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Infinitely Many Solutions for a Class of Fractional Boundary Value Problem

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Abstract. In this paper, by using the minmax methods in critical point theory, we give some new criteria to guarantee a class of fractional boundary value problem have infinite solutions under some adequate conditions. Recent results in the literature are significantly improved.

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1. Introduction

In this paper, we consider the fractional boundary value problem (BVP for short) of the following form

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \beta < 1$ respectively, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumptions:

(A) $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+), b \in L^1(0,T;\mathbb{R}^+)$, such that

$$|F(t,x)| \le a(|x|)b(t), \quad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. In particular, if $\beta = 0$, BVP (1.1) reduces to the standard second-order boundary value problem.

Fractional calculus and fractional differential equations can find many applications in various fields of physical science such as viscoelasticity, diffusion, control, relaxation processes and modeling phenomena in engineering, see [1,2,8,11,13,14,17,21,23,25,28,33].

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Recently, many results were obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by use of techniques of nonlinear analysis, such as fixed point theory (including Leray-Schauder nonlinear alternative) (see [4,30]), topological degree theory (including coincidence degree theory) (see [5, 15]) and comparison method (including upper and lower solutions methods and monotone iterative method) (see [18, 31]) and so on. However, it seems that the popular methods mentioned above are not appropriate for discussing BVP (1.1), for the equivalent integral equation is not easy to be obtained.

It should be noted that variational methods have turned out to be a very effective analytical tool in determining the existence of solutions for integer order differential equation. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical (stationary) points of a suitable energy functional defined on an appropriate function space. The classical critical point theory was developed in the sixties and seventies for C^1 functional. The celebrated and important result in the last 30 years has been Mountain Pass Theorem due to Ambrosetti and Rabinowitz [3] in 1973. Since then, a series of new theorems in the minimax form have appeared via various linking, category and index theories. Now these results become a wonderful tool in studying the existence of solutions to differential equations with variational structures. We refer readers to the books (or surveys) due to Mawhin and Willem [20], Rabinowitz [24], Brézis and Nirenberg [9], Nirenberg [22], Schechter [26], Willem [29], and the papers [10, 19, 27, 32] and the references therein.

Recently, Jiao and Zhou [16] introduced some appropriate function spaces as their working space and set up a variational functional for BVP (1.1), then they gave two existence results of solutions for problem (1.1) by using the least action principle and Mountain Pass Theorem in critical point theory. It is interesting to ask whether problem (1.1) has infinite solutions under suitable conditions. Motivated by the excellent results of [16], we give definite answer to this question by using Fountain Theorem and Dual Fountain Theorem. Fountain Theorem and its dual form were established by Bartsch in [6] and by Bartsch-Willem in [7] respectively. They are effective tools for studying the existence of infinitely many large or small energy solutions. It should be noted that the $(\mathrm{PS})_c$ condition and its variants play an important role in these theorems and their applications.

The structure of the paper is the following. In the next section, we present the necessary preliminary knowledge. In Section 3, using variational methods we prove the multiplicity results for the solutions of problem (1.1). Finally in Section 4, two examples are presented to illustrate our results.

2. Preliminaries and variational setting

In this section, we recall some related preliminaries and display the variational setting which has been established for our problem.

Definition 2.1. [17] *Let $f(t)$ be a function defined on $[a,b]$ and $\gamma > 0$. The left and right Riemann-Liouville fractional integrals of order γ for function $f(t)$ denoted by ${}_aD_t^{-\gamma} f(t)$ and ${}_tD_b^{-\gamma} f(t)$, respectively, are defined by*

$${}_aD_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)ds, \quad t \in [a,b],$$

and

$$ {}_tD_b^{-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_t^b (t-s)^{\gamma-1}f(s)ds, \quad t\in[a,b], $$

provided the right-hand sides are pointwise defined on $[a,b]$, *where* Γ *is the gamma function.*

Definition 2.2. [17] *Let* $f(t)$ *be a function defined on* $[a,b]$. *The left and right Riemann-Liouville fractional derivatives of order* γ *for function* $f(t)$ *denoted by* ${}_aD_t^{\gamma}f(t)$ *and* ${}_tD_b^{\gamma}f(t)$, *respectively, are defined by*

$$ {}_aD_t^{\gamma}f(t) = \frac{d^n}{dt^n}{}_aD_t^{\gamma-n}f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\left(\int_a^t (t-s)^{n-\gamma-1}f(s)ds\right) $$

and

$$ {}_tD_b^{\gamma}f(t) = (-1)^n\frac{d^n}{dt^n}{}_tD_b^{\gamma-n}f(t) = \frac{1}{\Gamma(n-\gamma)}(-1)^n\frac{d^n}{dt^n}\left(\int_t^b (s-t)^{n-\gamma-1}f(s)ds\right), $$

where $t\in[a,b]$, $n-1\le\gamma<n$ *and* $n\in\mathbf{N}$.

Property 2.1. [17] The left and right Riemann-Liouville fractional integral operators have the property of a semigroup, i.e.

$$ {}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2}f(t)) = {}_aD_t^{-\gamma_1-\gamma_2}f(t) \quad \text{and} \quad {}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2}f(t)) = {}_tD_b^{-\gamma_1-\gamma_2}f(t), \quad \forall\gamma_1,\gamma_2>0 $$

The left and right Caputo fractional derivatives are defined via the above Riemann-Liouville fractional derivatives. In particular, they are defined for the function belonging to the space of absolutely continuous functions, which we denote by $AC([a,b],\mathbb{R}^N)$. $AC^k([a,b],\mathbb{R}^N)(k=1,\cdots)$ is the space of functions f such that $f\in C^{k-1}([a,b],\mathbb{R}^N)$ and $f^{(k-1)}\in AC([a,b],\mathbb{R}^N)$. In particular, $AC([a,b],\mathbb{R}^N)=AC^1([a,b],\mathbb{R}^N)$.

Definition 2.3. [17] *Let* $\gamma\ge 0$ *and* $n\in\mathbf{N}$. *If* $\gamma\in[n-1,n)$ *and* $f(t)\in AC^n([a,b],\mathbb{R}^N)$, *then the left and right Caputo fractional derivative of order* γ *for function* $f(t)$ *denoted by* ${}_a^cD_t^{\gamma}f(t)$ *and* ${}_t^cD_b^{\gamma}f(t)$, *respectively, exist almost everywhere on* $[a,b]$. ${}_a^cD_t^{\gamma}f(t)$ *and* ${}_t^cD_b^{\gamma}f(t)$ *are represented by*

$$ {}_a^cD_t^{\gamma}f(t) = {}_aD_t^{\gamma-n}f^{(n)}(t) = \frac{1}{\Gamma(n-\gamma)}\left(\int_a^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds\right) $$

and

$$ {}_t^cD_b^{\gamma}f(t) = (-1)^n{}_tD_b^{\gamma-n}f^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\left(\int_t^b (s-t)^{n-\gamma-1}f^{(n)}(s)ds\right), $$

respectively, where $t\in[a,b]$.

Definition 2.4. [16] *Define* $0<\alpha\le 1$ *and* $1<p<\infty$. *The fractional derivative space* $E_0^{\alpha,p}$ *is defined by the closure of* $C_0^\infty([0,T],\mathbb{R}^N)$ *with respect to the norm*

$$ \|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^cD_t^{\alpha}u(t)|^p dt\right)^{1/p}, \quad \forall u\in E_0^{\alpha,p}, \tag{2.1} $$

where $C_0^\infty([0,T],\mathbb{R}^N)$ *denotes the set of all functions* $u\in C^\infty([0,T],\mathbb{R}^N)$ *with* $u(0)=u(T)=0$. *It is obvious that the fractional derivative space* $E_0^{\alpha,p}$ *is the space of functions* $u\in L^p(0,T;\mathbb{R}^N)$ *having an* α*-order Caputo fractional derivative* ${}_0^cD_t^{\alpha}u\in L^p(0,T;\mathbb{R}^N)$ *and* $u(0)=u(T)=0$.

Proposition 2.1. [16] *Let $0<\alpha\le 1$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable space.*

Proposition 2.2. [16] *Let $0<\alpha\le 1$ and $1<p<\infty$. For all $u\in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p}\le \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0^cD_t^\alpha u\|_{L^p}. \tag{2.2}$$

Moreover, if $\alpha>1/p$ and $1/p+1/q=1$, then

$$\|u\|_\infty\le \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\|{}_0^cD_t^\alpha u\|_{L^p}. \tag{2.3}$$

According to (2.3), we can consider $E_0^{\alpha,p}$ with respect to the norm

$$\|u\|_{\alpha,p}=\|{}_0^cD_t^\alpha u\|_{L^p}=\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\right)^{1/p}. \tag{2.4}$$

Proposition 2.3. [16] *Define $0<\alpha\le 1$ and $1<p<\infty$. Assume that $\alpha>1/p$ and the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e. $u_k\rightharpoonup u$. Then $u_k\to u$ in $C([0,T],\mathbb{R}^N)$, i.e. $\|u-u_k\|_\infty\to 0$, as $k\to\infty$.*

Making use of the Property 2.1 and the Definition 2.3, for any $u\in AC([0,T],\mathbb{R}^N)$, BVP (1.1) is equivalent to the following problem:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{\alpha-1}\left({}_0^cD_t^\alpha u(t)\right)-\dfrac{1}{2}\,{}_tD_T^{\alpha-1}\left({}_t^cD_T^\alpha u(t)\right)\right) \\ \qquad\qquad\qquad\qquad +\nabla F(t,u(t))=0, \quad \text{a.e. } t\in[0,T], \\ u(0)=u(T)=0, \end{cases} \tag{2.5}$$

where $\alpha=1-\beta/2\in(1/2,1]$.

In the following, we will treat BVP (2.5) in the Hilbert space $E^\alpha=E_0^{\alpha,2}$ with the corresponding norm $\|u\|_\alpha=\|u\|_{\alpha,2}$. It follows from [16, Theorem 4.1] that the functional φ given by

$$\varphi(u)=\int_0^T\left[-\frac{1}{2}\left({}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha u(t)\right)-F(t,u(t))\right]dt \tag{2.6}$$

is continuously differentiable on E^α. Moreover, for $\forall u,v\in E^\alpha$, we have

$$\begin{aligned}\langle\varphi'(u),v\rangle=&-\int_0^T\frac{1}{2}\left[\left({}_0^cD_t^\alpha u(t),{}_t^cD_T^\alpha v(t)\right)+\left({}_t^cD_T^\alpha u(t),{}_0^cD_t^\alpha v(t)\right)\right]dt\\ &-\int_0^T\left(\nabla F\left(t,u(t)\right),v(t)\right)dt.\end{aligned} \tag{2.7}$$

Definition 2.5. [16] *A function $u\in AC([0,T],\mathbb{R}^N)$ is called a solution of BVP* (2.5) *if*

(i) *$D^\alpha(u(t))$ is derivative for almost every $t\in[0,T]$, and*

(ii) *u satisfies* (2.5),

where $D^\alpha(u(t)):=1/2\,{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t))-1/2\,{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))$.

It is well known that the solutions for BVP (2.5) correspond to the critical points of the functional φ, see [16, Theorem 4.2].

Proposition 2.4. [16] *If* $1/2 < \alpha \le 1$, *then for any* $u \in E^\alpha$, *we have*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le -\int_0^T \left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right)dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{2.8}$$

3. Main results and proofs

Theorem 3.1. *Assume that* $F(t,x)$ *satisfies the condition* (A), *and suppose the following conditions hold:*

(A1) *there exist* $\kappa > 2$ *and* $r > 0$ *such that*

$$\kappa F(t,x) \le (\nabla F(t,x), x)$$

for a.e. $t \in [0,T]$ *and all* $|x| \ge r$ *in* $\mathbb{R}^N$;

(A2) *there exist positive constants* $\mu > 2$ *and* $Q > 0$ *such that*

$$\limsup_{|x|\to+\infty} \frac{F(t,x)}{|x|^\mu} \le Q$$

uniformly for a.e. $t \in [0,T]$;

(A3) *there exist* $\mu' > 2$ *and* $Q' > 0$ *such that*

$$\liminf_{|x|\to+\infty} \frac{F(t,x)}{|x|^{\mu'}} \ge Q'$$

uniformly for a.e. $t \in [0,T]$;

(A4) $F(t,x) = F(t,-x)$ *for* $t \in [0,T]$ *and all* x *in* $\mathbb{R}^N$.

Then BVP (1.1) *has infinite solutions* $\{u_n\}$ *on* E^α *for every positive integer* n *such that* $\|u_n\|_\infty \to +\infty$, *as* $n \to \infty$.

Theorem 3.2. *Assume that* $F(t,x)$ *satisfies the following assumption:*

(A5) $F(t,x) := a(t)|x|^\gamma$, *where* $a(t) \in L^\infty(0,T;\mathbb{R}^+)$ *and* $1 < \gamma < 2$ *is a constant.*

Then BVP (1.1) *has infinite solutions* $\{u_n\}$ *on* E^α *for every positive integer* n *with* $\|u_n\|_\alpha$ *bounded.*

Before giving the proof of our main results, we still state some necessary definition and theorems.

Definition 3.1. [20] *Let* H *be a real Banach space,* ψ: $H \to \mathbb{R}$ *is differentiable and* $c \in \mathbb{R}$. *We say that* ψ *satisfies the* $(PS)_c$ *condition if the existence of a sequence* $\{u_k\}$ *in* H *such that*

$$\psi(u_k) \to c, \quad \psi'(u_k) \to 0$$

as $k \to \infty$, *implies that* c *is a critical value of* ψ.

As E^α is a separable and reflexive Banach space, there exist (see [12]) $\{e_n\}_{n=1}^\infty \subset E^\alpha$ and $\{f_n\}_{n=1}^\infty \subset (E^\alpha)^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1, & n = m, \\ 0, & n \ne m, \end{cases}$$

$$E^\alpha = \overline{\operatorname{span}}\{e_n : n = 1,2,\cdots\} \quad \text{and} \quad (E^\alpha)^* = \overline{\operatorname{span}}^{W^*}\{f_n : n = 1,2,\cdots\}. \tag{3.1}$$

For $k=1,2,\ldots$, denote

$$X_k=\operatorname{span}\{e_k\},\quad Y_k=\bigoplus_{j=1}^{k}X_j,\quad Z_k=\overline{\bigoplus_{j=k}^{\infty}X_j}. \tag{3.2}$$

Theorem 3.3. (Fountain Theorem, see [6]). *Suppose*

(H1) *X is a Banach space, $\varphi\in C^1(X,\mathbb{R})$ is an even functional, the subspace X_k, Y_k and Z_k are defined by* (3.2)*;*

If for every $k\in\mathbf{N}$, there exist $\rho_k>r_k>0$ such that

(H2) $a_k:=\max_{\substack{u\in Y_k\\ \|u\|=\rho_k}}\varphi(u)\le 0$*;*

(H3) $b_k:=\inf_{\substack{u\in Z_k\\ \|u\|=r_k}}\varphi(u)\to\infty$*, as $k\to\infty$;*

(H4) *φ satisfies the (PS)$_c$ condition for every $c>0$.*

Then φ has an unbounded sequence of critical values.

Theorem 3.4. (Dual Fountain Theorem, see [7]). *Assume (H1) is satisfied, and there is a $k_0>0$ so as to for each $k\ge k_0$, there exist $\rho_k>r_k>0$ such that*

(H5) $d_k:=\inf_{\substack{u\in Z_k\\ \|u\|\le\rho_k}}\varphi(u)\to 0$*, as $k\to\infty$;*

(H6) $i_k:=\max_{\substack{u\in Y_k\\ \|u\|=r_k}}\varphi(u)<0$*;*

(H7) $\inf_{\substack{u\in Z_k\\ \|u\|=\rho_k}}\varphi(u)\ge 0$*;*

(H8) *φ satisfies the (PS)$_c^*$ condition for every $c\in[d_{k_0},0)$.*

Then φ has a sequence of negative critical values converging to 0.

Remark 3.1. φ satisfies the (PS)$_c^*$ condition means that: if any sequence $\{u_{n_j}\}\subset X$ such that $n_j\to\infty$, $u_{n_j}\in Y_{n_j}$, $\varphi(u_{n_j})\to c$ and $(\varphi|_{Y_{n_j}})'(u_{n_j})\to 0$, then $\{u_{n_j}\}$ contains a subsequence converging to critical point of φ. It is obvious that if φ satisfies the (PS)$_c^*$ condition, then φ satisfies the (PS)$_c$ condition.

The proof of Theorem 3.1 is organized as follows: first, we show the functional φ defined by (2.6) satisfies the (PS) condition, then we verify for φ the conditions in Theorem 3.3 item by item, then φ has an unbounded sequence of critical values.

Proof of Theorem 3.1. Let $\{u_n\}\subset E^\alpha$ such that $\varphi(u_n)$ is bounded and $\varphi'(u_n)\to 0$ as $n\to\infty$. First we prove $\{u_n\}$ is a bounded sequence, otherwise, $\{u_n\}$ would be unbounded sequence, passing to a subsequence, still denoted by $\{u_n\}$, such that $\|u_n\|_\alpha\ge 1$ and $\|u_n\|_\alpha\to\infty$, as $n\to\infty$. Noting that

$$\langle\varphi'(u_n),u_n\rangle=-\int_0^T\big[\big({}_0^cD_t^\alpha u_n(t),{}_t^cD_T^\alpha u_n(t)\big)+\big(\nabla F(t,u_n(t)),u_n(t)\big)\big]dt.$$

In view of the condition (A1) and (2.8) that

$$\begin{aligned}\varphi(u_n)-\frac{1}{\kappa}\langle\varphi'(u_n),u_n\rangle=&\left(\frac{1}{\kappa}-\frac{1}{2}\right)\int_0^T({}_0^cD_t^\alpha u_n(t),{}_t^cD_T^\alpha u_n(t))dt\\&+\int_0^T\left[\frac{1}{\kappa}(\nabla F(t,u_n(t)),u_n(t))-F(t,u_n(t))\right]dt\end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{1}{2}-\frac{1}{\kappa}\right)|\cos(\pi\alpha)|\|u_n\|_\alpha^2 \\
&\quad + \left(\int_{\Omega_1}+\int_{\Omega_2}\right)\left[\frac{1}{\kappa}\big(\nabla F(t,u_n(t)),u_n(t)\big)-F(t,u_n(t))\right]dt \\
&\geq \left(\frac{1}{2}-\frac{1}{\kappa}\right)|\cos(\pi\alpha)|\|u_n\|_\alpha^2 - C_1, \qquad (3.3)
\end{aligned}
$$

where $\Omega_1 := \{t\in[0,T]; |u_n(t)|\leq r\}$, $\Omega_2 := [0,T]\setminus\Omega_1$ and C_1 is a positive constant.

Since $\varphi(u_n)$ is bounded, there exists a positive constant C_2, such that $|\varphi(u_n)|\leq C_2$. Hence, we have

$$
\begin{aligned}
C_2 \geq \varphi(u_n) &\geq \left(\frac{1}{2}-\frac{1}{\kappa}\right)|\cos(\pi\alpha)|\|u_n\|_\alpha^2 + \frac{1}{\kappa}\langle\varphi'(u_n),u_n\rangle - C_1 \\
&\geq \left(\frac{1}{2}-\frac{1}{\kappa}\right)|\cos(\pi\alpha)|\|u_n\|_\alpha^2 - \frac{1}{\kappa}\|\varphi'(u_n)\|_\alpha\|u_n\|_\alpha - C_1,
\end{aligned}
\tag{3.4}
$$

so $\{u_n\}$ is a bounded sequence in E^α by (3.4).

Since E^α is a reflexive space, going to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in E^α, thus we have

$$
\begin{aligned}
\langle\varphi'(u_n)-\varphi'(u),\, u_n-u\rangle &= \langle\varphi'(u_n),\, u_n-u\rangle - \langle\varphi'(u),\, u_n-u\rangle \\
&\leq \|\varphi'(u_n)\|_\alpha\|u_n-u\|_\alpha - \langle\varphi'(u),\, u_n-u\rangle \to 0,
\end{aligned}
\tag{3.5}
$$

as $n\to\infty$. Moreover, according to (2.3) and Proposition 2.3, we have $\{u_n\}$ is bounded in $C([0,T],\mathbb{R}^N)$ and $\|u_n-u\|_\infty\to 0$ as $n\to\infty$.

Observing that

$$
\begin{aligned}
&\langle\varphi'(u_n)-\varphi'(u),\, u_n-u\rangle \\
&= -\int_0^T \big({}_0^cD_t^\alpha(u_n(t)-u(t)),{}_t^cD_T^\alpha(u_n(t)-u(t))\big)dt \\
&\quad -\int_0^T \big(\nabla F(t,u_n(t))-\nabla F(t,u(t)),u_n(t)-u(t)\big)dt \\
&\geq |\cos(\pi\alpha)|\|u_n-u\|_\alpha^2 - \left|\int_0^T \big(\nabla F(t,u_n(t))-\nabla F(t,u(t))\big)dt\right|\|u_n-u\|_\infty.
\end{aligned}
\tag{3.6}
$$

Combining this with (3.5), it is easy to verify that $\|u_n-u\|_\alpha\to 0$ as $n\to\infty$, and hence that $u_n\to u$ in E^α. Thus, $\{u_n\}$ admits a convergent subsequence.

For any $u\in Y_k$, let

$$
\|u\|_* := \left(\int_0^T |u(t)|^{\mu'}dt\right)^{1/\mu'}, \tag{3.7}
$$

and it is easy to verify that $\|\cdot\|_*$ defined by (3.7) is a norm of Y_k. Since all the norms of a finite dimensional normed space are equivalent, so there exists positive constant C_3 such that

$$
C_3\|u\|_\alpha \leq \|u\|_* \quad \text{for} \quad u\in Y_k. \tag{3.8}
$$

In view of (A3), there exist two positive constants M_1 and C_4 such that

$$
F(t,x)\geq M_1|x|^{\mu'} \tag{3.9}
$$

for a.e. $t \in [0,T]$ and $|x| \geq C_4$.

It follows (2.8), (3.8) and (3.9) that

$$
\begin{aligned}
\varphi(u) &= -\int_0^T \frac{1}{2}\left({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\right)dt - \int_0^T F(t,u(t))dt \\
&\leq \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \int_{\Omega_3} F(t,u(t))dt - \int_{\Omega_4} F(t,u(t))dt \\
&\leq \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - M_1\int_{\Omega_3} |u(t)|^{\mu'}dt - \int_{\Omega_4} F(t,u(t))dt \\
&= \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - M_1\int_0^T |u(t)|^{\mu'}dt + M_1\int_{\Omega_4} |u(t)|^{\mu'}dt - \int_{\Omega_4} F(t,u(t))dt \\
&\leq \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - C_3^{\mu'}M_1\|u\|_\alpha^{\mu'} + C_5,
\end{aligned}
$$

where $\Omega_3 := \{t \in [0,T];\ |u(t)| \geq C_4\}$, $\Omega_4 := [0,T] \setminus \Omega_3$ and C_5 is a positive constant.

Since $\mu' > 2$, then there exists positive constants d_k such that

$$
\varphi(u) \leq 0 \quad \text{for all} \quad u \in Y_k \quad \text{and} \quad \|u\|_\alpha \geq d_k. \tag{3.10}
$$

For any $u \in Z_k$, let

$$
\|u\|_\mu := \left(\int_0^T |u(t)|^\mu dt\right)^{1/\mu} \quad \text{and} \quad \beta_k := \sup_{\substack{u \in Z_k \\ \|u\|_\alpha = 1}} \|u\|_\mu, \tag{3.11}
$$

then we conclude $\beta_k \to 0$ as $k \to \infty$.

In fact, it is obvious that $\beta_k \geq \beta_{k+1} > 0$, so $\beta_k \to \beta \geq 0$ as $k \to \infty$. For every $k \in \mathbb{N}$, there exists $u_k \in Z_k$ such that

$$
\|u_k\|_\alpha = 1 \quad \text{and} \quad \|u_k\|_\mu > \beta_k/2. \tag{3.12}
$$

As E^α is reflexive, $\{u_k\}$ has a weakly convergent subsequence, still denoted by $\{u_k\}$, such that $u_k \rightharpoonup u$. We claim $u = 0$.

In fact, for any $f_m \in \{f_n : n = 1, 2, \cdots\}$, we have $f_m(u_k) = 0$, when $k > m$, so

$$
f_m(u_k) \to 0, \quad \text{as} \quad k \to \infty
$$

for any $f_m \in \{f_n : n = 1, 2, \cdots\}$, therefore $u = 0$.

By Proposition 2.3, when $u_k \rightharpoonup 0$ in E^α, then $u_k \to 0$ strongly in $C([0,T];\mathbb{R}^N)$. So we conclude $\beta = 0$ by (3.12).

In view of (A2), there exist two positive constants M_2 and C_6 such that

$$
F(t,x) \leq M_2|x|^\mu \tag{3.13}
$$

uniformly for a.e. $t \in [0,T]$ and $|x| \geq C_6$.

We have

$$\begin{aligned}
\varphi(u) &= -\int_0^T \frac{1}{2}\big({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\big)dt - \int_0^T F(t,u(t))dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \int_{\Omega_5} F(t,u(t))dt - \int_{\Omega_6} F(t,u(t))dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - M_2\int_{\Omega_5} |u(t)|^\mu dt - \int_{\Omega_6} F(t,u(t))dt \\
&= \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - M_2\int_0^T |u(t)|^\mu dt + M_2\int_{\Omega_6} |u(t)|^\mu dt - \int_{\Omega_6} F(t,u(t))dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - M_2{\beta_k}^\mu\|u\|_\alpha^\mu - C_7,
\end{aligned}$$

where $\Omega_5 := \{t \in [0,T];\ |u(t)| \geq C_6\}$, $\Omega_6 := [0,T] \setminus \Omega_5$ and C_7 is a positive constant.

Choosing $r_k = 1/\beta_k$, it is obvious that $r_k \to \infty$ as $k \to \infty$, then

$$b_k := \inf_{\substack{u\in Z_k \\ \|u\|_\alpha = r_k}} \varphi(u) \to \infty \quad \text{as} \quad k \to \infty, \tag{3.14}$$

that is, the condition (H3) in Theorem 3.3 is satisfied.

In view of (3.10), let $\rho_k := \max\{d_k, r_k + 1\}$, then

$$a_k := \max_{\substack{u\in Y_k \\ \|u\|_\alpha = \rho_k}} \varphi(u) \leq 0,$$

and this shows the condition of (H2) in Theorem 3.3 is satisfied.

We have proved the functional φ satisfies all the conditions of Theorem 3.3, then φ has an unbounded sequence of critical values $c_n = \varphi(u_n)$ by Theorem 3.3. We only need to show $\|u_n\|_\infty \to \infty$ as $n \to \infty$. In fact, since u_n is a critical point of the functional φ, that is

$$\langle \varphi'(u_n), u_n\rangle = -\int_0^T \big[({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t)) + (\nabla F(t,u_n(t)), u_n(t))\big]dt = 0. \tag{3.15}$$

Hence, we have

$$\begin{aligned}
c_n = \varphi(u_n) &= -\int_0^T \frac{1}{2}\big({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t)\big)dt - \int_0^T F(t,u_n(t)\big)dt, \\
&= \frac{1}{2}\int_0^T \big(\nabla F(t,u_n(t)), u_n(t)\big)dt - \int_0^T F\big(t,u_n(t)\big)dt, \\
&\leq \frac{1}{2}\int_0^T \big|\nabla F(t,u_n(t))\big||u_n(t)|dt + \int_0^T \big|F(t,u_n(t))\big|dt,
\end{aligned} \tag{3.16}$$

since $c_n \to \infty$, we conclude

$$\|u_n\|_\infty \to \infty \quad \text{as} \quad n \to \infty$$

by (3.16). In fact, if not, going to a subsequence if necessary, we may assume that

$$\|u_n\|_\infty \leq M_3,$$

for all $n \in \mathbf{N}$ and some positive constant M_3.

Combining assumption (A) and (3.16), we have

$$\begin{aligned} c_n &\le \frac{1}{2}\int_0^T \big|\nabla F(t,u_n(t))\big||u_n(t)|dt + \int_0^T \big|F\big(t,u_n(t)\big)\big|dt \\ &\le \frac{1}{2}(M_3+1)\max_{0\le s\le M_3} a(s)\int_0^T b(t)dt, \end{aligned}$$

which contradicts the unboundness of c_n. This completes the proof of Theorem 3.1. ∎

Proof of Theorem 3.2. Let us show that φ satisfies conditions in the Theorem 3.4 item by item. First, we show that φ satisfies the $(\mathrm{PS})_c^*$ condition for every $c\in\mathbb{R}$. Suppose $n_j\to\infty$, $u_{n_j}\in Y_{n_j}$, $\varphi(u_{n_j})\to c$ and $(\varphi|_{Y_{n_j}})'(u_{n_j})\to 0$, then $\{u_{n_j}\}$ is a bounded sequence, otherwise, $\{u_{n_j}\}$ would be unbounded sequence, passing to a subsequence, still denoted by $\{u_{n_j}\}$ such that $\|u_{n_j}\|_\alpha\ge 1$ and $\|u_{n_j}\|_\alpha\to\infty$. Note that

$$\langle\varphi'(u_{n_j}),u_{n_j}\rangle-\gamma\varphi(u_{n_j}) = \left(-1+\frac{\gamma}{2}\right)\int_0^T \big({}_0^cD_t^\alpha u_{n_j}(t),{}_t^cD_T^\alpha u_{n_j}(t)\big)dt \tag{3.17}$$

However, from (3.17), we have

$$-\gamma\varphi(u_{n_j}) \ge \left(1-\frac{\gamma}{2}\right)|\cos(\pi\alpha)|\|u_{n_j}\|_\alpha^2 - \|(\varphi|_{Y_{n_j}})'(u_{n_j})\|\|u_{n_j}\|_\alpha, \tag{3.18}$$

thus $\|u_{n_j}\|_\alpha$ is a bounded sequence in E^α. Going, if necessary, to a subsequence, we can assume that $u_{n_j}\rightharpoonup u$ in E^α. As $E^\alpha=\overline{\bigcup_{n_j} Y_{n_j}}$, we can choose $v_{n_j}\in Y_{n_j}$ such that $v_{n_j}\to u$. Hence

$$\begin{aligned} \lim_{n_j\to\infty}\langle\varphi'(u_{n_j}),u_{n_j}-u\rangle &= \lim_{n_j\to\infty}\langle\varphi'(u_{n_j}),u_{n_j}-v_{n_j}\rangle+\lim_{n_j\to\infty}\langle\varphi'(u_{n_j}),v_{n_j}-u\rangle \\ &= \lim_{n_j\to\infty}\langle(\varphi|_{Y_{n_j}})'(u_{n_j}),u_{n_j}-v_{n_j}\rangle = 0. \end{aligned}$$

So we have

$$\lim_{n_j\to\infty}\langle\varphi'(u_{n_j})-\varphi'(u),u_{n_j}-u\rangle = \lim_{n_j\to\infty}\langle\varphi'(u_{n_j}),u_{n_j}-u\rangle - \lim_{n_j\to\infty}\langle\varphi'(u),u_{n_j}-u\rangle = 0.$$

and

$$\begin{aligned} &\langle\varphi'(u_{n_j})-\varphi'(u),u_{n_j}-u\rangle \\ &= -\int_0^T \big({}_0^cD_t^\alpha(u_{n_j}(t)-u(t)),{}_t^cD_T^\alpha(u_{n_j}(t)-u(t))\big)\,dt \\ &\quad -\int_0^T \big(\nabla F(t,u_{n_j}(t))-\nabla F(t,u(t)),u_{n_j}(t)-u(t)\big)dt \\ &\ge |\cos(\pi\alpha)|\|u_{n_j}-u\|_\alpha^2 - \left|\int_0^T \big(\nabla F(t,u_{n_j}(t))-\nabla F(t,u(t))\big)dt\right|\|u_{n_j}-u\|_\infty\,, \end{aligned}$$

we can conclude $u_{n_j}\to u$ in E^α, furthermore, we have $\varphi'(u_{n_j})\to\varphi'(u)$. Let us prove $\varphi'(u)=0$ below. Taking arbitrarily $\omega_k\in Y_k$, notice when $n_j\ge k$, we have

$$\begin{aligned} \langle\varphi'(u),\omega_k\rangle &= \langle\varphi'(u)-\varphi'(u_{n_j}),\omega_k\rangle+\langle\varphi'(u_{n_j}),\omega_k\rangle \\ &= \langle\varphi'(u)-\varphi'(u_{n_j}),\omega_k\rangle+\langle(\varphi|_{Y_{n_j}})'(u_{n_j}),\omega_k\rangle. \end{aligned}$$

Going to limit in the right side of above equation reaches

$$\langle \varphi'(u), \omega_k \rangle = 0, \quad \forall \omega_k \in Y_k,$$

so $\varphi'(u) = 0$, this shows that φ satisfies the $(\mathrm{PS})_c^*$ for every $c \in \mathbb{R}$.

For any finite dimensional subspace $E \subset E^{\alpha}$, there exists $\varepsilon > 0$ such that

$$\text{meas}\{t \in [0,T] : a(t)|u(t)|^{\gamma} \geq \varepsilon \|u\|_{\alpha}^{\gamma}\} \geq \varepsilon, \quad \forall u \in E \backslash \{0\}. \tag{3.19}$$

Otherwise, for any positive integer n, there exists $u_n \in E\backslash\{0\}$ such that

$$\text{meas}\left\{t \in [0,T] : a(t)|u_n(t)|^{\gamma} \geq \frac{1}{n}\|u_n\|_{\alpha}^{\gamma}\right\} < \frac{1}{n}. \tag{3.20}$$

Set $v_n(t) := (u_n(t))/(\|u_n\|_{\alpha}) \in E\backslash\{0\}$, then $\|v_n\|_{\alpha} = 1$ for all $n \in \mathbb{N}$ and

$$\text{meas}\left\{t \in [0,T] : a(t)|v_n(t)|^{\gamma} \geq \frac{1}{n}\right\} < \frac{1}{n}. \tag{3.21}$$

Since $\dim E < \infty$, it follows from the compactness of the unit sphere of E that there exists a subsequence, denoted also by $\{v_n\}$, such that $\{v_n\}$ converges to some v_0 in E. It is obvious that $\|v_0\|_{\alpha} = 1$. By the equivalence of the norms on the finite dimensional space, we have $v_n \to v_0$ in $L^2([0,T];\mathbb{R}^N)$, i.e.

$$\int_0^T |v_n - v_0|^2 dt \to 0 \quad \text{as} \quad n \to \infty. \tag{3.22}$$

By (3.22) and Hölder inequality, we have

$$\begin{aligned} \int_0^T a(t)|v_n - v_0|^{\gamma} dt &\leq \left(\int_0^T a(t)^{\frac{2}{2-\gamma}} dt\right)^{\frac{2-\gamma}{2}} \left(\int_0^T |v_n - v_0|^2 dt\right)^{\frac{\gamma}{2}} \\ &= \|a\|_{\frac{2}{2-\gamma}} \left(\int_0^T |v_n - v_0|^2 dt\right)^{\frac{\gamma}{2}} \to 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.23}$$

Thus, there exist $\xi_1, \xi_2 > 0$ such that

$$\text{meas}\{t \in [0,T] : a(t)|v_0(t)|^{\gamma} \geq \xi_1\} \geq \xi_2. \tag{3.24}$$

In fact, if not, we have

$$\text{meas}\left\{t \in [0,T] : a(t)|v_0(t)|^{\gamma} \geq \frac{1}{n}\right\} = 0$$

for all positive integer n. It implies that

$$0 \leq \int_0^T a(t)|v_0|^{\gamma+2} dt < \frac{T}{n}\|v_0\|_{\infty}^2 \leq \frac{C_8^2 T}{n}\|v_0\|_{\alpha}^2 \to 0$$

as $n \to \infty$, where

$$C_8 := \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}$$

by (2.3). Hence $v_0 = 0$ which contradicts that $\|v_0\|_{\alpha} = 1$. Therefore, (3.24) holds.

Now let

$$\Omega_0 = \{t \in [0,T] : a(t)|v_0(t)|^{\gamma} \geq \xi_1\}, \quad \Omega_n = \left\{t \in [0,T] : a(t)|v_n(t)|^{\gamma} < \frac{1}{n}\right\},$$

and $\Omega_n^c = [0,T]\backslash\Omega_n = \{t \in [0,T] : a(t)|v_n(t)|^{\gamma} \geq 1/n\}$.

By (3.21) and (3.24), we have

$$\operatorname{meas}(\Omega_n \cap \Omega_0) = \operatorname{meas}(\Omega_0 \backslash (\Omega_n^c \cap \Omega_0) \geq \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_n^c \cap \Omega_0) \geq \xi_2 - \frac{1}{n}$$

for all positive integer n. Let n be large enough such that

$$\xi_2 - \frac{1}{n} \geq \frac{1}{2}\xi_2 \quad \text{and} \quad \frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n} \geq \frac{1}{2^\gamma}\xi_1,$$

then we have

$$\begin{aligned}\int_0^T a(t)|v_n - v_0|^\gamma dt &\geq \int_{\Omega_n \cap \Omega_0} a(t)|v_n - v_0|^\gamma dt \\ &\geq \frac{1}{2^{\gamma-1}} \int_{\Omega_n \cap \Omega_0} a(t)|v_0|^\gamma dt - \int_{\Omega_n \cap \Omega_0} a(t)|v_n|^\gamma dt \\ &\geq \left(\frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n}\right) \operatorname{meas}(\Omega_n \cap \Omega_0) \geq \frac{\xi_1}{2^\gamma} . \frac{\xi_2}{2} = \frac{\xi_1 \xi_2}{2^{\gamma+1}} > 0\end{aligned}$$

for all large n, which is a contradiction to (3.23). Therefore, (3.19) holds.

For any $u \in Z_k$, let

$$\|u\|_2 := \left(\int_0^T |u(t)|^2 dt\right)^{1/2} \quad \text{and} \quad \gamma_k := \sup_{\substack{u \in Z_k \\ \|u\|_\alpha = 1}} \|u\|_2,$$

then we conclude $\gamma_k \to 0$ as $k \to \infty$ in the same way as in the proof of Theorem 3.1.

$$\begin{aligned}\varphi(u) &= -\int_0^T \frac{1}{2}\left({}_0^c D_t^\alpha u(t), {}_t^c D_T^\alpha u(t)\right) dt - \int_0^T F(t, u(t)) dt, \\ &\geq \frac{1}{2}|\cos(\pi\alpha)| \|u\|_\alpha^2 - \int_0^T a(t)|u(t)|^\gamma dt \\ &\geq \frac{1}{2}|\cos(\pi\alpha)| \|u\|_\alpha^2 - \left(\int_0^T a(t)^{\frac{2}{2-\gamma}} dt\right)^{\frac{2-\gamma}{2}} \|u\|_2^\gamma \\ &\geq \frac{1}{2}|\cos(\pi\alpha)| \|u\|_\alpha^2 - \left(\int_0^T a(t)^{\frac{2}{2-\gamma}} dt\right)^{\frac{2-\gamma}{2}} \gamma_k^\gamma \|u\|_\alpha^\gamma.\end{aligned} \tag{3.25}$$

Let $\rho_k := \left((4c\gamma_k^\gamma)/(|\cos(\pi\alpha)|)\right)^{1/(2-\gamma)}$, where $c := \left(\int_0^T a(t)^{2/(2-\gamma)} dt\right)^{(2-\gamma)/2}$, it is obvious that $\rho_k \to 0$, as $k \to \infty$.

In view of (3.25), we conclude

$$\inf_{\substack{u \in Z_k \\ \|u\|_\alpha = \rho_k}} \varphi(u) \geq \frac{|\cos(\pi\alpha)|}{4} \rho_k^2 > 0,$$

so the condition (H7) in Theorem 3.4 is satisfied.

Furthermore, by (3.25), for any $u \in Z_k$ with $\|u\|_\alpha \leq \rho_k$, we have

$$\varphi(u) \geq -c\gamma_k^\gamma \|u\|_\alpha^\gamma.$$

Therefore,

$$-c\gamma_k^\gamma \rho_k^\gamma \leq \inf_{\substack{u \in Z_k \\ \|u\|_\alpha \leq \rho_k}} \varphi(u) \leq 0.$$

So we have

$$\inf_{\substack{u\in Z_k\\ \|u\|_\alpha\le\rho_k}} \varphi(u)\to 0$$

for ρ_k, $\gamma_k\to 0$, as $k\to\infty$. Hence (H5) in Theorem 3.4 is satisfied.

For any $u\in Y_k\backslash\{0\}$,

$$\begin{aligned}\varphi(u) &= -\int_0^T \frac{1}{2}\big({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\big)dt - \int_0^T F(t,u(t))dt,\\ &\le \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \int_0^T a(t)|u(t)|^\gamma dt\\ &\le \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \varepsilon\|u\|_\alpha^\gamma \mathrm{meas}(\Omega_u)\\ &\le \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \varepsilon^2\|u\|_\alpha^\gamma,\end{aligned}$$

where ε is given in (3.19), and $\Omega_u := \big\{t\in[0,T] : a(t)|u(t)|^\gamma \ge \varepsilon\|u\|_\alpha^\gamma\big\}$. Choosing $0<r_k<\min\{\rho_k, (|\cos(\pi\alpha)|\varepsilon^2)^{1/(2-\gamma)}\}$, we conclude

$$i_k := \max_{\substack{u\in Y_k\\ \|u\|_\alpha=r_k}} \varphi(u) < -\frac{1}{2|\cos(\pi\alpha)|}r_k^2 < 0, \quad \forall\, k\in\mathbb{N},$$

that is, the condition (H6) in Theorem 3.4 is satisfied. We have proved the functional φ satisfies all the conditions of Theorem 3.4, then φ has a bounded sequence of negative critical values $c_n=\varphi(u_n)$ converging to 0 by Theorem 3.4, we only need to show $\|u_n\|_\alpha$ is bounded as for every positive integer n.

$$\begin{aligned}c_n=\varphi(u_n) &= -\int_0^T \frac{1}{2}\big({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t)\big)dt - \int_0^T F(t,u_n(t))dt,\\ &= -\int_0^T \frac{1}{2}\big({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t)\big)dt - \int_0^T a(t)|u_n(t)|^\gamma dt,\\ &\ge \frac{|\cos(\pi\alpha)|}{2}\|u_n\|_\alpha^2 - a_0\|u_n\|_\infty^\gamma T,\\ &\ge \frac{|\cos(\pi\alpha)|}{2}\|u_n\|_\alpha^2 - a_0TC_8^\gamma\|u_n\|_\alpha^\gamma,\end{aligned} \tag{3.26}$$

where $a_0 =$ ess sup $\big\{a(t) : t\in[0,T]\big\}$. By Theorem 3.4, $c_n\to 0$ as $n\to\infty$. If $\|u_n\|_\alpha$ has an unbounded sequence, then c_n is unbounded by (3.26), which is a contradiction. The proof of Theorem 3.2 is completed. ∎

4. Examples

In this section, we give two examples to illustrate our results.

Example 4.1. In BVP (1.1), let $F(t,x)=|x|^4$, and choose

$$\kappa=4,\quad r=2,\quad \mu=\mu'=4 \quad \text{and} \quad Q=Q'=1,$$

so it is easy to verify that all the conditions (A1)–(A4) are satisfied. Then by Theorem 3.1, BVP (1.1) has infinite solutions $\{u_k\}$ on E^α for every positive integer k such that $\|u_k\|_\infty\to+\infty$, as $k\to\infty$.

Example 4.2. In BVP (1.1), let $F(t,x) = a(t)|x|^{3/2}$ where

$$a(t) = \begin{cases} T, & t = 0 \\ t, & 0 < t \leq T. \end{cases}$$

By Theorem 3.2, BVP (1.1) has infinite solutions $\{u_k\}$ on E^{α} for every positive integer k with $\|u_k\|_{\alpha}$ bounded.

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