretive mapping, $B_2 : C \to E^*$ be an $\omega$-inverse-strongly accretive mapping. Let $f : E \to E$ be an $\alpha$-contraction with coefficient $\alpha$ $(0 \leq \alpha < 1)$. Let $S : E \to E$ be an $\lambda$-strictly pseudo-contractive mapping with a fixed point. Define a mapping $S_k$ by $S_k x = kx + (1-k)Sx$, $\forall x \in E$. Suppose that*

$$\Theta := F(S) \cap VI(E,A_1,M_1) \cap VI(E,A_2,M_2) \cap VI(C,B_1) \cap VI(C,B_2) \neq \emptyset.$$

*Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} \langle B_1 x_n, y_1 - u_n^{(1)} \rangle + \frac{1}{r} \langle y_1 - u_n^{(1)}, J u_n^{(1)} - J x_n \rangle \geq 0, \ \forall y_1 \in C, \\ \langle B_2 x_n, y_2 - u_n^{(2)} \rangle + \frac{1}{s} \langle y_2 - u_n^{(2)}, J u_n^{(2)} - J x_n \rangle \geq 0, \ \forall y_2 \in C, \\ y_n = J_{M_2,\rho_2}(x_n - \rho_2 A_2 x_n), \\ v_n = J_{M_1,\rho_1}(x_n - \rho_1 A_1 x_n), \\ t_n = \mu_n^{(1)} S_k x_n + \mu_n^{(2)} v_n + \mu_n^{(3)} y_n + \mu_n^{(4)} u_n^{(1)} + \mu_n^{(5)} u_n^{(2)}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n t_n, \end{cases}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n^{(i)}\}$ are sequences in $(0,1)$, where $i = 1,2,3,4,5$, $\alpha_n + \beta_n + \gamma_n = 1$, $\rho_1 \in (0, \gamma_1/K^2]$, $\rho_2 \in (0, \gamma_2/K^2]$, $r \in (0, 2\rho)$ and $s \in (0, 2\omega)$. Assume that the control sequences satisfy the following restrictions:*

(C1) $\sum_{i=1}^{5} \mu_n^{(i)} = 1$,
(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(C4) $\lim_{n\to\infty} \mu_n^{(i)} = \mu^{(i)} \in (0,1)$, *where* $i = 1,2,3,4,5$.

*Then $\{x_n\}$ converges strongly to $x^* = Q_\Theta f(x^*)$, where $Q_\Theta$ is the sunny nonexpansive retraction from $E$ onto $\Theta$.*

*Proof.* Put $F_1(x,y) = F_2(x,y) = 0$ for all $x, y \in C$ in Theorem 3.1. Then, we can conclude the desired result easily. ∎

**Corollary 4.3.** *Let $C$ be a subset of uniformly convex and 2-uniformly smooth Banach space of $E$ with the smooth constant $K$. Let $M_1, M_2 : E \to 2^E$ be maximal accretive mappings and $A_1, A_2 : E \to E$ $\gamma_1, \gamma_2$-inverse-strongly accretive mappings, respectively. Let $B_1 : C \to E^*$ be an $\rho$-inverse-strongly accretive mapping, $B_2 : C \to E^*$ be an $\omega$-inverse-strongly accretive mapping and $D_1, D_2 : C \to E^*$ be mappings. Let $f : E \to E$ be an $\alpha$-contraction with coefficient $\alpha$ $(0 \leq \alpha < 1)$. Let $S : E \to E$ be an $\lambda$-strictly pseudo-contractive mapping with a fixed point. Define a mapping $S_k$ by $S_k x = kx + (1-k)Sx$, $\forall x \in E$. Suppose that*

$$F(S) \cap VI(E,A_1,M_1) \cap VI(E,A_2,M_2) \cap VI(C,D_2) \cap VI(C,D_2) \cap VI(C,B_1) \cap VI(C,B_2) \neq \emptyset.$$

*Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} \langle D_1 x_n, y_1 - u_n^{(1)} \rangle + \langle B_1 x_n, y_1 - u_n^{(1)} \rangle + \frac{1}{r} \langle y_1 - u_n^{(1)}, J u_n^{(1)} - J x_n \rangle \geq 0, \ \forall y_1 \in C, \\ \langle D_2 x_n, y_2 - u_n^{(2)} \rangle + \langle B_2 x_n, y_2 - u_n^{(2)} \rangle + \frac{1}{s} \langle y_2 - u_n^{(2)}, J u_n^{(2)} - J x_n \rangle \geq 0, \ \forall y_2 \in C, \\ y_n = J_{M_2,\rho_2}(x_n - \rho_2 A_2 x_n), \\ v_n = J_{M_1,\rho_1}(x_n - \rho_1 A_1 x_n), \\ t_n = \mu_n^{(1)} S_k x_n + \mu_n^{(2)} v_n + \mu_n^{(3)} y_n + \mu_n^{(4)} u_n^{(1)} + \mu_n^{(5)} u_n^{(2)}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n t_n, \end{cases}$$

*where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n^{(i)}\}$ *are sequences in* $(0,1)$, *where* $i = 1,2,3,4,5$, $\alpha_n + \beta_n + \gamma_n = 1$, $\rho_1 \in (0, \gamma_1/K^2]$, $\rho_2 \in (0, \gamma_2/K^2]$, $r \in (0, 2\rho)$ *and* $s \in (0, 2\omega)$. *Assume that the control sequences satisfy the following restrictions:*

(C1) $\sum_{i=1}^{5} \mu_n^{(i)} = 1$,
(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,
(C4) $\lim_{n\to\infty} \mu_n^{(i)} = \mu^{(i)} \in (0,1)$, *where* $i = 1,2,3,4,5$.

*Then* $\{x_n\}$ *converges strongly to* $x^* = Q_\Theta f(x^*)$, *where* $Q_\Theta$ *is the sunny nonexpansive retraction from* $E$ *onto* $\Theta$.

*Proof.* Put $F_1(x,y) = \langle D_1 x_n, y_1 - u_n^{(1)} \rangle$ and $F_2(x,y) = \langle D_2 x_n, y_2 - u_n^{(2)} \rangle$ for all $x, y \in C$ in Theorem 3.1. Then, we can conclude the desired result easily. ∎

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