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A Note on the Product of Element Orders of Finite Abelian Groups

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Abstract. Given a finite group G, we denote by $\psi'(G)$ the product of element orders of G. Our main result proves that the restriction of ψ' to abelian p-groups of order p^n is strictly increasing with respect to a natural order on the groups relating to the lexicographic order of the partitions of n. In particular, we infer that two finite abelian groups of the same order are isomorphic if and only if they have the same product of element orders.

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1. Introduction

Let G be a finite group and

$$\psi(G) = \sum_{x \in G} o(x),$$

where $o(x)$ denotes the order of $x \in G$. The starting point for our discussion is given by the papers [1, 2] which investigate the minimum/maximum of ψ on groups of the same order. Other properties of the function ψ have been studied in [10] for finite abelian groups.

In the current note we will focus on the function

$$\psi'(G) = \prod_{x \in G} o(x).$$

In contrast with ψ, this is not multiplicative, as shown by the following result.

Proposition 1.1. *Let $G_1, G_2, \dots, G_k$ be finite groups having coprime orders. Then*

$$\psi'\left(\bigtimes_{i=1}^{k} G_i\right) = \prod_{i=1}^{k} \psi'(G_i)^{n_i},$$

where $n_i = \prod_{j=1, j\neq i}^{k} \mid G_j \mid, i = 1, 2, \dots, k$. In particular, if G is a finite nilpotent group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $G_1, G_2, \dots, G_k$ are the Sylow subgroups of G and $n_i = n/p_i^{\alpha_i}, i = 1, 2, \dots, k$, then

$$\psi'(G) = \prod_{i=1}^{k} \psi'(G_i)^{n_i}.$$

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By Proposition A we infer that the computation of $\psi'(G)$ for nilpotent groups is reduced to p-groups and explicit formulas can be given in several particular cases. One of them consists of abelian groups, for which [9, Corollary 4.4] leads to the following theorem.

Theorem 1.1. *Let $G = \bigtimes_{i=1}^{k} \mathbb{Z}_{p^{\alpha_i}}$ be a finite abelian p-group, where $1 \le \alpha_1 \le \alpha_2 \le \dots \le \alpha_k$. Then*

$$\psi'(G) = p^{\alpha_k p^{\alpha_1+\alpha_2+\dots+\alpha_k} - \sum_{i=0}^{\alpha_k-1} p^i f_{(\alpha_1,\alpha_2,\dots,\alpha_k)}(i)}, \tag{1.1}$$

where

$$f_{(\alpha_1,\alpha_2,\dots,\alpha_k)}(i) = \begin{cases} p^{(k-1)i}, & \text{if } 0 \le i \le \alpha_1 \\ p^{(k-2)i+\alpha_1}, & \text{if } \alpha_1 \le i \le \alpha_2 \\ p^{(k-3)i+\alpha_1+\alpha_2}, & \text{if } \alpha_2 \le i \le \alpha_3 \\ \vdots & \\ p^{\alpha_1+\alpha_2+\dots+\alpha_{k-1}}, & \text{if } i \ge \alpha_{k-1}\, (\text{where } \alpha_{k-1} = 1 \ \text{if } \ k = 1). \end{cases}$$

We exemplify (1.1) by computing $\psi'(G)$ for cyclic p-groups and for rank two abelian p-groups.

Example 1.1. We have:

1. $\psi'(\mathbb{Z}_{p^\alpha}) = p^{(\alpha p^{\alpha+1}-(\alpha+1)p^\alpha+1)/(p-1)}$;
2. $\psi'(\mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{p^\beta}) = p^{(\beta p^{\alpha+\beta+2}-p^{\alpha+\beta+1}-(\beta+1)p^{\alpha+\beta}+p^{2\alpha+1}+1)/(p^2-1)}$.

Given a positive integer n, it is well-known that there is a bijection between the set of types of abelian groups of order p^n and the set $P_n = \{(x_1,x_2,\dots,x_n) \in \mathbb{N}^n \mid x_1 \ge x_2 \ge \dots \ge x_n, x_1+x_2+\dots+x_n = n\}$ of partitions of n. Namely,

$$\bigtimes_{i=1}^{k} \mathbb{Z}_{p^{\alpha_i}} (\text{with } 1 \le \alpha_1 \le \alpha_2 \le \dots \le \alpha_k \text{ and } \sum_{i=1}^{k} \alpha_i = n) \mapsto (\alpha_k,\dots,\alpha_1, \underbrace{0,\dots,0}_{n-k \text{ positions}}).$$

Moreover, recall that P_n is totally ordered under the lexicographic order $\preceq$, where $(x_1,x_2,\dots,x_n) \prec (y_1,y_2,\dots,y_n)$ if and only if $x_1 = y_1,\dots,x_m = y_m$ and $x_{m+1} < y_{m+1}$ for some $m \in \{0,1,\dots,n-1\}$. This induces a total order on the set of types of abelian p-groups of order p^n. Our next theorem shows that the restriction of ψ' on this set is strictly increasing.

Theorem 1.2. *Let $G = \bigtimes_{i=1}^{k} \mathbb{Z}_{p^{\alpha_i}}$ and $H = \bigtimes_{j=1}^{r} \mathbb{Z}_{p^{\beta_j}}$ be two finite abelian p-groups of order p^n. Then*

$$\psi'(G) < \psi'(H) \iff (\alpha_k,\dots,\alpha_1, \underbrace{0,\dots,0}_{n-k \text{ positions}}) \prec (\beta_r,\dots,\beta_1, \underbrace{0,\dots,0}_{n-r \text{ positions}}). \tag{1.2}$$

Since a strictly increasing function is injective, by Theorem 1.2 we infer the following corollary.

Corollary 1.1. *Two finite abelian p-groups of order p^n are isomorphic if and only if they have the same product of element orders.*

This can be extended to arbitrary finite abelian groups, according to Proposition 1.1.

Theorem 1.3. *Two finite abelian groups of the same order are isomorphic if and only if they have the same product of element orders.*

Remark 1.1. The above two results are not true for *arbitrary* finite abelian groups. For example, we have $\psi'(\mathbb{Z}_4 \times \mathbb{Z}_3^2) = \psi'(\mathbb{Z}_2^4 \times \mathbb{Z}_3)$, but obviously the groups $\mathbb{Z}_4 \times \mathbb{Z}_3^2$ and $\mathbb{Z}_2^4 \times \mathbb{Z}_3$ are not isomorphic.

Finally, we associate to a finite (abelian) group $G = \{x_1, x_2, ..., x_n\}$ the polynomial

$$P_G = \prod_{i=1}^{n} (X - o(x_i)) \in \mathbb{Z}[X].$$

Recall that if G and H are two finite abelian groups for which $P_G = P_H$ (that is, G and H have the same element orders with the same multiplicities), then $G \cong H$ by [8, Theorem 5]. In order to improve this result, we construct the quantities

$$\psi_k(G) = \sum_{1 \le i_1 < i_2 < ... < i_k \le n} o(x_{i_1}) o(x_{i_2}) \cdots o(x_{i_k}), \quad k = 1, 2, ..., n.$$

Obviously, explicit formulas for $\psi_k(G)$, $k = 1, 2, ..., n$, can be given by using [9, Corollary 4.4]. We also observe that $\psi_1(G) = \psi(G)$ and $\psi_n(G) = \psi'(G)$. Inspired by [10, Conjecture 6] and the above Theorem 1.3, we came up with the following conjecture, which we have verified by GAP for many values of k and n.

Conjecture 1.1. *Let G and H be two finite abelian groups of order n. Then for every $k \in \{1, 2, ..., n\}$, we have $G \cong H$ if and only if $\psi_k(G) = \psi_k(H)$.*

Most of our notation is standard and will not be repeated here. For basic notions and results of group theory we refer the reader to [4, 5]. Other interesting papers on the above topic are [3, 6, 7].

2. Proof of Theorem 1.2

We first remark that it suffices to verify (1.2) only for consecutive partitions of n, because P_n is totally ordered. Assume that $(\alpha_k, ..., \alpha_1, 0, ..., 0) \prec (\beta_r, ..., \beta_1, 0, ..., 0)$ and let $s \in \{1, 2, ..., r-1\}$ such that $\beta_1 = \beta_2 = \cdots = \beta_s < \beta_{s+1}$. We distinguish the following two cases.

Case 1. $\beta_1 \ge 2$

Then $(\alpha_k, ..., \alpha_1, 0, ..., 0)$ is of type $(\beta_r, ..., \beta_2, \beta_1 - 1, 1, 0, ..., 0)$, i.e. $k = r + 1$, $\alpha_1 = 1$, $\alpha_2 = \beta_1 - 1$ and $\alpha_i = \beta_{i-1}$ for $i = 3, 4, ..., r+1$. It is easy to see that $f_{(\alpha_1, \alpha_2, ..., \alpha_k)}(i) = f_{(\beta_1, \beta_2, ..., \beta_r)}(i)$ for all $i \ge \beta_1$. One obtains

$$\begin{aligned} \psi'(G) < \psi'(H) &\Longleftrightarrow \sum_{i=1}^{\beta_r - 1} p^i f_{(\beta_1, \beta_2, ..., \beta_r)}(i) < \sum_{i=1}^{\alpha_k - 1} p^i f_{(\alpha_1, \alpha_2, ..., \alpha_k)}(i) \\ &\Longleftrightarrow \sum_{i=1}^{\beta_1 - 1} p^i f_{(\beta_1, \beta_2, ..., \beta_r)}(i) < \sum_{i=1}^{\beta_1 - 1} p^i f_{(\alpha_1, \alpha_2, ..., \alpha_k)}(i) \end{aligned}$$

and the last inequality is true because

$$f_{(\beta_1, \beta_2, ..., \beta_r)}(i) = p^{(r-1)i} < p^{(r-1)i+1} = p^{(k-2)i+1} = f_{(\alpha_1, \alpha_2, ..., \alpha_k)}(i), \quad i = 1, 2, ..., \beta_1 - 1\,.$$

Case 2. $\beta_1 = 1$

Then $(\alpha_k,\dots,\alpha_1,0,\dots,0)$ is of type $(\beta_r,\dots,\beta_{s+1}-1,\beta'_t,\beta'_{t-1},\dots,\beta'_1,0,\dots,0)$, where $\beta_{s+1}-1\geq\beta'_t\geq\beta'_{t-1}\geq\dots\geq\beta'_1\geq 1$ and $\beta'_t+\beta'_{t-1}+\dots+\beta'_1=s+1$. We infer that $f_{(\alpha_1,\alpha_2,\dots,\alpha_k)}(i)=f_{(\beta_1,\beta_2,\dots,\beta_r)}(i)$ for all $i\geq\beta_{s+1}$. So, we can suppose that $s=r-1$, i.e.

$$(\alpha_k,\dots,\alpha_1,0,\dots,0)=(\beta_r-1,\beta'_t,\beta'_{t-1},\dots,\beta'_1,0,\dots,0).$$

One obtains

$$\psi'(G)<\psi'(H)\iff(\beta_r-1)p^n-\sum_{i=0}^{\beta_r-2}p^if_{(\alpha_1,\alpha_2,\dots,\alpha_k)}(i)<\beta_rp^n-\sum_{i=0}^{\beta_r-1}p^if_{(\beta_1,\beta_2,\dots,\beta_r)}(i)$$

$$\iff p^n-\sum_{i=0}^{\beta_r-1}p^if_{(\beta_1,\beta_2,\dots,\beta_r)}(i)+\sum_{i=0}^{\beta_r-2}p^if_{(\alpha_1,\alpha_2,\dots,\alpha_k)}(i)>0. \tag{2.1}$$

Since

$$f_{(\beta_1,\beta_2,\dots,\beta_r)}(i)=\begin{cases}p^{(r-1)i}, & \text{if } 0\leq i\leq 1\\ p^{r-1}, & \text{if } i\geq 1,\end{cases}$$

we easily get

$$\sum_{i=0}^{\beta_r-1}p^if_{(\beta_1,\beta_2,\dots,\beta_r)}(i)=1+p^{r-1}\frac{p^{\beta_r}-p}{p-1}=1+\frac{p^n-p^r}{p-1}<p^n$$

and therefore (2.1) is true.

Conversely, assume that $(\alpha_k,\alpha_{k-1},\dots,\alpha_1,0,\dots,0)\succeq(\beta_r,\beta_{r-1},\dots,\beta_1,0,\dots,0)$. If these partitions are equal, then $G\cong H$, so $\psi'(G)=\psi'(H)$. Otherwise, $(\alpha_k,\alpha_{k-1},\dots,\alpha_1,0,\dots,0)\succ(\beta_r,\beta_{r-1},\dots,\beta_1,0,\dots,0)$, and the first part of the proof implies $\psi'(G)>\psi'(H)$, as desired.

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