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Generalized Derivations and Multilinear Polynomials in Prime Rings

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Abstract. Let R be a prime ring with Utumi quotient ring U and extended centroid C, g a nonzero generalized derivation of R, I a nonzero right ideal of R, $f(r_1,\ldots,r_k)$ a multilinear polynomial over C and $n \geq 2$ be a fixed integer. If $g(f(r_1,\ldots,r_k)^n) = g(f(r_1,\ldots,r_k))^n$ for all $r_1,\ldots,r_k \in I$, then one of the following holds: (1) $IC = eRC$ for some idempotent $e \in soc(RC)$ and $f(x_1,\ldots,x_k)$ is central-valued on $eRCe$; (2) there exist $a,b \in U$ such that $g(x) = ax+xb$ for all $x \in R$ and $(a-\alpha)I = (0)$, $(b-\beta)I = (0)$ for some $\alpha,\beta \in C$ with $(\alpha+\beta)^{n-1} = 1$; (3) there exists $a \in U$ such that $g(x) = ax$ for all $x \in R$ with $aI = (0)$.

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1. Introduction

Let R be an associative prime ring with center $Z(R)$. Throughout this paper, U will denote the Utumi quotient ring of R and $C = Z(U)$, the center of U, which is called extended centroid of R. For $x,y \in R$, the symbol $[x,y]$ stands for the commutator $xy-yx$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x,y \in R$. The concept of derivation is extended to generalized derivation. The generalized derivation means an additive mapping $g : R \to R$ such that $g(xy) = g(x)y + xd(y)$ for all $x,,y \in R$, where d is a derivation of R. For some fixed $a,b \in R$, the maps $g(x) = ax+xb$ for all $x \in R$, is an example of generalized derivation. This kind of generalized derivations are called generalized inner derivations.

Let S be a nonempty set of R and $F : R \to R$ be an additive mapping. Then we say that F acts as homomorphism or anti-homomorphism on S if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x,y \in S$ respectively. The additive mapping F acts as a Jordan homomorphism on S if $F(x^2) = F(x)^2$ holds for all $x \in S$. Obviously, any additive mapping acting as homomorphism or anti-homomorphism is a surjective Jordan homomorphism, but the converse is not true in general. In [11, Theorem 3.1], Herstein proved that in a 2-torsion free prime ring, any Jordan homomorphism is either a homomorphism or an anti-homomorphism.

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In [2], Bell and Kappe proved that if a derivation d of a prime ring R acts as a homomorphism or anti-homomorphism on a nonzero right ideal of R, then $d = 0$ on R. Recently, Ali, Rehman and Ali in [1] proved a similar result to Lie ideal case. They proved that if R is a 2-torsion free prime ring, L a nonzero Lie ideal of R such that $u^2 \in L$ for all $u \in L$ and d acts as a homomorphism or anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z(R)$. In [22], Wang and You eliminated the assumption $u^2 \in L$ for all $u \in L$ and obtain the same conclusion of [1].

On the other hand, the authors developed above results, replacing the derivation d with a generalized derivation g of R. In [21], Rehman proved that the 2-torsion free prime ring R must be commutative, if there is a generalized derivation g admitting a nonzero associated derivation, that acts as homomorphism or anti-homomorphism on a nonzero ideal of R. Gusic in [10] showed that the result of Rehman is not in complete form. He proved the following: let R be a prime ring, I a nonzero ideal of R and d, g any two functions on R (not necessary to be additive and d not necessary to be a derivation) such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$. If g acts as a homomorphism or an anti-homomorphism on I, then $d = 0$ and either $g = 0$ or $g(x) = x$ for all $x \in R$; in addition, when g acts as an anti-homomorphism on I, then R must be commutative. In the same line of investigation, recently in [7] De Filippis studied the situation when generalized derivation g acts as a Jordan homomorphism on a noncentral Lie ideal L of R and on the set $[I, I]$, where I is a nonzero right ideal of a prime ring R. More precisely, De Filippis proved the following two theorems:

Theorem 1.1. *Let R be a prime ring, L a non-central Lie ideal of R and g a nonzero generalized derivation of R. If g acts as a Jordan homomorphism on L, then either $g(x) = x$ for all $x \in R$, or char$(R) = 2$, R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$, L is commutative and $u^2 \in Z(R)$ for any $u \in L$.*

Theorem 1.2. *Let R be a prime ring, I a nonzero right ideal of R and g a nonzero generalized derivation of R. If g acts as a Jordan homomorphism on the set $[I, I]$, then one of the following holds: (i) char $(R) = 2$ and I satisfies the identity $s_4(x_1, \ldots, x_4)x_5$; (ii) $[I, I]I = 0$; (iii) there exists $a \in R$ such that $g(x) = ax$ for all $x \in R$ and $aI = 0$; (iv) $g(x) = x$ for all $x \in I$; (v) there exists $q \in R$ such that $g(x) = xq$ and $qx = x$ for all $x \in I$.*

It is natural to generalize above results considering the generalized derivation g acts as Jordan homomorphism on the set $\{f(x_1, \ldots, x_k) | x_1, \ldots, x_k \in I\}$, where I is a nonzero right ideal of R and $f(x_1, \ldots, x_k)$ is a multilinear polynomial on R over C. In the present paper, our aim is to study this situation in more generalized form by considering n-power values.

Let R be a prime ring and U be the Utumi quotient ring of R and $C = Z(U)$, the center of U. Note that U is also a prime ring with C a field. Let $f(x_1, \ldots, x_k)$ be a multilinear polynomial over C. We can write it as

$$f(x_1, \ldots, x_k) = x_1 x_2 \ldots x_k + \sum_{I \neq \sigma \in S_k} \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(k)},$$

where S_k is the permutation group over k elements and any $\alpha_\sigma \in C$. We denote by $f^d(x_1, \ldots, x_k)$ the polynomial obtained from $f(x_1, \ldots, x_k)$ by replacing each coefficient α_σ with $d(\alpha_\sigma.1)$. In this way we have

$$d(f(x_1, \ldots, x_k)) = f^d(x_1, \ldots, x_k) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_k).$$

Now we include some facts which will be used to prove our theorems.

Fact 1. It is well known that any derivation of R can be uniquely extended to a derivation of U (see [17, Lemma 2]).

Fact 2. Let ρ be a nonzero right ideal of R. Then ρ, ρC, ρU satisfy the same generalized polynomial identities with coefficients in U (see [5]).

Fact 3. Let ρ be a nonzero right ideal of R. Then ρ, ρR and ρU satisfy the same differential identities with coefficients in U (see [17, Theorem 2]).

Fact 4. Let ρ be a nonzero right ideal of R. If ρ satisfies a nontrivial polynomial identity, then RC is a primitive ring with $soc(RC) \neq 0$ and $\rho C = eRC$ for some idempotent $e = e^2 \in soc(RC)$ (see [16, Proposition]).

Fact 5. Let R be a dense ring of linear transformations of a vector space V over a division ring D and $a \in R$. If for any $v \in V$, av and v are linearly D-dependent, then there exists a $\beta \in D$ such that $av = v\beta$ for all $v \in V$.

Proof. For any $v \in V$, $av = v\alpha_v$ for some $\alpha_v \in D$. Now we prove that α_v is independent of the choice of $v \in V$. Let u be a fixed vector of V. Then $au = u\alpha$. Let v be any vector of V. Then $av = v\alpha_v$, where $\alpha_v \in D$. If u and v are linearly D-dependent, then $u = v\beta$, for $\beta \in D$. In this case, we see that $u\alpha = au = av\beta = (v\alpha_v)\beta = (v\beta)\alpha_v = u\alpha_v$, implying $\alpha = \alpha_v$.

Now if u and v are linearly D-independent, then we have $(u+v)\alpha_{u+v} = a(u+v) = au + av = u\alpha + v\alpha_v$, which implies $u(\alpha_{u+v} - \alpha) + v(\alpha_{u+v} - \alpha_v) = 0$. Since u and v are linearly D-independent, we have $\alpha_{u+v} - \alpha = 0 = \alpha_{u+v} - \alpha_v$ and so $\alpha = \alpha_v$. Thus $av = v\alpha$ for all $v \in V$, where $\alpha \in D$ independent of the choice of $v \in V$. ∎

Fact 6. Let I be a nonzero right ideal of R and $a \in U$. If for every $x \in I$, ax and x are linearly C-dependent, then there exists $\alpha \in C$ such that $(a - \alpha)I = (0)$.

The proof of Fact 6 is similar to that of Fact 5, so we omit it here.

Remark 1.1. Now we mention a result of Lee in [15] which will be used to prove our main theorem. In [15], Lee extended the definition of generalized derivation as follows: generalized derivation means an additive mapping $g : \rho \to U$ such that $g(xy) = g(x)y + x\delta(y)$ for all $x, y \in \rho$, where ρ is a dense right ideal of R and δ ia a derivation from ρ into U. The author proved that every generalized derivation of R can be uniquely extended to generalized derivation of U and has the form $g(x) = ax + \delta(x)$ for all $x \in U$, where $a \in U$ and δ is a derivation of U [15, Theorem 3]. For more details about generalized derivations we refer to [3], [12], [15] and [18].

2. Main results

First we study the case when g is inner generalized derivation of R, that is, for some $a, b \in U$, $g(x) = ax + xb$ for all $x \in R$.

Lemma 2.1. *Let $R = M_m(F)$, $m \geq 2$, be the set of all $m \times m$ matrices over a field F and $f(x_1, \ldots, x_k)$ be a noncentral multilinear polynomial over F. If for some $a, b \in R$, $af(x_1, \ldots, x_k)^n + f(x_1, \ldots, x_k)^n b = (af(x_1, \ldots, x_k) + f(x_1, \ldots, x_k)b)^n$ for all $x_1, \ldots, x_k \in R$, then $a, b \in F.I_m$ with $(a+b)^n - (a+b) = 0$.*

Proof. Let $a = (a_{ij})_{m\times m}$, $b = (b_{ij})_{m\times m}$. Since $f(x_1,\ldots,x_k)$ is not central valued on R, by [19, Lemma 2, Proof of Lemma 3] there exists a sequence of matrices $r = (r_1,\ldots,r_k)$ in R such that $f(r_1,\ldots,r_k) = \gamma e_{ij}$ with $0 \neq \gamma \in F$ and $i \neq j$. Since the set $f(R) = \{f(x_1,\ldots,x_k), x_i \in R\}$ is invariant under the action of all inner automorphisms of R, for all $i \neq j$ there exists a sequence of matrices $r = (r_1,\ldots,r_k)$ such that $f(r) = \gamma e_{ij}$. Thus

$$af(x_1,\ldots,x_k)^n + f(x_1,\ldots,x_k)^n b = (af(x_1,\ldots,x_k) + f(x_1,\ldots,x_k)b)^n$$

gives $0 = (a\gamma e_{ij} + \gamma e_{ij}b)^n$ i.e., $0 = (ae_{ij} + e_{ij}b)^n$. Left multiplying by e_{ij} yields $a_{ji}^n = 0$ and right multiplying by e_{ij} yields $b_{ji}^n = 0$. Thus, we have $a_{ji} = 0$ and $b_{ji} = 0$ for any $i \neq j$, that is, a and b are diagonal matrices.

Now for any F-automorphism θ of R, we have

$$a^\theta f(x_1,\ldots,x_k)^n + f(x_1,\ldots,x_k)^n b^\theta = \left(a^\theta f(x_1,\ldots,x_k) + f(x_1,\ldots,x_k)b^\theta\right)^n$$

for all $x_1,\ldots,x_k \in R$. Then by above argument a^θ and b^θ must be diagonal. Write, $a = \sum_{i=0}^{m} a_{ii}e_{ii}$ and $b = \sum_{i=0}^{m} b_{ii}e_{ii}$; then for $s \neq t$, we have

$$(1+e_{ts})a(1-e_{ts}) = \sum_{i=0}^{m} a_{ii}e_{ii} + (a_{ss} - a_{tt})e_{ts}$$

diagonal and

$$(1+e_{ts})b(1-e_{ts}) = \sum_{i=0}^{m} b_{ii}e_{ii} + (b_{ss} - b_{tt})e_{ts}$$

diagonal, implying $a_{ss} = a_{tt}$, $b_{ss} = b_{tt}$ and so $a, b \in F.I_m$. Then our assumption

$$af(x_1,\ldots,x_k)^n + f(x_1,\ldots,x_k)^n b = \left(af(x_1,\ldots,x_k) + f(x_1,\ldots,x_k)b\right)^n$$

for all $x_1,\ldots,x_k \in R$, reduces to $((a+b)^n - (a+b))f(x_1,\ldots,x_k)^n = 0$. This implies either $(a+b)^n - (a+b) = 0$ or $f(x_1,\ldots,x_k)^n = 0$ for all $x_1,\ldots,x_k \in R$. But by [19, Corollary 5], $f(x_1,\ldots,x_k)^n = 0$ for all $x_1,\ldots,x_k \in R$, implies that $f(x_1,\ldots,x_k) = 0$ for all $x_1,\ldots,x_k \in R$, a contradiction. ∎

Proposition 2.1. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, and $f(r_1,\ldots,r_k)$ be a multilinear polynomial over C which is not central valued on R. If for some $a, b \in U$, $af(r)^n + f(r)^n b = (af(r) + f(r)b)^n$ for all $r = (r_1,\ldots,r_k) \in R^k$, where $n \geq 2$ is a fixed integer, then $a, b \in C$ with $(a+b)^n - (a+b) = 0$.*

Proof. Since R and U satisfy same generalized polynomial identity (see [5]), U satisfies

$$h(x_1,\ldots,x_k) = af(x_1,\ldots,x_k)^n + f(x_1,\ldots,x_k)^n b - (af(x_1,\ldots,x_k) + f(x_1,\ldots,x_k)b)^n = 0.$$

Suppose that $h(x_1,\ldots,x_k)$ is a trivial GPI for U. Let $T = U *_C C\{x_1,\ldots,x_k\}$, the free product of U and $C\{x_1,\ldots,x_k\}$, the free C-algebra in noncommuting indeterminates $x_1,\ldots,x_k$. Then,

$$af(x_1,\ldots,x_k)^n + f(x_1,\ldots,x_k)^n b - (af(x_1,\ldots,x_k) + f(x_1,\ldots,x_k)b)^n$$

is zero element in T. If $a \notin C$, then a and 1 are linearly independent over C. Then expanding the above identity, it will imply

$$af(x_1,\ldots,x_k)^n - af(x_1,\ldots,x_k)(af(x_1,\ldots,x_k) + f(x_1,\ldots,x_k)b)^{n-1} = 0$$

that is,

$$af(x_1,\ldots,x_k)\{f(x_1,\ldots,x_k)^{n-1} - (af(x_1,\ldots,x_k) + f(x_1,\ldots,x_k)b)^{n-1}\} = 0$$

in T. Again, since a and 1 are linearly independent over C, this implies that

$$af(x_1,\dots,x_k)\{af(x_1,\dots,x_k)(af(x_1,\dots,x_k)+f(x_1,\dots,x_k)b)^{n-2}\}=0$$

and so $(af(x_1,\dots,x_k))^n=0$, implying $a=0$, a contradiction. Hence, $a\in C$. Then our generalized polynomial identity (GPI) reduces to $f(x_1,\dots,x_k)^n(a+b)-(f(x_1,\dots,x_k)(a+b))^n=0$ in T. If $a+b\notin C$, then $a+b$ and 1 are linearly independent over C. Then by same argument as above, $(f(x_1,\dots,x_k)(a+b))^n=0$, which is a nontrivial generalized polynomial identity for R, a contradiction. Thus, $a+b\in C$ and hence $b\in C$. Then our GPI becomes $\{(a+b)-(a+b)^n\}f(x_1,\dots,x_k)^n=0$, which is trivial GPI for R, implying $(a+b)-(a+b)^n=0$.

Next suppose that $h(x_1,\dots,x_k)$ is a nontrivial GPI for R and so for U. In case C is infinite, we have $h(x_1,\dots,x_k)=0$ for all $x_1,\dots,x_k\in U\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U\otimes_C\overline{C}$ are prime and centrally closed [8, Theorems 2.5 and 3.5], we may replace R by U or $U\otimes_C\overline{C}$ according to C finite or infinite. Then R is centrally closed over C and $h(x_1,\dots,x_k)=0$ for all $x_1,\dots,x_k\in R$. By Martindale's theorem [20], R is then a primitive ring with nonzero socle $soc(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [13, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V=m$. By density of R, we have $R\cong M_m(C)$. Since $f(r_1,\dots,r_k)$ is not central valued on R, R must be noncommutative and so $m\geq 2$. In this case, by Lemma 2.1, we obtain our required conclusion.

Now, if V is infinite dimensional over C, then as in lemma 2 in [23], the set $f(R)$ is dense on R and so from

$$af(r_1,\dots,r_k)^n+f(r_1,\dots,r_k)^nb-(af(r_1,\dots,r_k)+f(r_1,\dots,r_k)b)^n=0$$

for all $r_1,\dots,r_k\in R$, we have $ar^n+r^nb-(ar+rb)^n=0$ for all $r\in R$. Let v and bv be linearly C-independent for some $v\in V$. Then by density there exists $r\in R$ such that $rv=0$, $rbv=v$. Therefore, we have $0=\{ar^n+r^nb-(ar+rb)^n\}v=-v$ for $n\geq 2$, contradiction. Hence, v and bv are linearly C-dependent for all $v\in V$. By Fact 5, we can write $bv=v\alpha$ for all $v\in V$ and $\alpha\in C$ fixed.

Now let $r\in R$, $v\in V$. Since $bv=v\alpha$,

$$[b,r]v=(br)v-(rb)v=b(rv)-r(bv)=(rv)\alpha-r(v\alpha)=0.$$

Thus $[b,r]v=0$ for all $v\in V$ i.e., $[b,r]V=0$. Since $[b,r]$ acts faithfully as a linear transformation on the vector space V, $[b,r]=0$ for all $r\in R$. Therefore, $b\in C$. Then we obtain $(a+b)r^n-((a+b)r)^n=0$ for all $r\in R$. Let v and $(a+b)v$ be linearly C-independent for some $v\in V$. By density, we may choose $r\in R$ such that $rv=v$, $r(a+b)v=0$. Then we get $0=\{(a+b)r^n-((a+b)r)^n\}v=(a+b)v$ for $n\geq 2$, a contradiction. Hence, v and $(a+b)v$ are linearly C-dependent for all $v\in V$, which implies as before that $a+b\in C$ and so $a\in C$. Therefore, $\{(a+b)^n-(a+b)\}r^n=0$ for all $r\in R$. Since V is infinite dimensional over C, $(a+b)^n-(a+b)=0$. ∎

Proposition 2.2. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, I a nonzero right ideal of R and $f(r_1,\dots,r_k)$ a multilinear polynomial over C. If for some $a,b\in U$, $af(r)^n+f(r)^nb=(af(r)+f(r)b)^n$ for all $r=(r_1,\dots,r_k)\in I^k$, then one of the following holds:*

(1) *$IC = eRC$ for some idempotent $e \in soc(RC)$ and $f(x_1,\dots,x_k)$ is central-valued on $eRCe$;*
(2) *there exist $\alpha,\beta \in C$ such that $(a-\alpha)I = (0)$ and $(b-\beta)I = (0)$ with $(\alpha+\beta)^{n-1} = 1$;*
(3) *$b \in C$ and $(a+b)I = (0)$.*

Proof. Let $u \in I$. Then R satisfies the GPI

$$af(ux_1,\dots,ux_k)^n + f(ux_1,\dots,ux_k)^n b = (af(ux_1,\dots,ux_k) + f(ux_1,\dots,ux_k)b)^n. \tag{2.1}$$

Now we consider following two cases:

Case I: R does not satisfy any nontrivial GPI
Then (2.1) is a trivial GPI for R, that is,

$$af(ux_1,\dots,ux_k)^n + f(ux_1,\dots,ux_k)^n b - (af(ux_1,\dots,ux_k) + f(ux_1,\dots,ux_k)b)^n \tag{2.2}$$

is zero element in $R *_C C\{x_1,\dots,x_k\}$. Suppose first that there exists $u \in I$ such that $\{bu,u\}$ are linearly C-independent. Then $b \notin C$, and hence above GPI implies that

$$f(ux_1,\dots,ux_k)^n b - (af(ux_1,\dots,ux_k) + f(ux_1,\dots,ux_k)b)^{n-1} f(ux_1,\dots,ux_k)b = 0.$$

Now since $\{bu,u\}$ are linearly C-independent, we see expanding the above expression that $(f(ux_1,\dots,ux_k)b)^n$ appears nontrivially, a contradiction. Hence bu and u are linearly C-dependent for all $u \in I$. Then by Fact 6, there exists $\beta \in C$ such that $(b-\beta)I = (0)$. Next suppose that there exists $u \in I$ such that $\{au,u\}$ are linearly C-independent. Then from above (2.2), we obtain that

$$af(ux_1,\dots,ux_k)^n - af(ux_1,\dots,ux_k)\{af(ux_1,\dots,ux_k) + f(ux_1,\dots,ux_k)b\}^{n-1} = 0. \tag{2.3}$$

Expanding the above expression we find that the term $\{af(ux_1,\dots,ux_k)\}^n$ appears nontrivially, a contradiction. Hence we conclude that au and u are linearly C-dependent for all $u \in I$. By Fact 6, there exists $\alpha \in C$ such that $(a-\alpha)I = (0)$.

Then (2.1) reduces to

$$f(ux_1,\dots,ux_k)^n(\alpha+b) = (f(ux_1,\dots,ux_k)(\alpha+b))^n. \tag{2.4}$$

Using $(b-\beta)I = (0)$, it follows that

$$f(ux_1,\dots,ux_k)^n(\alpha+b) = f(ux_1,\dots,ux_k)^n(\alpha+\beta)^{n-1}(\alpha+b) \tag{2.5}$$

that is

$$f(ux_1,\dots,ux_k)^n\{1-(\alpha+\beta)^{n-1}\}(\alpha+b) = 0. \tag{2.6}$$

Since this is trivial GPI for R, either $1-(\alpha+\beta)^{n-1} = 0$ or $b = -\alpha \in C$. These two cases gives conclusion (2) and (3) respectively.

Case II: R satisfy a nontrivial GPI
Now assume first that $[f(I),I]I = 0$, that is $[f(x_1,\dots,x_k),x_{k+1}]x_{k+2} = 0$ for all $x_1,x_2,\dots,x_{k+2} \in I$. Then by Fact 4, $IC = eRC$ for some idempotent $e \in soc(RC)$. Since $[f(I),I]I = 0$, we have $[f(IR),IR]IR = 0$ and hence $[f(IU),IU]IU = 0$ by [5, Theorem 2]. In particular, $[f(IC),IC]IC = 0$, or equivalently, $[f(eRC),eRC]eRC = 0$. Then $[f(eRCe),eRCe] = 0$, that is, $f(x_1,\dots,x_k)$ is central-valued on $eRCe$ and hence conclusion (1) is obtained.

So, we assume that $[f(I),I]I \neq 0$, that is, $[f(x_1,\dots,x_k),x_{k+1}]x_{k+2}$ is not an identity for I. In this case R is a prime GPI-ring and so is U (see [5]). Since U is centrally closed over C, it follows from [20] that U is a primitive ring with $H = Soc(U) \neq 0$. Then $[f(IH), IH]IH \neq 0$. For otherwise, $[f(IU), IU]IU = 0$ by [5], a contradiction. Choose $u_1, \ldots, u_{k+2} \in IH$ such that $[f(u_1, \ldots, u_k), u_{k+1}]u_{k+2} \neq 0$. Let $u \in IH$. Since H is a regular ring, there exists $e^2 = e \in H$ such that $eH = uH + u_1H + \cdots + u_{k+2}H$. Then $e \in IH$ and $u = eu$, $u_i = eu_i$ for $i = 1, \ldots, k+2$. Thus, we have $0 \neq [f(eH), eH]eH = [f(eHe), eHe]H$ i.e., $f(r_1, \ldots, r_k)$ is not central-valued in eHe.

By our assumption and by [5], we may also assume that

$$af(x_1, \ldots, x_k)^n + f(x_1, \ldots, x_k)^n b = (af(x_1, \ldots, x_k) + f(x_1, \ldots, x_k)b)^n$$

is an identity for IU. In particular,

$$af(x_1, \ldots, x_k)^n + f(x_1, \ldots, x_k)^n b = (af(x_1, \ldots, x_k) + f(x_1, \ldots, x_k)b)^n$$

is an identity for IH and so for eH. It follows that, for all $r_1, \ldots, r_k \in H$,

$$af(er_1, \ldots, er_k)^n + f(er_1, \ldots, er_k)^n b = (af(er_1, \ldots, er_k) + f(er_1, \ldots, er_k)b)^n. \tag{2.7}$$

We may write

$$f(x_1, \ldots, x_k) = \sum_i t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)x_i,$$

where t_i is a suitable multilinear polynomial in $k-1$ variables and x_i never appears in any monomials of t_i. Since $f(eHe) \neq 0$, there exists some t_i which does not vanish in eHe. Without loss of generality, we assume that $t_k(eHe) \neq 0$. Let $r \in H$. Then replacing r_k with $r(1-e)$ in (2.7), we have

$$0 = (at_k(er_1, \ldots, er_{k-1})er(1-e) + t_k(er_1, \ldots, er_{k-1})er(1-e)b)^n. \tag{2.8}$$

Left multiplying by $(1-e)$, we obtain $(1-e)(at_k(er_1, \ldots, er_{k-1})er(1-e))^n = 0$, that is, $\{(1-e)at_k(er_1, \ldots, er_{k-1})er\}^{n+1} = 0$ for all $r \in H$. By [9], $(1-e)at_k(er_1e, \ldots, er_{k-1}e) = 0$ for all $r_1, \ldots, r_{k-1} \in H$. Since eHe is a simple Artinian ring and $t_k(eHe) \neq 0$ is invariant under the action of all inner automorphisms of eHe, by [6, Lemma 2], $(1-e)ae = 0$. Now again right multiplying by e in (2.8), we obtain $(t_k(er_1, \ldots, er_{k-1})er(1-e)b)^n e = 0$ that is, $\{(1-e)bt_k(er_1, \ldots, er_{k-1})er)^{n+1} = 0$ for all $r \in H$, implying $(1-e)bt_k(er_1e, \ldots, er_{k-1}e) = 0$ for all $r_1, \ldots, r_{k-1} \in H$. By above argument we conclude that $(1-e)be = 0$.

In particular, from (2.7), we can write that H satisfies

$$\begin{aligned} &e\{af(er_1e, \ldots, er_ke)^n + f(er_1e, \ldots, er_ke)^n b \\ &\qquad - (af(er_1e, \ldots, er_ke) + f(er_1e, \ldots, er_ke)b)^n\}e = 0 \end{aligned} \tag{2.9}$$

and so using the facts $(1-e)ae = 0$ and $(1-e)be = 0$, we have, prime ring eHe satisfies

$$\begin{aligned} &eaef(r_1, \ldots, r_k)^n + f(r_1, \ldots, r_k)^n ebe \\ &\qquad - (eaef(r_1, \ldots, r_k) + f(r_1, \ldots, r_k)ebe)^n = 0. \end{aligned} \tag{2.10}$$

By Proposition 2.1, since $f(r_1, \ldots, r_k)$ is not central-valued in eHe, we conclude $eae, ebe \in Ce$ with $(eae + ebe)^n - (eae + ebe) = 0$. Therefore, $ae = eae \in Ce$ and $be = ebe \in Ce$. Thus $au = aeu = eaeu \in Cu$ and hence au, u are linearly C-dependent for each $u \in I$. So $(a - \alpha)I = (0)$ for some $\alpha \in C$. Similarly, $(b - \beta)I = (0)$ for some $\beta \in C$.

Thus our hypothesis $af(x_1, \ldots, x_k)^n + f(x_1, \ldots, x_k)^n b = (af(x_1, \ldots, x_k) + f(x_1, \ldots, x_k)b)^n$ for all $x_1, \ldots, x_k \in I$, implies that $f(x_1, \ldots, x_k)^n\{(\alpha+\beta)^{n-1} - 1\}(\alpha + b) = 0$ for all $x_1, \ldots, x_k \in I$. By Lemma 2 in [4], either $f(I)I = 0$ or $\{(\alpha+\beta)^{n-1} - 1\}(\alpha + b) = 0$. If $f(I)I = 0$, then by Fact 4, conclusion (1) is obtained. If $\{(\alpha+\beta)^{n-1}-1\}(\alpha+b)=0$, then either $(\alpha+\beta)^{n-1}=1$ or $b=-\alpha\in C$. Both cases imply conclusions (2) and (3) respectively. ∎

We are now ready to prove our main theorem.

Theorem 2.1. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, g a nonzero generalized derivation of R, I a nonzero right ideal of R, $f(r_1,\dots,r_k)$ a multilinear polynomial over C and $n\geq 2$ be a fixed integer. If $g(f(r_1,\dots,r_k)^n)=g(f(r_1,\dots,r_k))^n$ for all $r_1,\dots,r_k\in I$, then one of the following holds:*

(1) *$IC=eRC$ for some idempotent $e\in soc(RC)$ and $f(x_1,\dots,x_k)$ is central-valued on $eRCe$;*
(2) *there exist $a,b\in U$ such that $g(x)=ax+xb$ for all $x\in R$ and $(a-\alpha)I=(0)$, $(b-\beta)I=(0)$ for some $\alpha,\beta\in C$ with $(\alpha+\beta)^{n-1}=1$;*
(3) *there exists $a\in U$ such that $g(x)=ax$ for all $x\in R$ with $aI=(0)$.*

Proof. If g is inner generalized derivation of R, then result follows by Proposition 2.2. Assume that g is not U-inner. Then by Remark 1.1, we may assume that for all $x\in U$, $g(x)=ax+d(x)$, where $a\in U$ and d is a derivation of U. By our assumption, I satisfies $g(f(x_1,\dots,x_k)^n)=g(f(x_1,\dots,x_k))^n$. Since I and IU satisfy the same generalized polynomial identities (see [5]) as well as the same differential identities (see [17]), we may assume for $u_1,\dots,u_k\in I$ that U satisfies

$$\begin{aligned}&af(u_1x_1,\dots,u_kx_k)^n+d(f(u_1x_1,\dots,u_kx_k)^n)\\&=\{af(u_1x_1,\dots,u_kx_k)+d(f(u_1x_1,\dots,u_kx_k))\}^n\end{aligned}\tag{2.11}$$

that is,

$$\begin{aligned}&af(u_1x_1,\dots,u_kx_k)^n\\&+\sum_{i=0}^{n-1}f(u_1x_1,\dots,u_kx_k)^i d(f(u_1x_1,\dots,u_kx_k))f(u_1x_1,\dots,u_kx_k)^{n-i-1}\\&=\{af(u_1x_1,\dots,u_kx_k)+d(f(u_1x_1,\dots,u_kx_k))\}^n.\end{aligned}\tag{2.12}$$

Since g is not inner, d can not be inner derivation of U. Then we have from (2.12) that

$$\begin{aligned}&af(u_1x_1,\dots,u_kx_k)^n+\sum_{i=0}^{n-1}f(u_1x_1,\dots,u_kx_k)^i\Big\{f^d(u_1x_1,\dots,u_kx_k)\\&\quad+\sum_j f(u_1x_1,\dots,d(u_j)x_j+u_jd(x_j),\dots,u_kx_k)\Big\}f(u_1x_1,\dots,u_kx_k)^{n-i-1}\\&=\Big\{af(u_1x_1,\dots,u_kx_k)+f^d(u_1x_1,\dots,u_kx_k)\\&\quad+\sum_j f(u_1x_1,\dots,d(u_j)x_j+u_jd(x_j),\dots,u_kx_k)\Big\}^n.\end{aligned}\tag{2.13}$$

By Kharchenko's theorem [14], we have that U satisfies

$$\begin{aligned}
& af(u_1x_1,\dots,u_kx_k)^n+\sum_{i=0}^{n-1} f(u_1x_1,\dots,u_kx_k)^i\Big\{f^d(u_1x_1,\dots,u_kx_k) \\
&\quad +\sum_j f(u_1x_1,\dots,d(u_j)x_j+u_jy_j,\dots,u_kx_k)\Big\}f(u_1x_1,\dots,u_kx_k)^{n-i-1} \\
&= \Big\{af(u_1x_1,\dots,u_kx_k)+f^d(u_1x_1,\dots,u_kx_k) \\
&\quad +\sum_j f(u_1x_1,\dots,d(u_j)x_j+u_jy_j,\dots,u_kx_k)\Big\}^n.
\end{aligned} \tag{2.14}$$

In particular, putting $x_1=0$, we have that U satisfies

$$0=\{f(u_1y_1,\dots,u_kx_k)\}^n. \tag{2.15}$$

Since I and IU satisfy the same polynomial identities, we have that I satisfies $f(x_1,\dots,x_k)^n=0$. By [6, Main Theorem], $f(I)I=0$ and hence conclusion (2) is obtained by using Fact 4. Hence the theorem is proved. ∎

It is well known that if R is a prime ring and L is a non-central Lie ideal of R, then there exists a nonzero two-sided ideal I of R such that $0\neq[I,R]\subseteq L$, unless char $(R)=2$ and R satisfies the standard identity s_4. Thus from above theorem following corollary is straightforward.

Corollary 2.1. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, g a nonzero generalized derivation of R, L a noncentral Lie ideal of R and $n\geq 2$ be a fixed integer. If $g(u^n)=g(u)^n$ for all $u\in L$, then one of the following holds:*

(1) *char $(R)=2$ and R satisfies s_4, standard identity of four variables.*
(2) *there exists $\lambda\in C$ such that $g(x)=\lambda x$ for all $x\in R$ with $\lambda^{n-1}=1$.*

Now we prove our next corollary, which states that the restriction on char $(R)=2$ and R satisfies s_4 in the Theorem 1.2 can be omitted.

Corollary 2.2. *Let R be a prime ring with Utumi quotient ring U and extended centroid C, g a nonzero generalized derivation of R, I a nonzero right ideal of R and $f(r_1,\dots,r_k)$ be a multilinear polynomial over C. If $g(f(r_1,\dots,r_k)^2)=g(f(r_1,\dots,r_k))^2$ for all $r_1,\dots,r_k\in I$, then one of the following holds:*

(1) *$IC=eRC$ for some idempotent $e\in soc(RC)$ and $f(x_1,\dots,x_k)$ is central-valued on $eRCe$;*
(2) *there exists $a\in U$ such that $g(x)=xa$ for all $x\in I$ and $(a-1)I=(0)$;*
(3) *there exists $a\in U$ such that $g(x)=ax$ for all $x\in R$ with $aI=(0)$.*

Proof. By Theorem 2.1, we have only to consider the case when $g(x)=ax+xb$ for all $x\in R$ and $(a-\alpha)I=(0)$, $(b-\beta)I=(0)$ for some $\alpha,\beta\in C$ with $\alpha+\beta=1$. Then $g(x)=ax+xb=\alpha x+xb=x(\alpha+b)$ for all $x\in I$ with $(0)=(b-\beta)I=(b+\alpha-1)I$. Hence we obtain our conclusion (2). ∎

Corollary 2.3. *Let R be a prime ring with extended centroid C, g a nonzero generalized derivation of R and $f(r_1,\dots,r_k)$ a noncentral multilinear polynomial over C. If $g(f(r_1,\dots,r_k)^2)=g(f(r_1,\dots,r_k))^2$ for all $r_1,\dots,r_k\in R$, then $g(x)=x$ for all $x\in R$.*

Corollary 2.4. *Let R be a prime ring with extended centroid C, d a derivation of R and $f(r_1,\dots,r_k)$ a noncentral multilinear polynomial over C. If $d(f(r_1,\dots,r_k)^2) = d(f(r_1,\dots,r_k))^2$ for all $r_1,\dots,r_k \in R$, then $d = 0$.*

Example 2.1. Let Z be the set of all integers. Consider a ring $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | x,y \in Z \right\}$ and a multilinear polynomial $f(X,Y) = XY$ which is not central-valued on R. We define maps $g,d : R \to R$, by $g\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 2y \\ 0 & 0 \end{pmatrix}$ and $d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then g is a generalized derivation associated to the derivation d satisfying $g(f(X,Y)^2) = g(f(X,Y))^2$ for all $X,Y \in R$. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0$, R is not prime ring. Since g is not an identity mapping in R, the primeness hypothesis in Corollary 2.3 is essential.

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