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Metrizability of Rectifiable Spaces

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Abstract. A topological space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x,x) = (x,e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. In this paper, we firstly show that every submaximal rectifiable space G either has a regular G_δ-diagonal, or is a P-space. Then, we mainly discuss rectifiable spaces are determined by a point-countable cover, and show that if G is an α_4-rectifiable space determined by a point-countable cover $\mathscr{G}$ consisting of bisequential subspaces then it is metrizable, which generalizes a result of Lin and Shen's.

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1. Introduction

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product maps of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous. A topological space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x,x) = (x,e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a *rectification* on G. It is well known that rectifiable spaces and paratopological groups are all good generalizations of topological groups. In fact, for a topological group with the neutral element e, then it is easy to see that the map $\varphi(x,y) = (x, x^{-1}y)$ is a rectification on G. However, there exists a paratopological group which is not a rectifiable space; Sorgenfrey line [7, Example 1.2.2] is such an example. Also, the 7-dimensional sphere S_7 is rectifiable but not a topological group [17, § 3]. Further, it is easy to see that paratopological groups and rectifiable spaces are all homogeneous. Recently, the study of rectifiable spaces is an interesting topic in topological algebra, see [4, 5, 9, 11, 12].

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In [12], Lin and Shen proved that each bisequential rectifiable space is metrizable. In this paper, we shall generalize this result, and show that if G is an α_4-rectifiable space determined by a point-countable cover $\mathscr{G}$ consisting of bisequential subspaces then it is metrizable. Moreover, we also discuss the submaximal rectifiable spaces and obtain some interesting results.

2. Preliminaries

Let $\mathscr{P}$ be a family of subsets of a space X. The space X is *determined by* $\mathscr{P}$ if each subset F of X is closed if and only if $F \cap P$ is closed in P for each $P \in \mathscr{P}$. A space X is called an S_2-space (*Arens' space*) if $X = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$ and the topology is defined as follows: Each $x_n(m)$ is isolated; a basic neighborhood of x_n is $\{x_n\} \cup \{x_n(m) : m > k\}$ for some $k \in \mathbb{N}$; a basic neighborhood of ∞ is $\{\infty\} \cup (\bigcup\{V_n : n > k\})$ for some $k \in \mathbb{N}$, where V_n is a neighborhood of x_n. A space X is called S_ω if X is obtained by identifying all the limit points of ω many convergent sequences, where the identification is topological quotient.

Definition 2.1. *A space X is said to be Fréchet-Urysohn if, for each $x \in \overline{A} \subset X$, there exists a sequence $\{x_n\}$ such that $\{x_n\}$ converges to x and $\{x_n : n \in \mathbb{N}\} \subset A$. A space X is said to be strongly Fréchet-Urysohn if the following condition is satisfied*

(SFU) *For every $x \in X$ and each sequence $\eta = \{A_n : n \in \mathbb{N}\}$ of subsets of X such that $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there is a sequence $\zeta = \{a_n : n \in \mathbb{N}\}$ in X converging to x and intersecting infinitely many members of η.*

Obviously, a strongly Fréchet-Urysohn space is Fréchet-Urysohn. However, the space S_ω is Fréchet-Urysohn and non-strongly Fréchet-Urysohn.

Let X be a space. For $P \subset X$, the set P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P.

Definition 2.2. *Let $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ be a cover of a space X such that for each $x \in X$, (a) if $U, V \in \mathscr{P}_x$, then $W \subset U \cap V$ for some $W \in \mathscr{P}_x$; (b) the family $\mathscr{P}_x$ is a network of x in X, i.e., $x \in \bigcap \mathscr{P}_x$, and if $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathscr{P}_x$.*

The family $\mathscr{P}$ is called a weak base for X if, for every $A \subset X$, the set A is open in X whenever for each $x \in A$ there exists $P \in \mathscr{P}_x$ such that $P \subset A$. The space X is weakly first-countable if $\mathscr{P}_x$ is countable for each $x \in X$.

Definition 2.3. [3] *Let ζ and η be any family of non-empty subsets of X.*

1. *The family ζ is called a prefilter on a space X if, whenever P_1 and P_2 are in ζ, there is a $P \in \zeta$ such that $P \subset P_1 \cap P_2$;*
2. *A prefilter ζ on a space X is said to converge to a point $x \in X$ if every open neighborhood of x contains an element of ζ;*
3. *If $x \in X$ belongs to the closure of every element of a prefilter ζ on X, we say that ζ accumulates to x or a cluster point for ζ;*
4. *Two prefilters ζ and η are called to be synchronous if, for any $P \in \zeta$ and any $Q \in \eta$, $P \cap Q \neq \emptyset$;*
5. *A space X is called bisequential* [3] *if, for every prefilter ζ on X accumulating to a point $x \in X$, there exists a countable prefilter ξ on X converging to the same point x such that ζ and ξ are synchronous.*

Definition 2.4. *Let X be a topological space. We say that X is an α_4-space if for each countable family $\{S_n : n \in \mathbb{N}\}$ of sequences converging to some point $x \in X$ there is a sequence S converging to x such that $S_n \cap S \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.*

Theorem 2.1. [6, 9, 16] *A topological space G is rectifiable if and only if there exists $e \in G$ and two continuous maps $p : G^2 \to G$, $q : G^2 \to G$ such that for any $x \in G, y \in G$ the next identities hold:*

$$p(x,q(x,y)) = q(x,p(x,y)) = y, q(x,x) = e.$$

In fact, we can assume that $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$ in Theorem 2.1. Fixed a point $x \in G$, then $f_x, g_x : G \to G$ defined with $f_x(y) = p(x,y)$ and $g_x(y) = q(x,y)$, for each $y \in G$, are homeomorphism, respectively. We denote f_x, g_x with $p(x,G), q(x,G)$, respectively.

Let G be a rectifiable space, and let p be the multiplication on G. Further, we sometime write $x \cdot y$ instead of $p(x,y)$ and $A \cdot B$ instead of $p(A,B)$ for any $A, B \subset G$. Therefore, $q(x,y)$ is an element such that $x \cdot q(x,y) = y$; since $x \cdot e = x \cdot q(x,x) = x$ and $x \cdot q(x,e) = e$, it follows that e is a right neutral element for G and $q(x,e)$ is a right inverse for x. Hence a rectifiable space G is a topological algebraic system with operation p,q, 0-ary operation e and identities as above. It is easy to see that this algebraic system need not to satisfy the associative law about the multiplication operation p. Clearly, every topological loop is rectifiable.

All spaces are T_1 and regular unless stated otherwise. The notation $\mathbb{N}$ denotes the set of all positive natural numbers. The letter e denotes the neutral element of a group and the right neutral element of a rectifiable space, respectively. Readers may refer to [3, 7, 8] for notations and terminology not explicitly given here.

3. Submaximal rectifiable subspaces

In this section, we mainly show that a submaximal rectifiable space G is metrizable if it is locally countably compact or is of point-countable type, see Theorem 3.2. Let A be a subset of a rectifiable space G. The set A is called *a rectifiable subspace of G* if we have $p(A,A) \subset A$ and $q(A,A) \subset A$.

Lemma 3.1. [11] *Let G be a rectifiable space. If V is an open rectifiable subspace of G, then V is closed in G.*

Proposition 3.1. [11] *Let G be a rectifiable space. If H is a rectifiable subspace of G, then $\overline{H}$ is also a rectifiable subspace of G.*

The following Lemma 3.2 is well known.

Lemma 3.2. *If Y is a dense subspace of a regular space X, then $\chi(y,Y) = \chi(y,X)$ for each $y \in Y$.*

From Proposition 3.1 and Lemma 3.2, the proof of the following proposition is obvious, and hence we omitted the proof.

Proposition 3.2. *If H is a metrizable rectifiable subspace of a rectifiable space G, then the closure of H is also metrizable.*

A space X is a *submaximal space* if every subset A is open in $\overline{A}$. The following Proposition 3.3 and the corollaries were proved in [2] for topological groups.

Proposition 3.3. *Every rectifiable subspace of a submaximal rectifiable space G is closed.*

Proof. Let H be a rectifiable subspace of G, and let $Y = \overline{H}$. By Proposition 3.1, the space Y is a rectifiable space. Moreover, the set H is open in Y since G is submaximal. It follows from Lemma 3.1 that H is closed in Y. Therefore, H is closed in G. ∎

Corollary 3.1. *If A is dense subset of a submaximal rectifiable space G, then $G = \bigcup_{n\in\mathbb{N}}(A_n \cup B_n)$, where $A_1 = A, B_1 = q(A,e)\cup q(A,A), A_2 = p(A_1\cup B_1, A_1\cup B_1), B_2 = q(A_1\cup B_1, A_1\cup B_1)$, $A_{n+1} = p(A_n\cup B_n, A_n\cup B_n), B_{n+1} = q(A_n\cup B_n, A_n\cup B_n), n = 1,2,\cdots$.*

Proof. Since $q(A,A)\subset B_1$, we have $e\in B_1$. Therefore, it is easy to see that $A_n\cup B_n\subset A_{n+1}\cup B_{n+1}$ for each $n\in\mathbb{N}$. Put $B = \bigcup_{n\in\mathbb{N}}(A_n\cup B_n)$. Next we shall prove that B is a rectifiable subspace of G. Indeed, take arbitrary points $x,y\in B$. Then there exists an $n\in\mathbb{N}$ such that $x,y\in A_n\cup B_n$, and hence $p(x,y)\in A_{n+1}\cup B_{n+1}$ and $q(x,y)\in A_{n+1}\cup B_{n+1}$. Therefore, B is a rectifiable subspace of G. By Proposition 3.3, B is closed in G. Moreover, it is obvious that $A\subset B$, and thus $B = G$. ∎

Corollary 3.2. *The density of a submaximal rectifiable space G is equal to the cardinality of G.*

Corollary 3.3. *Every separable submaximal rectifiable space is countable.*

Remark 3.1. (1) The 7-dimensional sphere S_7 is a rectifiable space which is not submaximal. Indeed, S_7 is separable, and however, the cardinality of S_7 is uncountable.

(2) Mrowka-Isbell space is a separable submaximal Tychonoff uncountable space. Therefore, Corollary 3.3 is not necessarily true if G is just a Tychonoff space.

Question 3.1. Is every pseudocompact submaximal rectifiable space finite?

Lemma 3.3. *Every non-discrete subrectifiable H of a submaximal rectifiable space X is open.*

Proof. Since H is non-discrete, it follows from [2, Theorem 1.2] that H contains a non-empty open subset, which implies H is open in G since H is a rectifiable subspace. ∎

A space X is said to have a *regular G_δ-diagonal* if the diagonal $\Delta = \{(x,x) : x\in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of Δ in $X\times X$.

The following Lemma 3.4 generalizes Arhangel'skiĭ and Burke's result for Abelian paracompact groups.

Lemma 3.4. [11] *Each rectifiable space G with countable pseudocharacter has a regular G_δ-diagonal.*

Theorem 3.1. *If G is a submaximal rectifiable space, then, either G has a regular G_δ-diagonal, or G is a P-space (that is, all G_δ-set are open in G).*

Proof. For each countable family $\zeta = \{U_n : n\in\mathbb{N}\}$ of neighborhoods of the right neutral element e of the rectifiable space G, then there is a closed rectifiable subspace H of G, contained in the intersection of ζ, which is a G_δ-subset of the space G. Indeed, we first may assume that $U_{n+1}\subset U_n$ for each $n\in\mathbb{N}$. By induction, it is easy to see from the joint continuous of p and q that there exists a decreasing open neighborhoods $\{V_n : n\in\mathbb{N}\}$ of the right neutral element e such that the following conditions are satisfy:

(1) For each $n\in\mathbb{N}$, $p(\overline{V_{n+1}},\overline{V_{n+1}})\subset V_n$ and $q(V_{n+1},V_{n+1})\subset V_n$;

(2) For each $n\in\mathbb{N}$, $V_n\subset U_n$.

Put $H = \bigcap_{n\in\mathbb{N}} V_n$. Obviously, H is closed G_δ-subset in G and $H \subset \bigcap_{n\in\mathbb{N}} U_n$. Next, we shall show that H is a rectifiable subspace of G. Take arbitrary points $x,y \in H$. Then $x,y \in V_n$ for each $n \in \mathbb{N}$. Hence $p(x,y) \in V_n$ and $q(x,y) \in V_n$ for each $n \in \mathbb{N}$, and thus $p(x,y) \in H$ and $q(x,y) \in H$. Therefore, H is a closed rectifiable subspace of G. If H is discrete, then G has a regular G_δ-diagonal. In fact, since $(\bigcap_{n\in\mathbb{N}} V_n) \cap (G - (H - \{e\})) = \{e\}$, it follows from the homogeneity of G that every point of G is G_δ. By Lemma 3.4, G has a regular G_δ-diagonal. If H is non-discrete, then H is open by Lemma 3.3. Therefore, G is a P-space. ∎

Remark 3.2. We can not omit the assumption "G is rectifiable" in Theorem 3.1. Indeed, the Alexandroff one-point compactification X of an uncountable discrete space is a submaximal space which is not metrizable. It is well known that a compact space with a G_δ-diagonal is metrizable or a compact P-space is finite [8]. Then X is not a P-space and does not have any G_δ-diagonal since G is not metrizable.

Corollary 3.4. *Every connected submaximal rectifiable space G has a regular G_δ-diagonal.*

Proof. If G is finite, then G is discrete, hence it has a regular G_δ-diagonal. If G is infinite, then G is not a P-space since an infinite P-space is not connected. By Theorem 3.1, G has a regular G_δ-diagonal. ∎

Recall that a space X is of *point-countable type* if each point of X is contained in a compact subspace of X with a countable base of neighborhoods in X.

Theorem 3.2. *Let G be a submaximal rectifiable space. If G is locally countably compact or is of point-countable type, then G is metrizable.*

Proof. It follows from Theorem 3.1 that G has a regular G_δ-diagonal, or G is a P-space.

Case 1: G has a regular G_δ-diagonal. If G is locally countably compact, then G is discrete since a countably compact space with a G_δ-diagonal is metrizable [8]; If G is a P-space, then there exists a compact subset F of G such that $e \in F$ and F has a countable neighborhood base. Since compact space space with a G_δ-diagonal is metrizable [8], it is easy to see that G is first-countable, and hence G is metrizable by [9].

Case 2: G is a P-space. It is well known that a countably compact P-space is finite, and therefore G is metrizable. ∎

Note By remark 3.2, we know that the condition "G is rectifiable" is important.

4. Rectifiable spaces determined by a point-countable cover

Let $\mathscr{P}$ be a family of subsets of a space X. The family $\mathscr{P}$ is called a *k-network* [15] if whenever K is a compact subset of X and $K \subset U \in \tau(X)$, there is a finite subfamily $\mathscr{P}' \subset \mathscr{P}$ such that $K \subset \cup\mathscr{P}' \subset U$.

In [12], Lin and Shen posed the following question:

Question 4.1. [12] Is every sequential rectifiable space with a point-countable k-network a paracompact space?

In this section, we shall give some partial answer for Question 4.1. Firstly, we give some lemmas.

Lemma 4.1. [12] *Let G be a rectifiable space. Then G contains a (closed) copy of S_ω if and only if G has a (closed) copy of S_2.*

Lemma 4.2. [14] *Let G be a space determined by a point-countable cover $\mathscr{G}$. Suppose that finite unions of elements of $\mathscr{G}$ are either Fréchet-Urysohn or sequential spaces in which each point is a G_δ-set. If G contains neither a closed copy of S_ω nor a closed copy of S_2, then each point of G has an open neighborhood covered by some finite subfamily of $\mathscr{G}$.*

Lemma 4.3. [14] *A space determined by a cover consisting of sequential space is sequential.*

The following Proposition 4.1 corresponds to [14, Corollary 2.6].

Proposition 4.1. *Let G be a rectifiable space determined by a countable increasing cover $\mathscr{G} = \{G_n : n \in \mathbb{N}\}$ consisting of its subrectifiable spaces. Assume also that one of the following conditions holds:*

(1) *each G_n is Fréchet-Urysohn, or*
(2) *each G_n is sequential and the right neutral element e is a G_δ-subset of G_n.*

Then either

(i) G contains a closed copy of S_ω, or
(ii) there exists an $n \in \mathbb{N}$ such that G_n is both open and closed in G.

Proof. It is easy to see that the family $\mathscr{G}$ satisfies the assumptions of Lemma 4.2. Suppose that G contains no a closed copy of S_w. Then G contains no closed copy of S_2 by Lemma 4.1. It follows from Lemma 4.2 that we have $e \in \mathrm{Int}G_n$ for some $n \in \mathbb{N}$. Therefore, we have $G_n = \cup\{x \cdot \mathrm{Int}G_n : x \in G_n\}$ is open in G. By Lemma 3.1, the set G_n is also closed in G. ∎

Theorem 4.1. *Let G be a rectifiable space determined by a point-countable sequential subrectifiable subsets with a point-countable k-network. Then G is metrizable if G contains no closed copy of S_ω.*

Proof. Let G be determined by a point-countable sequential subrectifiable subsets $X_\alpha (\alpha \in \Gamma)$, where each X_α has a point-countable k-network. Since each X_α contains no closed copy of S_ω, then each X_α is weakly first-countable [13], and hence each X_α is first-countable and has a point-countable base [10]. Let $f : \bigoplus_{\alpha\in\Gamma} X_\alpha \to G$ be a obvious map. Then f is a quotient countable-to-one map. Hence G is a sequential space with a point-countable k-network. Therefore, G is weakly first-countable since G contains no closed copy of S_ω and G is metrizable. ∎

A prefilter $\mathscr{U}$ in a topological space X is called a *nest* [1] if $\mathscr{U}$ consists of open subsets of X and has the following property: For any $U, V \in \mathscr{U}$ either $U \subset V$ or $V \subset U$. A space X is *π-nested* at a point $x \in X$ [1] if there exists a nest in X converging to x, and X is *nested* at x [1] if there exists a nest in X which forms a local base for X at x. Finally, a space is nested if it is nested at each of its points.

Lemma 4.4. *If a rectifiable space G is π-nested at some point $a \in G$ then G is nested.*

Proof. Since G is homogeneous, we may assume that a is the right neutral element e of the rectifiable space G. Take a nest $\mathscr{A}$ converges to e. Put $\phi = \{q(U,U) : U \in \mathscr{A}\}$. Then ϕ is a nest which is a base for G at the point e. By the homogeneity of the space G we have G is nested. ∎

Lemma 4.5. [14] *Let U be a nonempty open subset of a space X, and let ϕ be a finite family of subsets of X such that $U \subset \cup\phi$. Then there exist $E \in \phi$ and a nonempty open subset V of X with $V \subset \overline{V \cap E}$.*

The following Lemma 4.6 corresponds to Theorem 2.9 in [14].

Lemma 4.6. *Assume that G is a rectifiable space determined by a point-countable cover $\mathscr{G}$ consisting of bisequential spaces. Assume also that*

(1) *$\cup\mathscr{G}'$ is Fréchet-Urysohn for each finite subfamily $\mathscr{G}' \subset \mathscr{G}$, and*
(2) *G contains no closed copy of S_ω.*

Then G is metrizable.

Proof. If G is discrete, then it is obvious that G is metrizable. Suppose that G is a non-discrete space. Obviously, $\mathscr{G}$ and G satisfy the assumptions of Lemma 4.2. Therefore, there are a finite subfamily $\mathscr{E} \subset \mathscr{G}$ and an open subset U with $e \in U \subset \cup\mathscr{E}$. It follows from Lemma 4.5 that there exist some $E \in \mathscr{E}$ and a non-empty open subset V such that $V \subset \overline{V \cap E}$. Take a point $x \in V \cap E$. Since $V \cap E$ is an open subspace of bisequential space E, the subspace $V \cap E$ is bisequential. Therefore, $V \cap E$ is π-nested at x by [1, Proposition 1]. It follows from [1, Lemma 20] that $\overline{V \cap E}$ is also π-nested at x. Since $x \in V \subset \overline{V \cap E}$, the set V is π-nested at x too. Therefore, we obtained that G is π-nested at x. Since G is a rectifiable space, the space G is nested by Lemma 4.4. By Lemma 4.3, the space G is sequential, and hence G contains a non-trivial convergent sequence. Therefore, the space G must have a G_δ-subset which is not open. Since G is nested, the space G is first-countable by [1, Lemma 10]. So G is metrizable. ∎

Lemma 4.7. [11] *Let G be a rectifiable space. Then G is an α_4-sequential space if and only if G is strongly Fréchet-Urysohn.*

The following Theorem 4.2 corresponds to [14, Theorem 3.1].

Theorem 4.2. *Let G be a rectifiable space determined by a point-countable cover $\mathscr{G}$ consisting of bisequential spaces. Then the following conditions are equivalent:*

(1) *The space G is an α_4-space;*
(2) *The space G is Fréchet-Urysohn;*
(3) *The space G is metrizable.*

Proof. Implication (3)⇒(2) is trivial, and for (2)⇒(1), see Lemma 4.7. It is suffice to show (1)⇒(3).

(1)⇒(3). From Lemmas 4.3 and 4.7, it follows that G is strongly Fréchet-Urysohn. In particular, we have $\cup\mathscr{G}'$ is Fréchet-Urysohn for each finite $\mathscr{G}' \subset \mathscr{G}$. Hence item (1) of Lemma 4.6 holds. Since closed subsets of a strongly Fréchet-Urysohn space are strongly Fréchet-Urysohn, the space G contains no closed copy of S_ω. It follows from Lemma 4.6 that G is metrizable. ∎

Lemma 4.8. [14] *If a space X is determined by a point-finite cover $\mathscr{U}$ consisting of α_4-spaces, then X is an α_4-space itself.*

Corollary 4.1. *If a rectifiable space G is determined by a point-finite cover $\mathscr{G}$ consisting of bisequential subspaces of G, then G is metrizable.*

Proof. Since bisequential spaces are α_4, it follows from Lemma 4.8 that G is α_4. By Theorem 4.2, the space G is metrizable. ∎

The following Theorem 4.3 corresponds to [14, Theorem 3.2].

Theorem 4.3. *Let G be a rectifiable space determined by a point-countable cover $\mathscr{G}$ consisting of closed bisequential spaces. Then the following conditions are equivalent:*

(1) *The space G contains no closed copy of S_ω;*
(2) *The space G contains no closed copy of S_2;*
(3) *The space G is metrizable.*

Proof. It follows from Lemma 4.1 that (1)⇔(2). (3)⇒(1) is obvious.

(1)⇒(3). Since every element $D \in \mathscr{G}$ is closed bisequential, we have $\cup\mathscr{G}'$ is Fréchet-Urysohn for each finite subfamily $\mathscr{G}' \subset \mathscr{G}$. It follows from Lemma 4.6 that G is metrizable. ∎

The following Theorem 4.4 corresponds to [14, Theorem 3.3].

Theorem 4.4. *Let G be a rectifiable space determined by a countable increasing cover $\mathscr{G} = \{G_n : n \in \mathbb{N}\}$ consisting of bisequential subspaces of G. Then the following conditions are equivalent:*

(1) *The space G contains no closed copy of S_ω;*
(2) *The space G contains no closed copy of S_2;*
(3) *The space G is metrizable.*

Proof. Note that each finite subfamily $\mathscr{G}' \subset \mathscr{G}$ contains in some G_n. Therefore, it is easy to see that our theorem holds. ∎

Remark 4.1. (1) [14, Example 3.5] demonstrates that in Theorems 4.2, 4.3 and 4.4 at least some countability restrictions on the cover are necessary.

(2) In Theorems 4.2, 4.3 and 4.4, we have to add some additional restrictions on G in order to G is metrizable. [14, Example 3.7] demonstrates that at least some additional restrictions on G are indeed necessary.

(3) We can not generalize Theorems 4.3 and 4.4 in the class of Maltsev spaces. Indeed, Sorgenfrey line $\mathscr{S}$ is a Maltsev space which is not metrizable, and $\mathscr{S}$ contains no copy of S_2 and S_ω since it is first-countable.

It follows from Theorem 4.3 or Theorem 4.4, we have the following corollary.

Corollary 4.2. [12] *Every bisequential rectifiable space is metrizable.*

A topological space X is said to be an *$\mathscr{M}_\omega$-space* if X is determined by a countable family $\mathscr{M}$ of closed metrizable subsets of X.

Corollary 4.3. *Let G be an $\mathscr{M}_\omega$-rectifiable space. If G contains no closed copy of S_ω, then the space G is metrizable.*

However, we cannot omit the condition "G contains no closed copy of S_ω" in Corollary 4.3. In fact, if X a convergent sequence containing the limit point then the free topological group $F(X)$ is an $\mathscr{M}_\omega$-space, but $F(X)$ is non-metrizable.

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